THE RESISTIVE-PIPE WAKE POTENTIALS FOR SHORT BUNCHES

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The wake field of a circular resistive pipe is calculated, to all multipole orders, with a particular attention paid to the short distance (or high frequency) range for which the well-known long distance asymptotic expressions are not correct. In the case of a point-like charge, the longitudinal wake oscillates twice between decelerating and accelerating values before it reaches its accelerating long-range behaviour. The wake fields and loss factors for a Gaussian bunch are derived, and their deviation from the known asymptotic expressions is studied for short bunches. An application is made for bunch lengths from $50 \mu m$ to $200 \mu m$ as foreseen for the future linear colliders.

1. INTRODUCTION

The calculation of the wake field generated by a charge distribution travelling in a metallic circular pipe with finite conductivity has been performed in 1, and since then has been presented in many lectures 2 as one of the few analytically solvable wake-potential problems. To our knowledge, however, all the results derived or reviewed in 1, 2, 3 for the longitudinal and transverse wake potentials, impedances and loss factors are asymptotic: they only apply to the long distance (or the low frequency) range of the wake potentials, namely when the distance $s$ behind the exciting charge or the bunch length $\sigma_z$ are large compared to a characteristic distance $s_0$ given by

$$s_0 = \left( \frac{a^2}{Z_0 \sigma} \right)^{\frac{1}{3}}$$

where $a$ is the pipe radius, $Z_0$ the vacuum impedance and $\sigma$ the metal conductivity.

In practice, these asymptotic expressions (summarized in the Appendix) encompass all accelerator applications since, as can be seen from Table 1 in the case of copper at room temperature, bunch lengths are usually much larger than $s_0$, even for large pipe radii. This situation may change with the future linear colliders 4, 5, 6 for which bunch lengths of the order of $100 \mu m$ are necessary in order to achieve a small $\beta^*$ in the final focus system and to avoid beam-strahlung and pair-creation problems during the collision. According to Table 1 for copper pipes, and even more so for metals of lower conductivity for which $s_0$ is larger, the use of the long-range approximation of the wake potentials is questionable for bunches so short, and going beyond this approximation might be necessary.
The aim of this paper is to derive the full expression of the wake potentials and loss factors, including their short distance behaviour, and to study how much they depart from their asymptotic behaviour. Impedances and wake potentials are calculated in Section 2. The loss factors are calculated in Section 3 for Gaussian charge distributions and the results are applied to the case of the very short bunches discussed above. The induced energy spread for a Gaussian bunch is also given. The range of validity of these results, based on the local Ohm’s law, for very short bunches is investigated in Section 4. Finally, the long-distance expressions of the wake potentials are recalled in the Appendix together with the basic definitions.

2. THE RESISTIVE IMPEDANCES AND WAKE POTENTIALS

We consider a charged particle moving inside and parallel to the axis of a pipe of radius \( a \). With respect to this axis and for a particle velocity \( v \), its cylindrical coordinates are \((r = r_0, \theta = 0, z = vt)\). The e.m. field excited by its charge \( q \) can be decomposed as follows

\[
\begin{align*}
E &= E^{(0)} + E^{(r)} \\
B &= B^{(0)} + B^{(r)}
\end{align*}
\]

where \((E^{(0)}, B^{(0)})\) are the e.m. fields corresponding to a pipe with infinite conductivity for which the wake potentials are zero. The additional fields \((E^{(r)}, B^{(r)})\) can therefore be identified with the wake fields generated by the finite conductivity \( a \).

Assuming that \((E^{(0)}, B^{(0)})\) derive from the potentials \((\Phi^{(0)}, A^{(0)})\) with the relation

\[
A^{(0)} = \frac{v}{c^2} \Phi^{(0)}
\]

which obeys Lorentz-gauge condition, one gets for the scalar potential the wave equation

\[
(\partial_{ct}^2 - \Delta) \Phi^{(0)}(r, \theta, z - vt) = \frac{q}{\epsilon_0} \delta^{(2)}(r - r_0) \delta(z - vt)
\]

