HAMILTONIAN TREATMENT OF SYNCHROBETATRON RESONANCES

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Abstract
Different mechanisms may couple the equations of betatron motion and synchrotron motion through the physical quantities of accelerators such as chromaticity and dispersion. They generate synchro-betatron resonances (SBR) that can be seen as sidebands of the pure betatron resonances. The importance of having convenient formulae to analyse their damaging effects, to make numerical evaluations, and to find possible corrections is emphasized. It is shown how the Hamiltonian formalism can be used to reach these goals. Direct applications to design studies or functioning accelerators are given, together with some observations.

1 INTRODUCTION

In circular accelerators and colliders, coupling between synchrotron (longitudinal) and betatron (transverse) motions gives rise to nonlinear resonances, termed synchro-betatron resonances (SBR). It generates a web of sidebands or satellites beside the more familiar betatron resonances associated with electromagnetic fields [1]. Different mechanisms may be at the origin of this coupling.

One possible source of coupling is linked to time variations of the betatron tunes, due to the simultaneous presence of momentum oscillations and finite chromaticity or tune dependence with energy. The treatment, based on unpublished work carried out by the author [2] during the design of the proton–antiproton facility in the Super Proton Synchrotron (SPS) at CERN, is given in detail for one-dimensional transverse oscillations in the presence of nonlinear electromagnetic fields, caused by magnetic multipoles or beam–beam forces. The corresponding field potential (see the Appendix) can be written, after an expansion to order $N$

$$H_1 = \sum_{N} \frac{1}{N!} K^{(N-1)} x^N,$$

with

$$K^{(N-1)} = \frac{R^2}{|B\rho|} \frac{\delta^{(N-1)} B_z}{\partial x^{(N-1)}}$$

for 2N–pole components,

and

$$K^{(N-1)} = \frac{1}{\beta c} \frac{R^2}{|B\rho|} \frac{\partial^N \Phi}{\partial x^N}$$

for beam–beam potential $\Phi$.

The stability of the motion is discussed analytically, and emphasis is given on how to use a few simple results for rapid numerical evaluations and practical applications.

Another source of synchrobetatron coupling finds its origin in the existence of finite dispersion at the RF accelerating cavities. The local change of energy induces a jump of
the equilibrium orbit, and consequently additional betatron excursions, while the variation of the orbit length provokes phase shifts of the beam with respect to the acceleration voltage. The summary which is presented below is based on the elegant Hamiltonian treatment of this mechanism published by T. Suzuki [3], and revised more recently by R. Baartman [4]. This treatment offers all the advantages linked to the Hamilton mechanics, i.e. the preservation of the symplectic conditions and the availability of many general results already established [1]. Analytical outcome is given in the form of a few formulae that make numerical evaluations easier, and provide useful clues on how to compensate or keep under control the effects of SBR caused by dispersion in cavities. Examples of application are given in order to illustrate the validity of the theory, and its usefulness in a design period to study the resonance effects and their compensations. Observations made in a running accelerator are also summarized.

For completeness, let us mention yet another mechanism, though not treated in this article, which is related to the longitudinal wakefields generated by a beam interacting electromagnetically with the cavities. Starting from the equations of the coupled synchrobetatron motion rather than from the Hamilton function, Ref. [5] gives analytic expressions of the vertical displacement and its derivative after n turns, for one particle in a gaussian bunch and a single RF cavity.

2 TUNE MODULATION AND SYNCHROBETRON RESONANCES

2.1 Time varying betatron tune

In this section, we shall deal with the one-dimensional betatron motion and explain how variations of the transverse tunes related to synchrotron oscillations (see the Appendix) may induce SBR. The perturbation treatment is applied, and the motion and its stability are investigated analytically, near the resonances.

The reasons for a time variation of betatron tunes are linked to a series of factors recalled hereafter:

- As it is well known, the magnetic deflection associated with a constant external field $B_0$ is inversely proportional to the energy (or momentum) of the particle.

- As a direct consequence of the first factor, the focusing strength of the quadrupoles depends on the actual energy of the particles inside the beam.

- The RF acceleration system generates energy oscillations (synchrotron motion) of the particles with a wave number per turn (synchrotron tune) $Q_s$.

- As a result of the two preceding points, i.e. energy oscillations and focusing variation with momentum, it follows that any of the two transverse tunes $Q$ ($Q_x$ or $Q_y$) may in general oscillate at a frequency given by the synchrotron tune $Q_s$.

- The $Q$-oscillation is not necessarily linear, since nonlinear magnetic fields (such as sextupoles or higher multipoles) generate a nonlinear dependence of $Q$ on the momentum $p$ or momentum deviation $\Delta p/p_0$.

Figure 1 gives an illustration of an arbitrary nonlinear chromatic dependence of one transverse tune $Q$ with the momentum deviation. The linear variation around the
origin \( Q = Q_0 \), and \( \Delta p = 0 \) corresponds to the finite slope of the curve \( Q(\Delta p/p_0) \), and is characterized by the so-called chromaticity in the accelerator jargon,

\[
\xi = \frac{dQ}{d(\Delta p/p_0)} = Q'.
\]  

(2)

![Graph showing the relationship between \( Q \) and \( \Delta p/p \).]  

Fig. 1: Example of chromatic dependence of the tune \( Q \) and of the slope at the origin that defines the chromaticity.

If the particle momentum oscillates and \( \Delta p/p_0 \) varies with time, then the betatron tune \( Q \) depends on the angular variable \( \theta \) via the function \( Q(\Delta p/p_0) \). Consequently, one likes to replace in the Floquet function (A.10) and the Hamiltonian (A.14) of the Appendix, the product \( Q\theta \) with \( Q \) constant by the integrated phase variation

\[
Q\theta \to \int_0^\theta Q(\theta)d\theta.
\]  

(3)

With this change in the perturbing Hamiltonian \( U \) (see the Appendix), the effects of the time varying betatron tune can be studied.

2.2 Modified Hamiltonian and synchrobetatron resonances

Using the notation as in the Appendix, the Hamiltonian of a perturbation due to electromagnetic fields can be written for one degree of freedom [1]

\[
U(a, \bar{a}, \theta) = \sum_N \sum_{j,k} \sum_P h_{jk-p}^{(N)} a^j \bar{a}^k \exp \left[ i(j - k) \int_0^\theta Q(\theta)d\theta \right] e^{-ip\theta},
\]  

(4)

where \( a \) is the transverse complex amplitude of the motion and \( h^{(N)} \) is by definition the Fourier component \( p \) of the perturbation of order \( N \)

\[
h_{jk-p}^{(N)} = \frac{1}{\pi} \int_{-\pi}^\pi \frac{\sqrt{\beta_z^{(j+k)}}}{(j+k)!} K^{(N-1)}(\theta) \exp \left[ i(j - k)(\mu_z - Q\theta) + ip\theta \right].
\]  

(5)
For small (near linear) synchrotron oscillations around the stable fixed point of the RF bucket (Appendix), one can write after linearization

\[
\frac{\Delta p}{p_0} = A_p \cos Q_s \theta .
\]

Consequently, for \(\Delta p/p_0 \ll 1\), one obtains via the chromaticity (2)

\[
Q(\theta) = Q_0 + \Delta Q \cos Q_s \theta
\]

and

\[
\int_0^\theta Q(\theta) d\theta = Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta ,
\]

with \(\Delta Q = \xi A_p\), where \(A_p\) is the peak amplitude of the momentum variation. \(Q_s\) is the synchrotron tune and \(Q_0\) the transverse tune on momentum.

