On three trigonometric integrals of \( \ln \Gamma(x) \)
or its derivative.

K.S. Kölblig

ABSTRACT. Three definite integrals of products of \( \ln \Gamma(x) \) or \( \psi(x) \) with trigonometric functions have been wrongly believed to be zero for nearly a century and are given in this form even in the most recent tables. A simple result is given for one of these integrals, the result for the two others being expressable in terms of an infinite series.
In the most recent edition of the well-known integral table of Gradsheyn–Ryzhik [2, No. 6.443 4, 6.465 1, 6.469 2], one can find the three integrals

\[
\int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x \, dx = 0 \quad (n \in \mathbb{N}_0),
\]

\[
\int_0^1 \psi(x) \sin \pi x \, dx = 0,
\]

\[
\int_0^1 \psi(x) \sin 2n\pi x \, dx = 0 \quad (n \in \mathbb{N}),
\]

where, as usual, \( \Gamma(x) \) is the gamma function and \( \psi(x) \) its logarithmic derivative.

These integrals, in a slightly different form, are also given in the table of Prudnikov et al [5, No. 2.2.3.7, 2.3.1.7, 2.3.1.9]. As has been indicated in [2], they go back to Nielsen [3, pp. 203–204].

In spite of the fact that these integrals have been transmitted in this vanishing form for nearly a century, a numerical check reveals immediately that they are incorrect, and we shall give correct expressions for them in this note. However, only the integral (3) has been found in closed form. The result for the related integrals (1) and (2) contains an infinite series which does not seem to be expressible in terms of well-known functions.

We start by showing that (1) should be replaced by

\[
\int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x \, dx =
\]

\[
\frac{2}{\pi^2} \left[ \frac{1}{(2n+1)^2} (\gamma + \ln 2\pi) + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - (2n+1)^2} \right] \quad (n \in \mathbb{N}_0),
\]

where \( \gamma = 0.57721 \ldots \) is the Euler constant.

To show this, we use the Fourier expansion [1, p. 406]

\[
\ln \Gamma(x) = \ln \sqrt{2\pi} + \sum_{k=1}^{\infty} \left[ \frac{1}{2k} \cos 2\pi k x + \frac{1}{k\pi} (\gamma + \ln 2\pi k) \sin 2\pi k x \right]
\]

which is uniformly convergent in any closed interval inside \( 0 < x < 1 \). We may therefore [1, pp. 406–407] integrate term by term, to obtain

\[
\int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x \, dx = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \left[ \frac{\gamma + \ln 2\pi}{4k^2 - (2n+1)^2} + \frac{\ln k}{4k^2 - (2n+1)^2} \right].
\]

which, using [4, No. 5.1.25.4],

\[
\sum_{k=1}^{\infty} \frac{1}{4k^2 - (2n+1)^2} = \frac{1}{2} \frac{1}{(2n+1)^2},
\]

yields the result stated.
For the integral (2), we have by partial integration
\[ 
\int_0^1 \psi(x) \sin(2n + 1)\pi x \, dx = -(2n + 1)\pi \int_0^1 \ln \Gamma(x) \cos(2n + 1)\pi x \, dx, 
\]
which, on setting \( n = 0 \) in (4), gives
\[ 
\int_0^1 \psi(x) \sin \pi x \, dx = -\frac{2}{\pi} \left[ \gamma + \ln 2\pi + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - 1} \right]. 
\]
Finally, for the integral (3), we have for \( n \in \mathbb{N} \)
\[ 
\int_0^1 \psi(x) \sin n\pi x \, dx = \begin{cases} 
\frac{1}{2} (\gamma + \ln 2\pi) & (n = 1) \\
\frac{1}{2} \ln \frac{n - 1}{n + 1} & (n \text{ > 1 odd}) \\
\frac{n}{1 - n^2} & (n \text{ even}) 
\end{cases}
\]
The results for \( n \) odd are given correctly in [2, No. 6.468, 6.469 2] and in [5, No. 2.3.1.9, 10]. To derive the result for \( n \) even, we use a simple trigonometric identity and obtain by partial integration, following Nielsen [3, p. 202],
\[ 
\frac{2}{\pi} \int_0^1 \psi(x) \sin \pi x \sin 2m\pi x \, dx = (2m - 1) \int_0^1 \ln \Gamma(x) \sin(2m - 1)\pi x \, dx - (2m + 1) \int_0^1 \ln \Gamma(x) \sin(2m + 1)\pi x \, dx.
\]
Use of the formula [2, No. 6.443 2]
\[ 
\int_0^1 \ln \Gamma(x) \sin(2m + 1)\pi x \, dx = \frac{1}{(2m + 1)\pi} \left[ \ln \frac{\pi}{2} + \frac{1}{2m + 1} + 2 \sum_{j=1}^{m} \frac{1}{2j - 1} \right]
\]
(the corresponding formula in [3, p. 202] is incorrect), leads to the result stated. It is remarkable that this integral of transcendental functions has a value which is rational for \( n \) even and irrational for \( n \) odd.

References

der Wissenschaften, Berlin, 1987).