The expression of \(\Phi^{(0)}\) can be conveniently calculated by using the set of orthonormal eigen-functions of the 3-d Laplace operator \(\Delta\)

\[
\Lambda_{(m,q,k)}(r, \theta, z) = \frac{1}{2\pi} e^{im\theta} J_m(qr) e^{ikz}
\]

associated with the eigen-values

\[
\lambda_{(m,q,k)} = -(k^2 + q^2)
\]
and fulfilling the completeness identity

$$\frac{1}{(2\pi)^2} \sum_{m=-\infty}^{+\infty} \int_{0}^{\infty} q dq \int_{-\infty}^{+\infty} dk \ e^{im\theta} J_m(qr) J_m(qr_0) e^{ik(z-z_0)} = \delta^{(2)}(r-r_0) \delta(z-z_0)$$

In the ultra-relativistic limit where $\gamma \to \infty$, one gets inside of the pipe ($r \leq a$)

$$\Phi^{(0)}(r, z-ct) = \frac{q}{2\pi\varepsilon_0} \delta(z-ct) \left\{ \ln(a/r_>) + \sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m} \left[ \left( \frac{r_<}{r_>} \right)^m - \left( \frac{r_{0r}}{a^2} \right)^m \right] \right\}$$

with $r_> = \text{sup}(r, r_0)$ and $r_< = \text{inf}(r, r_0)$, and $\Phi^{(0)} = 0$ for $r > a$.

According to Eq.(2), the resistive wake-fields $(E^{(r)}, B^{(r)})$ are solutions of the homogeneous Maxwell’s equations in the vacuum for the region inside of the pipe ($r < a$), and with a metallic current given by Ohm’s law

$$j = \sigma E$$

for the region ($r > a$) in the metal. At the metallic interface ($r = a$) between the two regions, the boundary conditions are fulfilled by imposing the continuity of the tangential components of $E$ and $B$ which is equivalent to imposing the continuity of $E_z^{(r)}, E_\theta^{(r)}, B_z^{(r)}$ and $(B_\theta^{(r)} + B_\theta^{(0)})$, where

$$B_\theta^{(0)} = -\frac{1}{c} \partial_\theta \Phi^{(0)}$$

is the only non-zero (and discontinuous) tangential component of $(E^{(0)}, B^{(0)})$ on the boundary.

2.1. The impedances per unit length

Solving Maxwell’s equations and the continuity conditions for the impedances as defined by Eqs.(46) is mostly a problem of algebra. It leads to the following mode expansion for the longitudinal impedance

$$Z_z(r, \theta, \omega) = \sum_{m=0}^{\infty} \cos(m\theta) \ r^m Z_{z(m)}^{(r)}(\omega)$$

where

$$Z_{z(m)}^{(r)}(\omega) = -\frac{i\omega Z_0}{\pi c} \left( \frac{r_0}{a^2} \right)^m (m+1) H_m(Ka) \left[ m(m+1) - \frac{\omega^2 a^2}{c^2} \right] H_m(Ka)$$

$$+ 2(m+1) \frac{\omega^2 a^2}{K c^2} H_{m+1}(Ka) + \frac{\alpha_m}{2} Ka (H_{m+1}(Ka) - H_{m-1}(Ka))^{-1}$$

with the following notations

$$K = (\pm 1 + i) \sqrt{\frac{\omega|Z_0\sigma}{2c}} \quad \text{for} \quad \pm \omega \geq 0$$
and

\[ \alpha_m = \begin{cases} 
2 & \text{for } m = 0 \\
 m + 1 & \text{for } m \geq 1 
\end{cases} \quad (14) \]

The Bessel function \( H_m = (J_m + iY_m) \) is such that \(^7\)

\[ H_m(z) \sim \sqrt{\frac{2}{\pi z}} e^{iz} e^{-i(2m+1)\pi/4} \text{ for } |z| \to \infty \quad (15) \]

From the above asymptotic behaviour, the expression of the impedance simplifies considerably if one assumes that the skin penetration-depth \( \delta \) is much smaller than the pipe radius \( a \), in which case the argument of the Bessel functions is large

\[ Ka = (\pm 1 + i) a/\delta \gg 1 \quad (16) \]

If one further restricts oneself to the first modes such that

\[ Ka \gg m \quad (17) \]

one finally gets

\[ Z_z(m)(\omega) = \frac{1 - i}{\alpha_m \pi a} \sqrt{\frac{Z_0\omega}{2\sigma c}} \left( \frac{r_0}{a^2} \right)^m (m + 1) \left[ 1 - \frac{1 + i}{\sqrt{2\alpha_m}} \left( \frac{\omega s_0 c}{\ell} \right)^{3/2} \right]^{-1} \quad (\omega \geq 0) \quad (18) \]

In practice this assumption is quite justified since it is valid for distances

\[ (s \sim c/\omega) \ll (L = a^2 Z_0 \sigma) \quad (19) \]

where, for instance, \( L \sim 1000 \) km for a copper pipe with 1 cm radius.