When introducing (7) into the equation (4) of \(U\), the phase term gives

\[
\exp \left\{ i (j - k) \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} = \\
= \cos \left\{ (j - k) \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} + i \sin \left\{ (j - k) \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} .
\]

Replacing \((j - k)\) by \(n\) for simplification and developing these trigonometric functions provides the following relations

\[
\cos \left\{ (j - k) \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} = \cos \left\{ n \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} = \\
= \cos nQ_0 \theta \cos \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) - \sin nQ_0 \theta \sin \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) ,
\]

\[
\sin \left\{ n \left[ Q_0 \theta + \frac{\Delta Q}{Q_s} \sin Q_s \theta \right] \right\} = \\
= \sin nQ_0 \theta \cos \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) + \cos nQ_0 \theta \sin \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) ,
\]

where the arguments of the underlined trigonometric functions are periodic functions with a period \(2\pi/Q_s\). Auspiciously, trigonometric functions of sines and cosines have closed-form Fourier’s expansions involving Bessel’s functions of the first kind [6]

\[
\cos \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) = J_0 \left( \frac{n\Delta Q}{Q_s} \right) + 2 \sum_{\ell=1}^{\infty} J_{2\ell} \left( \frac{n\Delta Q}{Q_s} \right) \cos 2\ell Q_s \theta
\]

\[
\sin \left( \frac{n\Delta Q}{Q_s} \sin Q_s \theta \right) = 2 \sum_{\ell=0}^{\infty} J_{2\ell+1} \left( \frac{n\Delta Q}{Q_s} \right) \sin (2\ell + 1) Q_s \theta .
\]
The use of Eqs. (10) and of the basic functional relations for the sine and cosine products that figure in the expressions (9) gives the entire phase term of the Hamiltonian $U$ (4)

$$\exp[i \int_0^\theta Q(\theta)d\theta] = J_0 \exp[i nQ_0\theta] + \sum_{\ell = -\infty}^{+\infty} J_{2\ell} \exp[i(nQ_0 + 2\ell Q_s)\theta] + \sum_{\ell = -\infty}^{+\infty} J_{2\ell+1} \exp\{i[nQ_0 + (2\ell + 1)Q_s]\theta\},$$  

(11)

where the argument of the Bessel functions is $n\Delta Q/Q_s$.

According to the principles recalled in the Appendix, the new Hamiltonian of the perturbation can be written as follows,

$$U = \sum_N \sum_{j,k-p} h^{(N)}_{jk-p} a^j a^k \sum_{q=-\infty}^{\infty} J_q \frac{\Delta Q}{Q_s} \exp\{i[(j-k)Q_0 + qQ_s - p]\theta\},$$  

(12)

in which the index $q$ replaces the indices $2\ell$ or $(2\ell + 1)$ of Eq. (11).

In order to push further the analysis of the SBR due to a time varying tune, it is necessary to introduce here a basic assumption: the low frequency part of the function $U$ gives the important variations of the amplitude $a$ of the perturbed motion. This requires extracting from $U$ the pure low-frequency terms obtained by equalling to zero the square-bracket of the phase term in (12); there are two possible ways of satisfying this condition:

1) $j - k = 0$ and $q = p = 0$, simultaneously.

The corresponding terms, sometimes called stabilizing coefficients, are the frequency shifts with amplitude that arise in the presence of nonlinear forces.

2) $j - k = N$, and the following relation holds

$$NQ_0 + qQ_s - p = 0.$$  

(13)

Since $j + k = N$ by construction, this implies together with $j - k = N$ that $j = N$ and $k = 0$. The relation (13) is the characteristic of a so-called synchrobetatron resonance of order $N$, while the integer $q$ is the side-band number, and $p$ the harmonic number of the driving force.

When wishing to study the motion near one of these resonances of order $N$ (at a side-band $q$), one has to consider the following 'single-resonance' Hamiltonian,

$$U = \sum_{\nu} h^{(2\nu)}_{\nu\nu} (a\tilde{a})^{\nu} + \kappa a^N \cos \psi_1.$$  

(14)

which can be rewritten with different variables, as defined below

$$U = \sum_{\nu} h^{(2\nu)}_{\nu\nu} r^{2\nu} + 2|\nu| r^N \cos \psi.$$  

(15)
where,

\[ h_{\nu \nu_0}^{(2\nu)} \] are the stabilizing coefficients with \( j = k = \nu \) associated with \( 2 \times 2\nu \)-pole magnetic components,

\[ \kappa = J_q \left[ N(\Delta Q/Q_s) \right] h_{N0-p}^{(N)} \] is the excitation coefficient or driving term of the resonance [with \( h^{(N)} \) defined by Eq. (5)],

and the complex amplitude \( a \) of the betatron motion has been replaced by a real amplitude, and a phase \( \varphi \) according to

\[
\begin{align*}
a &= r e^{i\varphi} & r, \varphi &\in \mathbb{R} \\
\bar{a} &= r e^{-i\varphi} & r^2 &= a \bar{a} = Re^{i\varphi}/2\pi
\end{align*}
\]

(16)

if \( \epsilon \) is the emittance associated with a particle trajectory.

The phase \( \psi_1 \) gives the 'distance' from the SBR at given tunes and harmonic number, and \( \psi \) includes in addition the phase \( \varphi \) of the motion,

\[
\begin{align*}
\psi_1 &= (NQ_0 + q Q_s - p) \theta \\
\psi &= \psi_1 + N \varphi
\end{align*}
\]

(17)

The Hamiltonian \( U \) (14) in the presence of the perturbation, i.e. the time variation of the betatron tune due to synchrotron oscillations, is the starting point to study separately each SBR defined by Eq. (13).

2.3 Analysis of the motion near resonance

As explained in the Appendix, the canonical equations associated with \( U \) (15) give the equations of the perturbed motion,

\[
\begin{align*}
\frac{d\varphi}{d\theta} &= \frac{\partial U}{\partial r} = 2|\kappa| Nr^N \sin \psi \\
\frac{d^2 r}{d\theta^2} &= \frac{\partial U}{\partial r} = \sum_{\nu=1}^{\infty} h_{\nu \nu_0}^{(2\nu)} \nu^{2(\nu-1)} + N|\kappa| r^{(N-2)} \cos \psi
\end{align*}
\]

(18)

and the first relation can be simplified to

\[
\frac{dr}{d\theta} = N|\kappa| r^{(N-1)} \sin \psi.
\]

(19)

At the resonance, where the 'distance' (13) vanishes and \( \psi_1 = 0 \), the equations of the motion become autonomous, i.e. independent of \( \theta \). Many properties of autonomous equations can be derived from the fixed points (in the phase space) that are defined by

\[
\begin{align*}
\frac{dr}{d\theta} (r_0, \varphi_0) &= \frac{d\varphi}{d\theta} (r_0, \varphi_0) = 0,
\end{align*}
\]

(20)

where \( (r_0, \varphi_0) \) are the coordinates of these points. For a SBR of order \( N \), there are different possible solutions to the conditions (20).
1) The trivial solution, \( r = 0 \) (any \( \varphi \)), corresponds to the origin of the phase space.

2) Non-trivial solutions result from the first condition \((20) \ r' = 0 \), and the expression \((19) \) taken at \( \psi_1 = 0 \),

\[
N|\kappa| r^{(N-1)} \sin N\varphi = 0 .
\]

They correspond to particular phases given by

\[
N\varphi_0 = m\pi \quad \text{or} \quad \varphi_0 = \frac{m\pi}{N} \quad m = 0, 1, 2, 3, \ldots, (2N - 1) ,
\]

which define \( 2N \) directions in the phase space. Then, the second condition \((20) \) that \( \varphi' = 0 \) gives the following relation

\[
\sum h^{(2\nu)}_{\nu,\varphi_0} r^{(2\nu-1)} + N |\kappa| r^{(N-2)} (-1)^m = 0 ,
\]

in which \( \cos \psi = \cos N\varphi \) has been replaced by \((-1)^m\) because of the particular values taken by \( \varphi_0 \) \((22)\). Equation \((23)\) is a polynomial in \( r \) that provides the amplitudes \( r_0 \) of the \( 2N \) fixed points \((r_0, \varphi_0)\) as solutions. In the phase space, the normalized coordinates of the fixed points are simply given by

\[
x^{(n)} = r_0 \cos \varphi_0 \\
p^{(n)}_2 = r_0 \sin \varphi_0 .
\]

Figure 2 gives an illustration of how the phase space and the invariants of the betatron motion near one single resonance defined by \( N = 4 \) and \( q = 0 \) look. It is possible to recognize the eight privileged directions corresponding to the eight non-trivial fixed points expected from the analysis made above. The trivial solution is of course visible at the origin, around which the motion is locally stable.