As for the transverse impedance, it can be calculated from the longitudinal one by using the Panofsky-Wenzel theorem leading to

\[ Z_\perp(\omega) = \sum_{m=1}^{\infty} m^{-m-1} \cos(m\theta) e_r - \sin(m\theta) e_\theta \frac{ic}{\omega} Z_z(m)(\omega) \quad (20) \]

2.2. **The wake potentials per unit length**

The wake potentials for Gaussian bunches can be computed from the inverse Fourier transform of the impedances and the convolution with Gaussian charge distributions, according to the definitions recalled in the Appendix. As a result, the expression of the longitudinal potential as derived from the impedance (18), obeys the following expansion

\[ W_z(r, \theta, s, \sigma_z) = \frac{2Z_0 c}{\pi^2 a^2} \sum_{m=0}^{\infty} \left( \frac{r}{a} \right)^m \left( \frac{r_0}{a} \right)^m \cos(m\theta) w_z^{(m)}(s/s_0, \sigma_z/s_0) \quad (21) \]

where the dimensionless function \( w_z^{(m)} \) is given by the integral

\[ w_z^{(m)}(u, v) = \frac{m + 1}{3} \int_{-1}^{\infty} dx \frac{e^{-\phi_m^2/2}}{x^2 + 1} \frac{x \sin(\theta_m) + \cos(\theta_m)}{x^2 + 1} \quad (22) \]
with
\[ \theta_m = u \left[ \frac{\alpha_m}{\sqrt{2}}(x + 1) \right]^\frac{3}{2} \quad \text{and} \quad \phi_m = v \left[ \frac{\alpha_m}{\sqrt{2}}(x + 1) \right]^\frac{3}{2} \] (23)

The long distance limit \((s \gg s_0)\) of the potential, given for \(m = 0\) by Eq.(44), follows from the asymptotic behaviour of \(w_z^{(m)}\)

\[ w_z^{(m)}(u, v) \quad u \gg (1, v) \begin{cases} -\left(\sqrt{\pi}/8\right) u^{-\frac{3}{2}} & \text{for } m = 0 \\ -\left(\sqrt{\pi}/4\right) u^{-\frac{3}{2}} & \text{for } m = 1 \end{cases} \] (24)

It is interesting to derive from Eqs.(21,22) the large bunch-length limit \((\sigma_z \gg s_0)\) of the longitudinal potential for \(m = 0\), given by Eq.(50), in order to estimate the magnitude of the correction terms. In this limit the parameter \(v\) in Eq.(22) is large in such a way that the exponential term in the integral damps rapidly the integrand for \(x > -1\). One can thus expand the integrand into powers of \((x + 1)\) which, after integration, are converted into powers of \(v^{-3/2}\). The leading term of the expansion, leading to Eq.(50) through an integral form \(8\) of the parabolic cylinder function \(D_{+\frac{1}{2}}\), is therefore corrected by higher-order integer powers of \((\sigma_z/s_0)^{-3/2}\).

The analysis of the transverse potential follows the same lines as for the longitudinal one. Its mode expansion reads

\[ W_{\perp}(r, \theta, s, \sigma_z) = \frac{2Z_0 e s_0}{\pi^2 a^3} \sum_{m=1}^{\infty} m \left( \frac{r}{a} \right)^{m-1} \left( \frac{r_0}{a} \right)^m (\cos(m\theta) e_r - \sin(m\theta) e_\theta) \times w_{\perp}^{(m)}(s/s_0, \sigma_z/s_0) \] (25)

where the dimensionless function \(w_{\perp}^{(m)}\) is given by

\[ w_{\perp}^{(m)}(u, v) = \left[ \frac{2(m + 1)}{3} \right]^3 \int_{-1}^{\infty} dx \ e^{-x^2/2} \sin(\theta_m) \frac{x \cos(\theta_m)}{(x + 1)^{2/3}(x^2 + 1)} \] (26)

with \(\theta_m\) and \(\phi_m\) given by Eq.(23). The long distance limit of the transverse potential, given by Eq.(45) for \(m = 1\), follows from the asymptotic behaviour of \(w_{\perp}^{(m)}\)

\[ w_{\perp}^{(m)}(u, v) \quad u \gg (1, v) \sim \left(\sqrt{\pi}/2\right) u^{-\frac{3}{2}} \] (27)

Finally, in the limit where \(\sigma_z \gg s_0\), one can show, as for the longitudinal potential, that for \(m = 1\) the transverse potential takes the form given by Eq.(51) and that the higher order corrections are powers of \((\sigma_z/s_0)^{-3/2}\).

The longitudinal and transverse potentials – actually the dimensionless functions \(w_z^{(0)}\) and \(w_{\perp}^{(1)}\) – are plotted in Figs.(1-4). Fig.1 shows the wake potentials for a point-like charge \((\sigma_z = 0)\), also called Green’s functions. The longitudinal potential oscillates twice between positive (decelerating) and negative (accelerating) values before reaching its accelerating long distance limit for \(s \gg s_0\). The value of \(w_z^{(0)}\) for \(s \to 0^+\) can be calculated to be \(\pi/2\) so that

\[ W_z(s \to 0^+, \sigma_z = 0) = \frac{Z_0 e}{\pi a^2} \] (28)
The transverse Green’s function joins its asymptote also in the same region after passing through a well-marked local minimum. Fig.2 shows how the discontinuity of the longitudinal Green’s function develops at \( s = 0 \) by plotting the longitudinal Gaussian potential for values of \( \sigma_z \) approaching zero. One sees the non-uniform convergence of \( W_z(s, \sigma_z) \) for \( s = 0 \) when \( \sigma_z \to 0 \), responsible for the well-known fact that

\[
W_z(s = 0) = \frac{1}{2} W_z(s \to 0^+) \quad \text{for} \quad \sigma_z = 0
\]  

(29)

In Fig.3 and Fig.4, the longitudinal and transverse wake potentials for Gaussian distributions are compared, for various values of \( \sigma_z/s_0 \), to their limiting form when \( \sigma_z \gg s_0 \).

3. THE GAUSSIAN LOSS FACTORS

In this section, the loss factors of Gaussian bunches are expressed under the form of one-dimensional integrals. An application is made for the very short bunches envisaged in the future linear colliders.

With the definitions given in the Appendix, the loss-factors per unit length can be calculated by using the impedances given in Eq.(18) and Eq.(20). The longitudinal and, for obvious symmetry reasons, the purely radial transverse loss-factors of a Gaussian bunch with radial offset \( r \), can be decomposed as

\[
k_z(r, \sigma_z) = \frac{2Z_0 c}{\pi^2 a^2} \sum_{m=0}^{\infty} \left( \frac{r}{a} \right)^{2m+1} \kappa_z^{(m)}(\sigma_z/s_0)
\]  

(30)

\[
k_r(r, \sigma_z) = \frac{2Z_0 c s_0}{\pi^2 a^3} \sum_{m=1}^{\infty} m \left( \frac{r}{a} \right)^{2m} \kappa_r^{(m)}(\sigma_z/s_0)
\]  

(31)

where the dimensionless functions \( \kappa_z^{(m)} \) and \( \kappa_r^{(m)} \) are given by the following integrals

\[
\kappa_z^{(m)}(v) = \frac{m + 1}{3} \int_{-1}^{\infty} dx \ e^{-\phi_m^2} \frac{1}{x^2 + 1}
\]  

(32)

\[
\kappa_r^{(m)}(v) = - \frac{[2(m + 1)]^{1/3}}{3} \int_{-1}^{\infty} dx \ e^{-\phi_m^2} \frac{x}{(x + 1)^{2/3}(x^2 + 1)}
\]  