Looking at the non-trivial fixed points (with finite \( r_0 \)), it is fairly obvious that there are two different kinds (Fig. 2):

- There are Stable Fixed Points at phases \( \varphi_0 \) equal to \( \pi/4, \ 3\pi/4, \ 5\pi/4, \ \text{and} \ 7\pi/4, \) with elliptic motion around them. Particles sweep closed invariants corresponding to oscillation around the point \((r_0, \varphi_0)\).

- There are however Unstable Fixed Points at phases \( \varphi_0 \) equal to \( 0, \ \pi/2, \ \pi, \ \text{and} \ 3\pi/2, \) with hyperbolic motion going away from \((r_0, \varphi_0)\) to possibly large amplitudes. Particles moving along these invariants may either circle the whole stable island area (as in Fig. 2) or diverge to infinity, depending on the initial conditions and the perturbation strength.

This duality of the fixed points can be established analytically from the linearization of the motion around \((r_0, \varphi_0)\), assuming

\[
\begin{align*}
\Delta r &= r_0 + \Delta r \\
\Delta \varphi &= \varphi_0 + \delta \varphi .
\end{align*}
\]
Fig. 2: Invariants in the betatron phase space near a fourth-order resonance, with the corresponding fixed points.

Developing the equations of motion (18) and (19) around one arbitrary fixed point, using (25), and keeping the terms of first order in $\Delta r$ and $\Delta \varphi$, we can derive new equations valid for small-amplitude motion near $(r_0, \varphi_0)$

$$\Delta r' = N^2 |\kappa| r_0^{(N-1)} (-1)^m \Delta \varphi$$

$$\Delta \varphi' = \left[ \sum_{\nu=2}^{\infty} 2\nu(\nu-1)h^{(2\nu)}_{2\nu0} r_0^{(2\nu-3)} + N(N-2)|\kappa| r_0^{(N-3)} (-1)^m \right] \Delta r,$$  \hspace{1cm} (26)

with $\varphi_0 = m\pi/N$, and therefore $\cos N\varphi_0 = (-1)^m$. In order to make the discussion about the stability of the fixed point easier, it is helpful to assume that among all the stabilizing coefficients $h^{(2\nu)}$, the one which dominates is of lowest order $\nu = 2$ (i.e. the octupole term), and that the driving term $\kappa$ of the resonance is small such as $\kappa^2 \ll 1$. With these assumptions, the second equations (26) can be simplified by neglecting the term with $\kappa$ and keeping only the term $\nu = 2$ of the sum, and the two first-order equations (26) can then be combined in a single second-order differential equation as follows,

$$\Delta \varphi'' = \left[ 4N^2 h^{(4)}_{220} |\kappa| r_0^N \right] (-1)^m \Delta \varphi.$$ \hspace{1cm} (27)

This is the key equation describing small excursions around a fixed point, now characterized by its amplitude $r_0$ and its 'direction' given by $m$ [Eq. (22)]. The condition of stability for these small motions is obviously

$$C_f (-1)^m < 0 \quad \text{with} \quad C_f = 4N^2 h^{(4)}_{220} |\kappa| r_0^N.$$ \hspace{1cm} (28)
Hence, the kind of fixed points observed depends on the sign of $h^{(4)}_{220}$ (octupole coefficient):

\[ h^{(4)}_{220} > 0 \quad \text{Stable Fixed Points for } m = 1, 3, 5, 7, \ldots \]
\[ h^{(4)}_{220} < 0 \quad \text{Stable Fixed Points for } m = 0, 2, 4, 6, \ldots \]
\[ h^{(4)}_{220} > 0 \quad \text{Unstable Fixed Points for } m = 0, 2, 4, 6, \ldots \]
\[ h^{(4)}_{220} < 0 \quad \text{Unstable Fixed Points for } m = 1, 3, 5, 7, \ldots \]

For stable fixed points, the small excursions given by (27) are periodic oscillations of frequency

\[ Q_f = 2N\sqrt{h^{(4)}_{220} |\kappa|^p} \tau_0^N. \quad (29) \]

The four stability islands of Fig. 2 (for $N = 4$) are delimited by separatrices passing by and crossing at the four unstable fixed points ($m$ even). The corresponding phase space configuration can therefore be represented schematically by these separatrices that are defined as the outermost invariants touching the nearby unstable fixed points. They can be determined numerically in specific cases by searching for the extreme ‘closed trajectory’ around each stable fixed point: as a consequence, the picture in the phase space will be reduced to four (or $N$) island contours defined with a precision that depends on the accuracy of the numerical calculations. Such families of $N$ unstable fixed points, and $N$ elliptic invariants (island contours) around the stable fixed points exist, and can be drawn for each side-band $q$ ($q > 1$) of a synchrobetatron resonance of order $N$. The width of the separatrix or of the island contour and the frequency spacing between different SBR side-bands (according to the expressions (17) of $\psi$ and $\psi_1$) are given respectively by

\[ W_f = 4Q_f \]
\[ S_f = \frac{Q_s}{N}. \quad (30) \]

Consequently, the elliptic islands of the sidebands of one SBR of particular order $N$ will overlap when $W_f > S_f$, and be separated when $W_f < S_f$. Taking into account the presence of stochastic layers of finite width along the separatrices, the elliptic islands do not overlap, if and only if

\[ W_f < \frac{2}{\pi} S_f \quad \rightarrow \quad Q_f < \frac{1}{2\pi} \frac{Q_s}{N}. \quad (31) \]

This condition of stability is known under the name of Chirikov’s criterion, after its author [7]. In the particular case of a time-varying betatron tune and a dominant stabilizing octupole, the criterion of stability (31) can be rewritten by using both Eq. (29) and the definition of $\kappa$ following Eq. (15),

\[ 4\pi N^2 \sqrt{|h^{(4)}_{220}| J_q \left( \frac{N\Delta Q}{Q_s} \right) h^{(N)}_{N0-p} \tau_0^N} < Q_s \quad (32) \]

with

\[ h^{(N)}_{N0-p} = \left( \frac{2\pi}{R} \right)^{N/2-1} \frac{\Delta e}{2N^2 e_b^{(N/2-1)}}. \quad (33) \]
Equation (33) is a reminder of the relation between $h^{(N)}$ and the total bandwidth $\Delta e$ of Ref. [1], $\epsilon_t$ being the transverse beam emittance [see Eq. (16)].

The passage from a stable situation to an unstable situation in the presence of a time-varying tune developing synchrobetatron sidebands is illustrated for $N = 4$ in three successive figures (Figs. 3, 4, 5) resulting from numerical solutions of the equations (18) for $\varphi$, and (19) for $r$, with increasing resonance strength $\kappa$ (in arbitrary units);

i) Figure 3 describes a stable case. The four island contours are drawn successively from inside to outside for every sideband between $q = 1$ (mostly inside) and $q = 6$ (mostly outside). With $\kappa = 0.05$, the resonance strength is weak enough for the sideband islands to remain separated; the particles trapped in these islands stay there, and their motion is therefore stable.

ii) Figure 4 shows a case where the excursion of the particles with initially small amplitudes may increase. The same contours are drawn for sideband islands from $q = 1$ to $q = 6$, but now with $\kappa = 0.20$; the first four families of islands overlap while the following ones are still separated. Particles with small amplitudes may travel from island to island and far enough to reach the largest radius of the separatrix of the $q = 4$ sideband. The particles remain confined within this finite amplitude, but there is a possible amplitude growth for particles near the centre of the beam.

iii) Figure 5 illustrates a case of instability for $\kappa = 0.75$. The island contours are now drawn for the sidebands $q = 1$ to $q = 5$, and then all the islands overlap. Furthermore, islands corresponding to higher $q$ values would also overlap with the ones which are represented. This means that particles near the phase space origin can wander from island to island (sideband $q$ to sideband $q + 1$), and diffuse to infinite amplitudes. Therefore, the betatron motion is unstable and particles may be lost in the direction indicated in Fig. 5.

The fact that the separatrices of each sideband of Figs. 3, 4, and 5 do not exactly join on the main horizontal and vertical axis as they should is simply due to the accuracy and step size of the numerical search for the outermost invariants.

The analytical treatment described above, and applied to the SBR mechanism illustrated in Figs. 3 to 5, provides the approximate but convenient formulae (32) and (33), which can be used to establish the stability or instability of the motion under given conditions, and in the presence of a time-varying betatron tune. In this paper, this time variation is supposed to come from the momentum oscillations caused by synchrotron motion in conjunction with finite chromaticity. The same analysis is however directly valid for other sources of tune modulation such as power supply ripple, provided the synchrotron tune $Q_s$ is replaced, from equation (6) onwards, by the wave number $Q_m$ of the modulation. The term 'synchrobetatron resonances' should then be replaced by something like 'modulation betatron resonances', involving also sidebands and possible instabilities.
Fig. 3: Typical island contours of six synchrotron sidebands of a fourth-order resonance, in a stable case.

Fig. 4: Typical island contours of six synchrotron sidebands of a fourth-order resonance, with limited amplitude growth.

Fig. 5: Typical island contours of five synchrotron sidebands of a fourth-order resonance, in an unstable case.
2.4 Practical application to accelerators

During the design study of the proton–antiproton colliding beam facility based on modifications and additions to the Super Proton Synchrotron (SPS) at CERN, the theoretical model presented above for SBR was developed [2] with a view to analysing some of the effects caused by beam–beam interactions of colliding bunches. The design [8] was based on a pp colliding facility with 6 bunches per beam, two low-beta insertions, and for a flat-top energy of 270 GeV. The bunch length was taken as 0.63 m, the emittances were 0.08 and 0.04 μrad m in the two planes, respectively, the assumed chromaticity ξ was ~ 0.1, the energy spread amounted to 0.7%, and the fractional parts of the betatron tunes were chosen near 0.88 for the study.

In this case, the dominant forces driving resonances are the beam–beam forces which were estimated first. In other words, the excitation coefficients $h_{jk}^{(N)}$ (5) and the total bandwidths $\Delta e$ of resonances of order 2 to 10 have been calculated for beam intensities corresponding to a beam–beam tune shift of 0.003 per crossing, and for beam off-centering of ~ 0.035 mm. Some of the results are briefly summarized in Table 1.

<table>
<thead>
<tr>
<th>Order</th>
<th>Range of $\Delta e$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$0.17 \cdot 10^{-3}$ to $0.17 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$0.90 \cdot 10^{-4}$ to $1.08 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>7</td>
<td>$0.50 \cdot 10^{-4}$ to $0.50 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>$0.40 \cdot 10^{-5}$ to $2.00 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>9</td>
<td>$1.10 \cdot 10^{-6}$ to $2.50 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>10</td>
<td>$0.50 \cdot 10^{-6}$ to $7.20 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

The next step consisted in looking at the SBR effects and the stability criterion (32) in order to decide which resonances can effectively be tolerated within the beam. The detuning or self-stabilizing effect of the strong octupole component of the beam–beam field has to be taken into account; the corresponding stabilizing coefficient is proportional to the linear beam–beam tune shift $\Delta Q_{bb}$ [2],

$$h_{220}^{(4)} \approx \frac{3\pi}{4} \frac{\Delta Q_{bb}}{\epsilon_b}, \quad (34)$$

using the notation of the previous section, the coefficient of proportionality depending on the beam aspect ratio (close to 1 in the present case). Expressing the criterion of stability at an amplitude $\tau_0$ given by the beam emittance $\epsilon_b$, Eq. (32) can be rewritten as follows

$$\pi N \sqrt{3\Delta Q_{bb} \Delta e J_q \left(\frac{N\Delta Q}{Q_s}\right)} < Q_s, \quad (35)$$

for dominant beam–beam forces characterized by $\Delta Q_{bb}$, and the bandwidths $\Delta e$ (Table 1). Using $\Delta Q_{bb} = 0.02$, the $\Delta e$-values of Table 1, an amplitude modulation $\Delta Q$ of 0.002, and a synchrotron tune $Q_s$ of 0.005, the left term of the relation (35) has been evaluated for each order $N$ of the beam–beam perturbation. Testing on the one hand the corresponding criterion (35) for each synchrotron satellite $q$ of the $N$-order resonance, it is possible
to determine the number $n_s$ of satellite islands that overlap, while for higher $q$-values they remain separated (as in Fig. 4). On the other hand, superposing the phase-space circle corresponding to the emittance of a gaussian beam onto an island pattern similar to the one in Fig. 4, the calculation of the number $n_t$ of satellite islands that lie inside the r.m.s. beam emittance can be carried out. Comparing these two numbers gives an indication of the probability of a steady beam growth (at a resonance $N$) as particles migrate from one side band to another. Both $n_s$ and $n_t$ are given in Table 2 for several resonances $N$ in the case of the $p\bar{p}$ collider design at CERN [8] (with the quoted assumptions).

The results in Table 2 show that resonances of order greater than $N = 10$ produce no growth, since no satellites overlap. A small amplitude growth detunes the particles from their resonance condition (because of $\Delta Q_{\beta\beta} \neq 0$). For resonances from 10th order down, the beam–beam tune spread is equal or greater than the satellite spacing, and several islands (of low $q$-values) overlap within the emittance limits. For orders $N$ below 7, even all the satellites lying inside the emittance overlap. For all these resonances with $N$ below 10, blow-ups can be expected, and growth rates can be estimated from the resonance multicrossing theory [1]. Consequently, at the time of this study it was recommended to avoid that particles cross resonances of order lower than and equal to 10, by choosing judiciously the betatron tunes of beams which collide (e.g. below an integer where resonances are sparse).

<table>
<thead>
<tr>
<th>Order $N$</th>
<th>$n_s$ (see text)</th>
<th>$n_t$ (see text)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>22</td>
</tr>
</tbody>
</table>

This conclusion was found to be in remarkably good agreement with the observations of beam–beam effects subsequently made when the SppS hadron collider was running. The antiproton beam lifetime was indeed seen to critically depend on the working point (set of betatron tunes); when the beam–beam tune shift (spread) induced the working tune diamond to touch the 10th order resonance, the lifetime fell dramatically by a factor of approximately 20 with three bunches per beam, and even more with six bunches per beam [9]. Decreasing the sensitivity of the emittance growth to the tune adjustment while running with six bunches per beam required beam separation at the unwanted crossing points, in order to reduce the total beam–beam tune shift (spread), and keep the beam away from all resonances of order lower than 13 (which comes directly from the fact that, in the working region of the tune diagram, the higher order mode next to the 10th order one is of order 13). This experience gives a positive indication about the usefulness in the design phase of the analytical model described above and suggests that, during the study of future circular colliders such as the Large Hadron Collider (LHC) at CERN, similar analyses of SBR and/or modulation betatron resonances could provide interesting results.
3 DISPERSION IN RF CAVITIES AND SYNCHROBETATRON RESONANCES

3.1 Hamiltonian formalism in the presence of dispersion

As mentioned in the introduction, this is a summary of fairly recent theories [3, 4] on SBR driven by dispersion at RF cavity locations. Since they have mostly been published in journals, the following reminder will be brief with some emphasis on the key points. We will begin directly with the expression of the Hamiltonian, for the analysis referred to is based on this particular formalism. As it is well known, and as it can also be retrieved from the equations of motion (A.8) and (A.17) (betatron and synchrotron motion, respectively) given in the Appendix, the pertinent Hamiltonian for the study of synchrobetatron coupling is as follows:

\[ H = \frac{R \rho_0}{2} \left( K x^2 + p_z^2 p_{x_o}^2 \right) - \frac{1}{2} \frac{h^2}{R \rho_0} \left( \frac{D_x}{\rho} - \frac{1}{\gamma^2} \right) W^2 - \frac{R}{\hbar c} \sum_i V_i e \delta(\theta - \theta_i)(\cos \psi + \psi \sin \varphi_x), \quad (36) \]

in which one has to replace the \( \eta \) of Eq. (A.17) by the actual function \(- (D/\rho - 1/\gamma^2)\), since \( \eta \) is the one turn average of it, and the RF voltage \( \tilde{V} \) by \( 2\pi \sum_i V_i \delta(\theta - \theta_i) \), since in an accelerator there are a certain number of discrete cavities at fixed angular coordinates \( \theta_i \) (or positions \( s_i \)). The definitions of the quantities entering Eq. (36) are given in the Appendix, but it is useful to recall here the meaning of \( W \) (different from the usual definition)

\[ W = -\frac{\Delta E}{\omega_{rf}} \quad \text{with} \quad \Delta E = E - E_s. \quad (37) \]