(33)

with \( \phi_m \) given by Eq.(23). The leading terms of the mode expansions, namely \( \kappa_z^{(0)} \) and \( \kappa_r^{(1)} \), are represented in Fig.5 together with their asymptotes for \( \sigma_z \gg s_0 \), given by Eqs.(54,55) in agreement with the following asymptotic behaviour of the functions \( \kappa_z^{(m)} \) and \( \kappa_r^{(m)} \)

\[
\kappa_z^{(m)}(v) \approx 1 \begin{cases} \left( \Gamma \left( \frac{3}{4} \right) / 8\sqrt{2} \right) v^{-\frac{3}{2}} & \text{for } m = 0 \\ \left( \Gamma \left( \frac{5}{4} \right) / 4\sqrt{2} \right) v^{-\frac{3}{2}} & \text{for } m \geq 1 \end{cases}
\]  

(34)

\[
\kappa_r^{(m)}(v) \approx 1 \begin{cases} \frac{\Gamma(1/4)}{4\sqrt{2}} v^{-\frac{1}{2}} & \text{for } m = 0 \\ \left( \Gamma \left( \frac{5}{4} \right) / 4\sqrt{2} \right) v^{-\frac{3}{2}} & \text{for } m \geq 1 \end{cases}
\]  

(35)

One sees that both loss factors depart significantly from their long-bunch limit for \( \sigma_z < 2s_0 \), with correction terms of the order of \( (\sigma_z/s_0)^{-3/2} \) like for the potentials. Nevertheless, the
FIGURE 1  The wake potentials of a point-like charge ($\sigma_z = 0$).

FIGURE 2  The longitudinal wake-potential $w_z^{(m=0)}$ for very short bunches. [$v = \sigma_z/s_0$].
FIGURE 3  The Gaussian longitudinal wake-potentials $w_{z}^{(m=0)}$. $[v = \sigma_z/s_0]$. 

FIGURE 4  The Gaussian transverse wake-potentials $w_{\perp}^{(m=1)}$. $[v = \sigma_z/s_0]$. 
order of magnitude of the longitudinal loss-factor is correctly given by its asymptote down to $\sigma_z \sim (0.4 \, s_0)$. This is shown also in Fig.6 where the longitudinal loss-factors are plotted as functions of the beam-pipe radius $a$ for bunch lengths ranging from $25 \, \mu m$ to $200 \, \mu m$. In the region of small $a$, corresponding to the large-$v$ region in Fig.5, the loss factors are well represented by their asymptote. On the other hand, the discrepancy is never larger than a factor 2 for values of $a$ up to $20 \, cm$. The cross-over visible on Fig.5 only appears, below $20 \, cm$, for the very small bunch-length of $25 \, \mu m$ which, as discussed in the next section, is close to the limit of validity of the calculation.

For a point-like charge ($\sigma_z = 0$), the integral in Eq.(32) can be calculated analytically and leads to

$$\kappa_z^{(m)}(v = 0) = (m + 1) \frac{\pi}{4}$$

(36)

As a consequence, the longitudinal loss-factor of particle on axis ($r = 0$) is given by

$$k_z(\sigma_z = 0) \equiv W_z(s = 0, \sigma_z = 0) = \frac{Z_0 c}{2\pi a^2}$$

(37)

in agreement with the value of longitudinal Green’s function $W_z(s \to 0^+, \sigma_z = 0)$ given by Eq.(28) and related to it through Eq.(29).

Finally, we indicate the result obtained for the rms energy-spread $\sigma_E$ induced, per unit length, by the resistive walls on a Gaussian bunch travelling on axis ($r = 0$). For $\sigma_z \gg s_0$, it is given by

$$\sigma_E(\sigma_z) \equiv N q^2 \left( \int_{-\infty}^{+\infty} ds \, \lambda(s) \, W_z^2(s, \sigma_z) - k_z^2(\sigma_z) \right)^{\frac{1}{2}} = \frac{0.46 N q^2 c}{2\pi^2 a \sigma_z^{3/2}} \sqrt{\frac{Z_0}{\sigma}}$$

(38)
where $N$ is the number of particles in the bunch.

4. DOMAIN OF VALIDITY

The domain of validity of the results presented in this paper is limited by the assumptions leading to Eq.(19) for the large values of $s = |z - ct|$ for the potentials, and, for the small values of $s$ or $\sigma_z$, by the validity of the local Ohm’s law, expressed in Eq.(9), which relies on the two following assumptions\textsuperscript{9, 10} that:

- the frequency-dependent Ohm’s law given by
  \[ j = \frac{\sigma}{1 + i\omega \tau} E \]  
  where $\tau$ is the relaxation time of the conduction electrons is well approximated by its static form used in Eq.(9)

- the skin penetration-depth $\delta = \sqrt{2\epsilon_0 \sigma / \omega Z_0}$ is much larger than the mean free-path $l = v_F \tau$ of the electrons in the metal, where the $v_F$ is the Fermi velocity, thus avoiding the anomalous skin effect.