The phase \( \psi \) is equal to the RF phase \( \varphi \) in the absence of the perturbation owing to finite horizontal dispersion, but is modified when \( D_x \neq 0 \) at cavities because of path length differences and consequent phase variations

\[ \psi = \varphi - \frac{h}{R} \left( D_x \frac{p_x}{p_0} - D_x x \right). \quad (38) \]

The mechanism of the SBR clarifies the form (38) of \( \psi \) and is of the twofold type:

i) The acceleration \( \Delta p \) does not change the instantaneous position and angle of the particle, but the equilibrium orbit, with respect to which the betatron oscillation takes place, suddenly changes with \( \Delta p \), and an additional betatron excursion is excited.

ii) The betatron oscillations lead to a change of the orbit length per revolution and thus to a variation of the beam position with respect to the phase of the acceleration voltage. Hence, betatron excursions affect the synchrotron motion.
This double mechanism can be treated as a perturbation, following the symplectic theory recalled in the Appendix. The Hamiltonian $H$ (36) can indeed be written in the required form $H_0 + H_1$ (A.1),

$$
H_0 = \frac{R p_0}{2} \left( K x^2 + \frac{p_z^2}{p_0^2} \right) - \frac{1}{2} \frac{h^2}{R p_0} \left( \frac{D_z}{\rho} - \frac{1}{\gamma^2} \right) W^2 - \frac{R}{\hbar \beta c} (\cos \varphi + \varphi \sin \varphi_\ast \sum_i e V_i \delta (\theta - \theta_i)) \\
H_1 = -\frac{\sin \varphi - \sin \varphi_\ast}{\beta c} \sum_i e V_i \delta (\theta - \theta_i) \left( D_{z_i} \frac{p_z}{p_0} - D'_{z_i} x \right),
$$

(39)

according to (38), and separating the contributions of $\varphi$ and $D_z$.

To single out each nonlinear term of power $q$ in $\Delta \varphi = \varphi - \varphi_\ast$ (see Appendix), $H_1$ can be expanded by using

$$
\sin \varphi - \sin \varphi_\ast \equiv \cos \varphi_\ast \Delta \varphi - \frac{\sin \varphi_\ast}{2} \Delta \varphi^2 - \frac{\cos \varphi_\ast}{3!} \Delta \varphi^3 + ...

\overset{\text{def}}{=} \sum_q \frac{\alpha_q}{q!} (\Delta \varphi)^q,
$$

(40)

which leads to the form

$$
H_1 = \sum_q \frac{\alpha_q (\Delta \varphi)^q}{2 \pi q!} \sum_i e V_i \left( -D_{z_i} \frac{p_z}{c \beta p_0} + D'_{z_i} \frac{x}{c \beta} \right),
$$

(41)

with

$$
\alpha_q = \begin{cases} 
(1)^{q/2} \sin \varphi_\ast & \text{if } q \text{ even} \\
(1)^{(q-1)/2} \cos \varphi_\ast & \text{if } q \text{ odd}
\end{cases}
$$

Equation (41) shows that the perturbation considered will only give rise to SBR of the form $Q_z \pm q Q_s = p$ (only the linear term of betatron motion in $H_0$).

The difficulty at this point, correctly pointed out in Ref. [4], is that the longitudinal motion is not simple harmonic (Fig. 6) when the maximum stable phase $\varphi_{\text{max}}$ is large and the terms $q > 1$ become important. This means that a consistent treatment requires not only the term $\sin \varphi - \sin \varphi_\ast$ of $H_1$ to be expanded, but also the term $\cos \varphi + \varphi \sin \varphi_\ast$ of $H_0$. The proposed way to do this consists in finding the solutions, formally written as

$$
\varphi = \sum_q a_q \cos q (Q_s \theta + \psi),
$$

(42)

of the differential equation (see the Appendix)

$$
\varphi'' + \frac{Q_s^2}{\cos \varphi_\ast} (\sin \varphi - \sin \varphi_\ast) = 0,
$$

(43)

in which $Q_s$ is in general different from the value $Q_{s0}$ valid at the small amplitude, where the motion is almost a simple harmonic (Fig. 6),

$$
Q_{s0} = \frac{h \eta V e}{2 \pi \beta p_0 c} \cos \varphi_\ast.
$$

(44)

It is very difficult to get closed expressions for $a_q$ at $\varphi_\ast \neq 0$ and Ref. [4] proposes an approximation at $\varphi_\ast = 0$ and $q$ odd (even orders do not occur), with $\varphi_{\text{max}}$ as the amplitude of the phase modulation

$$
a_q = \frac{8}{q} \frac{(2 \chi)^{q/2}}{2^q + \chi^q}
$$

(45)
with \( \lambda = \frac{1 - \sqrt{\cos \varphi_{\text{max}}/2}}{1 + \sqrt{\cos \varphi_{\text{max}}/2}} \) and \( \frac{Q_z}{Q_{s0}} = \frac{1}{(1 + \lambda)^2} \).

The canonical angle is \( \psi \) and the associated action \( I_s \) is hidden in the coefficient \( a_q \); the latter is proportional to the area of the closed unperturbed orbit in phase space.

Fig. 6: Invariants of the longitudinal motion with \( \varphi_s = 150^\circ \), in the phase space defined by the coordinates \( \varphi \) and \( \varphi/\Omega_s \).

3.2 Analysis of the perturbed dynamics

In order to analyse further the perturbed motion, it is convenient to use the canonical action-angle variables for both the betatron and synchrotron motion. The link between these variables and physical quantities results from the forms of the equations and solutions of the unperturbed motions recalled in the Appendix (with \( a_1 \sim \sqrt{I_z} \exp(i\varphi_z) \) and \( a_1 \tilde{a}_1 \sim I_z \), in the betatron case and similarly in the synchrotron case using \( I_s \) and \( \psi \))

\[
\begin{align*}
  x &= \left( \frac{2\beta_z I_z}{p_0} \right)^{1/2} \cos (\mu_z + \varphi_z) \\
  p_x &= -\left( \frac{2p_0 I_z}{\beta_z} \right)^{1/2} \left[ \alpha_z \cos (\mu_z + \varphi_z) + \sin(\mu_z + \varphi_z) \right] \\
  W &= -\left( \frac{2c p_0 \beta Q_s I_z}{\hbar^2 |\eta| \omega_0} \right)^{1/2} \sin (\pm Q_s \theta + \psi) \\
  \Delta \varphi &= \left( \frac{2\hbar^2 |\eta| \omega_0 I_z}{c p_0 \beta Q_s} \right)^{1/2} \cos (\pm Q_s \theta + \psi) .
\end{align*}
\]

(46)

Most of the symbols used in these equations are defined in the Appendix and \( \omega_0 \) is the angular revolution frequency. The quantity \( \eta \) that changes sign across the transition is taken as its absolute value, but in the last two relations, the sign + must be retained below transition energy (\( \eta > 0 \)) and the sign − above (\( \eta < 0 \)).
The next step consists in finding the transformed Hamiltonians \( H_0 \) and \( H_1 \) for the new set of variables \( \{I_x, \varphi_x, I_s, \psi\} \), and this is done by performing the transformations defined by (46). The same basic assumption is then introduced, as for the case of the time varying tune in Section 2.2: i.e. the low frequency part of \( H_1 \) gives the important contribution to the perturbation and the resonant terms are the only ones kept. The resonance condition can now be written (with positive integers \( q \) and \( p \))

\[
Q_x \pm q Q_s - p = 0 ,
\]

and is of first order in \( Q_x \) since only the linear betatron motion is included. However, the sideband number \( q \) can be as large as desired, for the nonlinear terms of the synchrotron motion have been taken into account. Close to the resonance (47), i.e. \( Q_x \pm q Q_s - p = c \), the canonical transformations give eventually,

\[
H_0 = \pm \frac{\varepsilon}{q} I_s ,
\]

\[
H_1 = \frac{\cos \varphi_s}{4\pi c^3} \left( \frac{Q_s}{Q_{s0}} \right)^2 q^2 a_q \left( \frac{2I_x}{\beta_z p_0} \right)^{1/2} \times [(D_{c,p} - F_{s,p}) \sin(\varphi_x \pm q\psi) + (D_{s,p} + F_{c,p}) \cos(\varphi_x \pm q\psi)],
\]

in which the consistent development (42) of \( \varphi \) [4] has been included and the contribution of the finite horizontal dispersion is given by Fourier’s expansions [3]

\[
D_{s,p} = \sum_i eV_i D_{z_i} \sin p\varphi_{z_i},
\]

\[
D_{c,p} = \sum_i eV_i D_{z_i} \cos p\varphi_{z_i},
\]

\[
F_{s,p} = \sum_i eV_i (D'_{z_i} \beta_{z_i} + D_{z_i} \alpha_{z_i}) \sin p\varphi_{z_i},
\]

\[
F_{c,p} = \sum_i eV_i (D'_{z_i} \beta_{z_i} + D_{z_i} \alpha_{z_i}) \cos p\varphi_{z_i}.
\]

These expansions sum the contributions of all the discrete cavities positioned at \( \theta = \theta_i \), and \( \alpha_{z} \) is related to the derivative of \( \beta_z \) by \( \beta'_z = -2\alpha_z \).

The perturbed motion is now entirely described by \( H_1 \) (48) and the perturbation theory (Appendix) tells us that

\[
\frac{dI_x}{d\theta} = -\frac{\partial H}{\partial \varphi_x} \quad \text{and} \quad \frac{dI_s}{d\theta} = -\frac{\partial H}{\partial \psi}.
\]

From the expression of \( H_1 \) and the form of the phase term involved, i.e. \( \varphi_x \pm q\psi \), the following relation holds

\[
\frac{dI_x}{d\theta} = \pm \frac{1}{q} \frac{dI_s}{d\theta},
\]

and after integration this gives the form of one invariant of the motion,

\[
qI_x \pm I_s = \text{constant} \quad \text{for} \quad Q_x \pm qQ_s \quad \text{above transition},
\]

\[
\text{for} \quad Q_x = qQ_s \quad \text{below transition}.
\]
The role of the sum and difference resonances are indeed interchanged above and below transition. According to (52) the amplitude growth is limited below transition for a difference resonance and above transition for a sum resonance. Knowing the initial emittances, Eq. (52) indicates where this limit is, though the growth in betatron amplitude can be important because $I_s$ is usually much larger than $I_x$.

The first equation of motion (50) makes it possible to calculate the growth rate per revolution of the beam size. It results from the Hamiltonian (48),

$$I'_x = -\frac{\cos \varphi_s}{4\pi \beta_c} \left( \frac{Q_s}{Q_{s0}} \right)^2 q^2 a_q \left( \frac{2I_z}{\beta_z p_0} \right)^{1/2} \times [\left( D_{c,p} - F_{s,p} \right)^2 + (D_{s,p} + F_{c,p})^2]^{1/2} \sin(\varphi_z \pm q\psi + \psi_0),$$

where $\psi_0$ is a constant phase, and the prime denotes differentiation with respect to $\theta$. Since the change $\delta I_x$ per revolution of $I_x$ is simply,

$$\delta I_{x,rev} = 2\pi I'_x \overset{\text{def}}{=} \frac{p_0}{\pi} \delta \epsilon_x,$$

the maximum change per revolution of the emittance $\epsilon_x$ is obtained by combining Eqs. (53) and (54),

$$\delta(\epsilon_x)_{\text{max}} = \frac{\cos \varphi_s}{\beta_c p_0} \left( \frac{Q_s}{Q_{s0}} \right)^2 q^2 a_q \left[ \frac{\epsilon_x}{\beta_z} \left( \frac{\left( D_{c,p} - F_{s,p} \right)^2 + (D_{s,p} + F_{c,p})^2 \right)^{1/2}}{\left( D_{c,p} - F_{s,p} \right)^2 + (D_{s,p} + F_{c,p})^2} \right].$$

Similarly, the second equation of motion (50) and the relation between $I_s$ and the energy spread give the maximum growth of $\Delta E/E$ per revolution [3, 4],

$$\delta(\frac{\Delta E}{E})_{\text{max}} = \frac{\beta^3 \cos \varphi_s}{2c_0 R} \left( \frac{Q_s}{Q_{s0}} \right)^2 q^2 a_q \left[ \frac{\epsilon_x}{\beta_z} \left( \frac{\left( D_{c,p} - F_{s,p} \right)^2 + (D_{s,p} + F_{c,p})^2 \right)^{1/2}}{\left( D_{c,p} - F_{s,p} \right)^2 + (D_{s,p} + F_{c,p})^2} \right].$$

These growth rates are given here at the resonance, but usual treatments of standard effects like multiple, fast-crossing of the resonance [1] also apply to the particular phenomenon discussed here.

### 3.3 Comparison with simulation and application

We shall first summarize briefly the discussion presented in Refs. [3, 4] about the two rings, PETRA and the TRIUMF booster, which were designed with finite horizontal dispersion in the accelerating cavities. Secondly, the observations made at LEP (Large Electron-Positron storage ring at CERN), a ring with zero dispersion at cavities by design, are also presented.

The PETRA ring of DESY (Hamburg) has been retained [3] to compare the Hamiltonian theory with numerical simulations. The most relevant parameters are an energy of 23 GeV, a synchronous phase of 38°, a synchrotron tune of 0.125, and a horizontal dispersion of 2 m. The latter value indicates that the SBR effects are probably going to be strong. Simulations were published in Ref. [10] and the rise times of the betatron amplitude for a 6σx beam size and a 6σE energy spread were computed analytically [4] on the basis of the development described in Section 3.2. Table 3 gives the results for a few sidebands and shows that the agreement between simulations and theory is quite good, including that for $q = 5$. 


Table 3
Betatron amplitude rise-times in PETRA

<table>
<thead>
<tr>
<th>Sideband $q$</th>
<th>Simulated ($\mu$sec)</th>
<th>Theoretical ($\mu$sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>48</td>
<td>42</td>
</tr>
<tr>
<td>3</td>
<td>312</td>
<td>260</td>
</tr>
<tr>
<td>4</td>
<td>1020</td>
<td>1250</td>
</tr>
<tr>
<td>5</td>
<td>1065</td>
<td>1800</td>
</tr>
</tbody>
</table>

This comparison shows that we may have confidence in the Hamiltonian theory and the few closed expressions such as (55) and (56) which are convenient for rapid numerical evaluations. Therefore, working on the design of a 3 GeV booster ring of the TRIUMF Kaon Factory project, people used the theory to get some insight about synchrotron effects and their compensation. The dispersion in the cavities was large (3–4 m) because a lattice with high transition energy was desired [11]. The tune $Q_x$ was 4.24, the superperiodicity of the machine five, and three of the five long straight sections were filled with cavities. The effects were then expected to be inherently strong at injection energy ($T = 458$ MeV) and for about 1 ms (or 1300 revolutions approximately) after injection, while $Q_x$ was reaching a maximum of 0.04. The corresponding increase $\Delta I_x/I_x$ was calculated [3] using Eq. (53) and multiple resonance-crossing estimates [1]. This amplitude growth was found to reach a factor 13 for $q = 1$, 10% for $q = 2$ and 50% for $q = 3$. For higher sidebands with $q$ values larger than 4, it remains however below 4%. This effect is consequently unacceptable up to $q = 3$, and there are basically only two possible remedies:

i) The most obvious is to make the dispersion and its derivative zero at the cavities.

ii) Another solution consists of using the phase relations (49) that depend on the positions of the cavities and the phase advances between them. The contribution of the dispersion may vanish if the cavities are placed symmetrically, and the integer part of the tune chosen to be different from the ring superperiodicity.