Both conditions set upper bounds on the frequency $\omega$. Considering that the potentials and loss factors receive, in the integrals given by Eqs.(22,26,32,33), their main contribution from the low frequencies, namely

\[ \omega < c/s \quad \text{and} \quad \omega < c/\sigma_z \]  

(40)
it is easy to translate these bounds into lower bounds for either of the parameters $s$ or $\sigma_z$:

$$s, \sigma_z > s'_0 = \left( \frac{l}{\sigma} \right) \frac{c}{v_F} \sigma$$

(41)

for the first condition, and

$$s, \sigma_z > s''_0 = \frac{1}{2} \left( \frac{l}{\sigma} \right)^2 Z_0 \sigma^3$$

(42)

for the second one. The ratio $(\sigma/l)$ is a characteristic constant independent of the temperature. For copper at 300 $K$, $(\sigma/l) = 1.54 \times 10^{15} \, \Omega^{-1} m^{-2}$ and $v_F = 1.57 \times 10^6 \, m/s$ leading to

$$s'_0 = 7 \, \mu m \quad \text{and} \quad s''_0 = 16 \, \mu m$$

(43)

so that the second condition is more constraining.

Finally it is worth noticing that translating these lower bounds in terms of the dimensionless variables $u = s/s_0$ and $v = \sigma_z/s_0$ leads to bounds which scales with the pipe radius like $a^{2/3}$. Therefore the small-$u$ and small-$v$ features of the wake potentials and loss factors, exhibited for instance in Figs. (1,2) and not accounted for by the asymptotic formulas of the Appendix, are physical for large enough beam-pipe radius.

5. CONCLUSIONS

In this paper, the resistive wall wake-potentials for a circular pipe have been studied beyond the usual approximation where the distance $|z - ct|$ behind the exciting charge is large compared to the characteristic distance $s_0$ given by Eq.(1). As a consequence the expressions of the wake potentials and loss factors for Gaussian bunches which have been derived to all multipole orders, apply to very short bunches for which the usually known expressions cannot, in principle, be extended.

Some peculiar short distance features have been found: in particular the longitudinal Green’s functions exhibits a kind of damped oscillatory behaviour in such a way that an extra decelerating region exists around $|z - ct| = 4s_0$, a property already observed for parallel resistive walls 11. Also the short-bunch limit of the Gaussian loss factors is precisely calculated. In this limit the radial loss-factor tends to zero and therefore differs violently from its $\sigma_z^{-1/2}$ asymptotic behaviour for large bunch-length $\sigma_z$. On the other hand, there is a cross over between the actual expression of the longitudinal loss-factor and its $\sigma_z^{-3/2}$ asymptotic large-$\sigma_z$ behaviour. A consequence of this cross-over is that, as seen in Fig.6, the discrepancy between the two expressions is never bigger than 100% even for small bunch-lengths, in the range of 100 $\mu m$, and for reasonable beam-pipe radius. The discrepancy increases for large pipe-radius for which however the absolute magnitude of the wake potentials tends to zero.

REFERENCES

A APPENDIX. THE LONG DISTANCE LIMIT OF THE RESISTIVE WAKE POTENTIALS

In this appendix are reviewed the expressions of the resistive wake potentials, impedances and loss factors, known from a long time in the case of a cylindrically symmetric pipe with finite conductivity $\sigma$, when the distances or bunch lengths are large compared to the characteristic length $s_0$ given by Eq.(1) (see also Table 1).

The longitudinal and transverse wake potentials, per unit length, of a point-like charge $q$ (also called Green's functions) have been calculated for $s \gg s_0$ [1,2]

$$W_z(s) \equiv -\frac{1}{q} \int dz \frac{E_z(r,z,t) = (z+s)/c}{4\pi a} \sqrt{\frac{Z_0}{\pi \sigma}} s^{-3/2} \quad (m = 0) \quad (44)$$

and

$$W_\perp(s) \equiv \frac{1}{q} \int dz \frac{(E_\perp + v \times B)(r,z,t) = (z+s)/c}{r_0 \sigma/3 \sqrt{\frac{Z_0}{\pi \sigma}} s^{-1/2} \quad (m = 1) \quad (45)}$$

where $r_0$ is the radial offset of the exciting charge. Both $W_z$ and $W_\perp$ at this order in $m$, do not depend on the radial offset $r$ of the test particle.