For the TRIUMF booster, one suggestion [12] was for instance to adopt a superperiodicity of 6, a threefold symmetry in the cavity positioning, and a tune above 5, for it is not a multiple of three.

LEP at CERN is a collider designed with zero dispersion and dispersion derivative at the cavities. Effects of SBR related to cavities are therefore only caused by spurious dispersion that was measured to be of the order of 20 to 40 cm. Consequently, the synchrotron coupling may appear as well in the vertical plane as in the horizontal one. Moreover, non-zero chromaticity may also contribute to the excitation of such resonances. Systematic investigations of SBR were carried out to study their possible role in intensity limitations, in relation with the spurious dispersion observed, and dispersion bumps generated in the RF cavities [13]. With tune integers equal to $70/76$ and $Q_x \approx 0.086$, two different scans were performed. Firstly, the coherent tunes were measured as functions of the beam current by scraping the beam, initially at 612 $\mu$A. The incoherent tunes stay constant while the coherent ones increase, therefore crossing resonance satellites as the
current decreases. For instance, vertical bunch instability was observed when the vertical coherent tunes were slightly below the resonance $Q_y = 3Q_z$. Secondly, the vertical emittance was measured as a function of $Q_y$, at constant $Q_z$, and for a beam current of $\sim 200 \, \mu A$. It shows sharp peaks at $Q_y$-values just below $3Q_z$ and $2Q_z$ (Fig. 7). As the incoherent vertical tunes are far away ($\sim 0.023$) from SBR, these observed resonances have to be coherent in contrast to the single-particle description made above that concerns incoherent effects. An interesting fact is that the lifetime shows dips corresponding to the emittance peaks (shifted towards lower $Q_z$ values, maybe because of the tune spread in the bunch). If this correlation is true, it could be a possible mechanism which limits the intensity in LEP; for higher currents, the split between coherent and incoherent tunes increases and the tune diagram-region free from SBR is reduced. For $Q_y$ close to 0.2, a vertical beam pulsation at intervals of a few seconds was noticeable, and this phenomenon could perhaps be explained by the coupling resonance $Q_z - Q_y = Q_s$. Note that if $Q_s$ was in fact slightly below 0.086, the observations would agree better with the resonance line positions. The remedies for the instability of the above-mentioned mechanism are the same as those given in previous sections for incoherent resonances. On the one hand, the residual dispersion has to be controlled using either local bumps or tilted quadrupoles, on the other hand, the betatron tune has to be optimized in order to avoid damaging SBR.

Fig. 7: Synchrobetatron resonances in the tune diagram and observed vertical emittance growth in LEP.
Beside these coherent resonances due to a residual dispersion different from zero in the cavities, SBR associated with the synchrotron motion have also been observed; they can be driven during energy ramping by tune and chromaticity excursions. Though SBR are present in LEP, measuring and correcting these excursions as well as controlling dispersion and tunes circumvent their effects, so that the beam intensity is rather limited by other mechanisms such as long range beam–beam forces and transverse mode coupling.

References


APPENDIX A: SUMMARY OF PERTURBATION PRINCIPLES IN ACCELERATORS

A.1 Brief reminder of perturbation theory

The method of the perturbation of the constants in the formalism of the classical Hamiltonian treatment is described elsewhere [1] in some detail and applied to a different source of linear motion disturbance. However, since it will be used extensively in this paper for the synchrobetatron coupling, its main points are briefly recalled below for the sake of completeness.

The motion is defined as usual by the total Hamiltonian $H$, which is the sum of the Hamiltonian $H_0$ of the unperturbed motion, and of the perturbing Hamiltonian $H_1$, i.e.

$$H = H_0(p,q,\theta) + H_1(p,q,\theta),$$  \hspace{1cm} (A.1)

where $\theta$ is the independent variable and $p,q$ are the $2N$ canonical variables in an $N$-dimensional space.

The perturbation treatment requires four basic steps:

i) To solve the canonical equations of the unperturbed motion described by $H_0$

\[
\begin{align*}
\dot{q}_\rho^{(0)} &= \frac{\partial H_0}{\partial p_\rho^{(0)}} \\
\dot{p}_\rho^{(0)} &= -\frac{\partial H_0}{\partial q_\rho^{(0)}}
\end{align*}
\Rightarrow \begin{align*}
q_\rho^{(0)} &= q_\rho^{(0)}(a_j,\theta) & \rho &= 1, \ldots, N \\
p_\rho^{(0)} &= p_\rho^{(0)}(a_j,\theta) & j &= 1, \ldots, 2N
\end{align*}
\]

(A.2)

where $a_j$ are the $2N$ arbitrary constants along the unperturbed trajectories.

ii) As a basic principle, the perturbed motion in the space $q - p - \theta$ can be followed by a set of $a_j$-values that change with the variable $\theta$ and describe the effect of the perturbation $H_1$

$$q_\rho = q_\rho^{(0)}(a_j(\theta),\theta) \quad p_\rho = p_\rho^{(0)}(a_j(\theta),\theta).$$

(A.3)

However, the solution of the perturbed motion keeps the form of the solution (A.2) of $H_0$ with time-dependent $a_j$. Since $a_j(\theta)$ become the new canonical variables, the Hamiltonian $H_1$ must be rewritten as a function of $a_j$ and $\theta$

$$H_1(p_\rho, q_\rho, \theta) = H_1[a_j(\theta), \theta] \overset{\text{def}}{=} U(a_j, \theta).$$

(A.4)

The last equation is also a definition of the function $U$.

iii) To solve the equations of the unperturbed motion i) for the constants $a_j$

$$a_j = a_j(p_\rho, q_\rho, \theta).$$

(A.5)

These functions are determined by the form of $H_0$ and form a system of coordinates in the phase space.

iv) Find out the expressions $a_j = f_j(\theta)$ by solving the differential equations that result from the presence of a perturbation $H_1$ [1],
\[
\frac{da_j}{d\theta} = [a_j, H_1] = \sum_{m=1}^{2N} [a_j, a_m] \frac{\partial U}{\partial a_m},
\]  
(A.6)

using the following definition of the Poisson bracket
\[
[a_j, a_m] = \sum_{p=1}^{N} \left[ \frac{\partial a_j}{\partial q_p} \frac{\partial a_m}{\partial p_p} - \frac{\partial a_j}{\partial p_p} \frac{\partial a_m}{\partial q_p} \right].
\]  
(A.7)

The $2N$ first-order equations (A.6) give the explicit solution of the perturbed motion and so far everything is exact.

**A.2 Specific case of single-particle motion**

Let us first consider the transverse oscillations of the betatron motion which satisfy Hill’s equation in the absence of perturbation, e.g. in the horizontal plane
\[
x'' + K(\theta)x = 0 \quad \text{with} \quad K(\theta) = K(\theta + 2\pi).
\]  
(A.8)

In order to show how to apply the perturbation theory in this case, when some perturbation is present, let us follow the four steps recalled in the preceding section:

i) The solution of the unperturbed motion results from the use of Floquet’s theorem which states that any Hill’s equation of the form (A.8) has a general solution which looks like
\[
x^{(0)}(\theta) = a_1 u(\theta)e^{iQ_x\theta} + \bar{a}_1 \bar{u}(\theta)e^{-iQ_x\theta},
\]
\[
p_z^{(0)}(\theta) = a_1 (u' + iQ_x u)e^{iQ_x\theta} + \bar{a}_1 (\bar{u}' - iQ_x \bar{u})e^{-iQ_x\theta},
\]  
(A.9)

where $x^{(0)}$, $p_z^{(0)}$ stand for $q_1^{(0)}$, $p_1^{(0)}$ of the previous section, $a_1$ and $\bar{a}_1$ are complex constants equivalent at the $a_j$s of Section A.1, and $u(\theta)$ is a complex periodic function with the same period as the focusing function $K(\theta)$,
\[
u(\theta) = \sqrt{\frac{\beta_x(\theta)}{2R}} \exp[i(\mu_x - Q_x\theta)],
\]  
(A.10)

in which $\beta_x(\theta)$ is the familiar horizontal betatron amplitude. The horizontal phase advance $\mu_x(\theta)$ and the horizontal wave number $Q_x$ (number of betatron oscillations per turn or tune) are defined by
\[
\mu_x(\theta) = \int_0^\theta \frac{Rd\xi}{\beta_x(\xi)}, \quad Q_x = \frac{\mu_x(2\pi)}{2\pi}.
\]  
(A.11)

In the case of two-dimensional motion (horizontal and vertical), the same description applies two times with the following equivalence in the notation:
\[
u(\theta) \longleftrightarrow v(\theta), \quad \beta_x(\theta), \mu_x(\theta), Q_x \longleftrightarrow \beta_z(\theta), \mu_z(\theta), Q_z
\]
\[a_1, \bar{a}_1 \longleftrightarrow a_2, \bar{a}_2
\]
\[x^{(0)}, p_z^{(0)} \longleftrightarrow z^{(0)}, p_z^{(0)}.
\]  
(A.12)

This provides the description of the form (A.2) for the betatron oscillations with $N = 2$ and with four arbitrary constants.
ii) In the presence of a perturbation in the two-dimensional betatron dynamics, the perturbed motion is described accordingly to Eq. (A.3), by

\[
\begin{align*}
  x &= x^{(0)} [a_1(\theta), \bar{a}_1(\theta), \theta] \\
  p_x &= p_x^{(0)} [a_1(\theta), \bar{a}_1(\theta), \theta] \\
  z &= z^{(0)} [a_2(\theta), \bar{a}_2(\theta), \theta] \\
  p_z &= p_z^{(0)} [a_2(\theta), \bar{a}_2(\theta), \theta].
\end{align*}
\]  

(A.13)

where \( a_1, \bar{a}_1, a_2 \) and \( \bar{a}_2 \) are then supposed to vary with \( \theta \) and to become the new variables \( (a_1 \sim \sqrt{I_z} \exp(i\varphi_z), a_2 \sim \sqrt{I_z} \exp(i\varphi_z)) \).

The Hamiltonian \( H_1 \) has to be rewritten as a function of the constants \( a \) and this can be done formally if the function \( H_1 \) satisfies some assumptions, though the explicit form of \( U \) depends on the problem treated. In general, the form of \( U \) is subordinated to the following properties:

a) \( H_1 \) is made of homogeneous polynomials of degree \( N \) in the coordinates \( x, z \) and momentum conjugates \( p_x, p_z \).

b) The solutions (A.13) are linear functions of the four constants \( a_1, \bar{a}_1, a_2, \bar{a}_2 \) and contain oscillatory terms with frequencies \( Q_z \) and \( Q_x \), by virtue of the form of equation (A.9).

c) For circular accelerators and storage rings, the perturbation is obviously periodic in \( \theta \) with period \( 2\pi \).

Introducing the linear equations (A.9) in a polynomial of degree \( N \) generates a sum of terms with the same degree and all possible combinations of powers for the four constants \( a \), the coefficients of these terms being termed \( h(\theta) \). The periodicity of the perturbation suggests that the coefficients \( h(\theta) \) be developed into Fourier's series. All this provides the following form for the Hamiltonian \( U = H_1 \),

\[
U(a_1, \bar{a}_1, a_2, \bar{a}_2, \theta) = \sum_N \sum_{j,k,\ell,m=0}^N \sum_{q=-\infty}^{\infty} h_{j,k,\ell,m}^{(N)}(\theta) a_1^j \bar{a}_1^k a_2^\ell \bar{a}_2^m \times \exp\{i[(j - k)Q_z + (\ell - m)Q_x + q]\theta\},
\]  

(A.14)

All the coefficients \( h \) depend on the Floquet functions \( u \) and \( v \) by virtue of (A.9) and on the perturbation strengths that multiply linearly each polynomial of degree \( N \).

iii) The equations (A.9) (and their equivalent ones in the vertical plane) can be solved for obtaining expressions of \( a_1 \) and \( \bar{a}_1 \) as functions of \( x \) and \( p_x \),

\[
\begin{align*}
  a_1 &= i[(u' - iQ_z u)x - u p_x]e^{-iQ_z \theta} \\
  \bar{a}_1 &= -i[(u' + iQ_z u)x - u p_x]e^{iQ_z \theta},
\end{align*}
\]  

(A.15)

while similar equations hold for \( a_2, \bar{a}_2 \) as functions of \( z, p_z, v \) and \( Q_z \).
iv) Using the equations (A.6), (A.7), and (A.15) (plus their equivalent ones in the vertical plane), as well as the Floquet functions \( u \) (A.10) and \( v \), the differential equations can be given for the \( a_j \)'s explicitly and simplify remarkably,

\[
\frac{d\hat{a}_1}{d\theta} = -i \frac{\partial U}{\partial a_1} \quad \frac{da_1}{d\theta} = i \frac{\partial U}{\partial \hat{a}_1} \\
\frac{d\hat{a}_2}{d\theta} = -i \frac{\partial U}{\partial a_2} \quad \frac{da_2}{d\theta} = i \frac{\partial U}{\partial \hat{a}_2} .
\]  
(A.16)

Let us next consider the longitudinal oscillations of the synchrotron motion which satisfy the two following equations, when \( \theta \) is the independent variable

\[
\frac{dW}{d\theta} = -\frac{e\tilde{V}}{2\pi \omega_f} (\sin \varphi - \sin \varphi_s) \\
\frac{d\varphi}{d\theta} = \frac{h^2 \eta}{R p_0} W .
\]  
(A.17)

It must be noted at this point that the definition of \( W \) used in these equations, is \( W = -\Delta E/\omega_f \), where \( \Delta E \) is the energy deviation from the synchronous value, and \( \omega_f \) is the RF angular frequency; and this definition differs from the one commonly quoted in the literature by a factor of \(-1/2\pi\hbar\), where \( \hbar \) is the RF harmonic number. The other quantities have their usual meaning; \( \varphi \) is the RF phase, \( \varphi_s \) the synchronous RF phase, \( \tilde{V} \) the RF voltage, \( \eta \) the change of frequency with momentum deviation, \( R \) the average radius (of the synchronous particle), and \( p_0 \) the central momentum. Combining these two equations (A.17), we obtain a second-order equation as expected

\[
\frac{d}{d\theta} \left[ \frac{R p_0}{h^2 \eta} \frac{d\varphi}{d\theta} \right] + \frac{e\tilde{V}}{2\pi \omega_f} (\sin \varphi - \sin \varphi_s) = 0 ,
\]  
(A.18)

which becomes, when considering time \( t \) and using \( \theta = \omega_f t \) and \( \omega_f = h \omega_s \)

\[
\frac{d}{dt} \left[ \frac{R p_0}{\hbar \omega_s} \frac{d\varphi}{dt} \right] + \frac{e\tilde{V}}{2\pi} (\sin \varphi - \sin \varphi_s) = 0 .
\]  
(A.19)

The parameters in the square brackets either being constant or varying slowly, it is possible to write the following equations, the second of which is an approximation of the first when \( \Delta \varphi = \varphi - \varphi_s \) is small

\[
\dot{\varphi} + \frac{h \eta \omega_s e\tilde{V}}{2\pi R p_0} (\sin \varphi - \sin \varphi_s) = 0 \\
or \
\dot{\varphi} + \Omega_s^2 \Delta \varphi = 0 ,
\]  
(A.20)

where \( \Omega_s^2/\cos \varphi_s \) is equal to the bunch of constants entering the first equation. The last equation is very similar to the equation (A.8) for betatron oscillations. It describes the synchrotron oscillations of non-synchronous particles in terms of the synchrotron angular frequency \( \Omega_s \) or synchrotron tune \( Q_s \) (when one keeps \( \theta \) instead of using time \( t \)) given by \( R \Omega_s/\beta c \) and of an amplitude a \( \sim \sqrt{T_s} \exp(i\psi) \) in complex notation, similar to \( a_1 \) of Eqs. (A.9). Starting from this, perturbation theory can be applied to synchrotron motion, just as described above for betatron motion.