Green's functions $W(s)$ derives from impedances $Z(\omega)$ through a Fourier transform:

$$W(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \; Z(\omega) \exp\left(-\frac{i\omega s}{c}\right) \quad (46)$$

The longitudinal and transverse impedances per unit-length are given, in the asymptotic range $|\omega| \ll c/s_0$ conjugate to $s \gg s_0$, by

$$Z_z(\omega) = \frac{1}{2\pi a} \sqrt{\frac{Z_0 \omega}{2\sigma c}} \left(1 - i\right) \quad (47)$$

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1 Units are S.I. Notice that, in Eq.(44), a factor 1/2 is missing in the second and third of Ref.2, and a factor 2 in the last of Ref.2 for Eqs.(45,48)

2 We do not follow the usual convention to divide by $i$ the transverse impedance
and

$$Z_{\perp}(\omega) = \frac{r_0}{\pi a^3} \sqrt{\frac{Z_0 c}{2\sigma \omega}} (1 + i)$$  \hspace{1cm} (48)$$

for $\omega \geq 0$, completed for $\omega \leq 0$ by the usual parity relation

$$Z(-\omega) = Z^*(\omega)$$  \hspace{1cm} (49)$$

The wake potentials of a normalized Gaussian charge distribution $\lambda(s)$ with rms-length $\sigma_z$, are also known $^2, ^3$ when $\sigma_z \gg s_0$:

$$W_z(s, \sigma_z) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} ds' W_z(s - s') \lambda(s')$$

$$= -\frac{c}{2\pi a^2 \sigma_z^2} \sqrt{\frac{Z_0}{2\pi \sigma}} \exp\left(-\frac{s^2}{4}\right) D_{\frac{1}{2}}(-\hat{s}) \quad (m = 0)$$  \hspace{1cm} (50)$$

and

$$W_\perp(s, \sigma_z) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} ds' W_\perp(s - s') \lambda(s')$$

$$= \frac{r_0 c}{\pi a^3 \sigma_z^2} \sqrt{\frac{Z_0}{2\pi \sigma}} \exp\left(-\frac{s^2}{4}\right) D_{-\frac{1}{2}}(-\hat{s}) \quad (m = 1)$$  \hspace{1cm} (51)$$

where $\hat{s} = s/\sigma_z$. The parabolic cylinder functions $D_{\pm \frac{1}{2}}$ can be expressed $^1, ^2$ as the following linear combinations of modified Bessel functions:

$$D_{\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{4} |x|^{3/2} (I_{-\frac{3}{4}} - I_{\frac{1}{4}} - \epsilon_x I_{\frac{3}{4}} + \epsilon_x I_{-\frac{1}{4}})$$  \hspace{1cm} (54)$$

$$D_{-\frac{1}{2}}(x) = \frac{\sqrt{\pi}}{2} |x|^{1/2} (I_{-\frac{3}{4}} - \epsilon_x I_{\frac{3}{4}})$$  \hspace{1cm} (55)$$

where the argument of the Bessel functions is $x^2 / 4$ and $\epsilon_x$ is the sign of $x$. The longitudinal and radial loss-factors deriving from the above expressions are given by

$$k_z(\sigma_z) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} ds W_z(s, \sigma_z) \lambda(s) = \frac{c}{4\pi^2 a^3 \sigma_z^{3/2}} \sqrt{\frac{Z_0}{2\sigma}} \Gamma(3/4) \quad (m = 0)$$  \hspace{1cm} (56)$$

and

$$k_r(r, \sigma_z) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} ds W_r(s, \sigma_z) \lambda(s) = \frac{c}{2\pi^2 a^3 \sigma_z^{1/2}} \sqrt{\frac{Z_0}{2\sigma}} \Gamma(1/4) r \quad (m = 1)$$  \hspace{1cm} (57)$$

where $r$ is the radial offset of the bunch.