STATISTICAL DESCRIPTION OF NONLINEAR PARTICLE MOTION IN CYCLIC ACCELERATORS

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Abstract

We study the problem of beam emittance growth in particle accelerators due to nonlinear perturbations and external noise in the framework of statistical mechanics, using one particle distribution function approach. The basic model for our analysis is a one-dimensional FODO cell. We introduce other modifications in order to include the effects of tune modulation, external noise and adiabatic variation of the nonlinearity strength.

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I. INTRODUCTION.

The understanding of beam losses due to slow particle transport has become of great importance in view of commissioning the next generation of particle accelerators like LHC and SSC. In such superconducting machines high multipole field errors can not be avoided in principle. Additionally other perturbations are commonly present, the source of which can be the beam-beam interaction, ground waves, intrabeam scattering, rf noise, power supply ripple etc.

An individual particle moving in a circular machine will experience amplitude growth in both transverse directions whenever a perturbing force acts on it. and this eventually could cause a loss of the particle itself. On a macroscopic scale this increase of the amplitude of each particle can not be detected as an isolated event, but rather as a blow up of beam size. The latter is the relevant macroscopic variable that one can measure by means of a macroscopic device. Therefore a single particle tracking code is not the most appropriate tool to study such transport phenomena. One possible way to describe them is through the use of one particle distribution function in phase space, which gives us the entire information about the properties of the beam, considered as an ensemble of particles.

An exhaustive theoretical basis in this field is far from being established at the moment. Nevertheless there are two main approaches to study the problem. The first one represents the time variation of the action variable in a certain region in phase space under the influence of a non-stationary perturbation (for example modulation of the linear tune) as a random walk [1-4], while the second is a phenomenological approach that tries to describe the evolution of particle distribution function as a diffusion process with a properly chosen diffusion coefficient [5].

In this paper we make an attempt to describe transport phenomena in a global way, using the powerful tools of statistical mechanics. We study the dynamics of a very simple model that is an ensemble of non interacting particles moving in a one-dimensional FODO cell where the sextupole is considered in the kick approximation. In addition we have a tune modulation and external noise.

II. EXACT SOLUTION OF THE MASTER EQUATION IN TERMS OF PATH INTEGRALS.

In a previous paper [6] we introduced the master equation

\[
\frac{\partial W(X,P;\theta)}{\partial \theta} = -\chi \left( P \frac{\partial}{\partial X} - X \frac{\partial}{\partial P} \right) W(X,P;\theta) + \frac{\beta G X}{R} \frac{\partial W(X,P;\theta)}{\partial P} + \\
+ \frac{\beta D}{2} \frac{\partial^2 W(X,P;\theta)}{\partial P^2} + \delta_x(\theta) [ W(X,P + K_X,\theta) - W(X,P;\theta) ],
\]

(2.1)

which governs the evolution of the conditional probability

\[ W(X,P;\theta) = W(X,P;\theta|X_0,P_0;\theta) \]
to find a particle with relative deviation from the closed orbit

\[ X = \frac{x}{\sqrt{\beta}} \]

and a relative momentum

\[ P = p\sqrt{\beta} - \frac{\beta x}{2R\sqrt{\beta}} , \]

at "time" \( \theta \) given the initial values \( X_0 \) and \( P_0 \) respectively at some initial \( \theta_0 \). Here \( x \) is the actual deviation of particle's coordinate from the closed orbit in the horizontal plane, \( p \) is the relative momentum, \( R \) is the mean machine radius and \( \beta \) is the horizontal beta-function. The first term on the r.h.s. of eq. (2.1) corresponds to particle motion in the linear machine; \( \dot{x} \) being the derivative of the phase advance with respect to the independent variable \( \theta \). In addition to it we assume a ripple in the power supply of linear machine elements to be present, which is described by the second term on the r.h.s. of eq. (2.1). The third term stands for the effect of some phenomenological random force with intensity \( D \), which tends to distort the closed orbit. Finally the last term describes the momentum jumps, due to a localized sextupolar nonlinearity with strength

\[ K_0 = \frac{\lambda_0 \beta^{1/2}(0)}{2R^2} . \]

One can not attack directly the problem of solving the master equation (2.1), nevertheless it has a simple structure in the "time" interval between successive jumps. That is why it is convenient to define the Fourier transform of the conditional probability, which is precisely the characteristic function as

\[ \Phi(\xi_x, \xi_p; \theta) = \int X dP W(X, P; \theta) \exp\left[ i\left( X\xi_x + P\xi_p \right) \right] , \tag{2.2} \]

and then try to derive an equation for it. It will turn out that the equation we are looking for is a first order partial differential equation, apart from a term that should be considered only in an interval of vanishing length on both sides of successive jump locations. For that purpose we multiply both sides of eq. (2.1) by \( \exp\left[ i\left( X\xi_x + P\xi_p \right) \right] \) and perform the integration in \( X \) and \( P \). The integral containing \( W(X, P + K_0X^2; \theta) \) as an integrand can be handled easily if one notices that

\[ W(X, P + K_0X^2; \theta) = \exp\left( K_0X^2 \frac{\partial}{\partial P} \right) W(X, P; \theta) , \]

and secondly that the partial integration in \( P \) in the presence of the operator \( \partial / \partial P \) yields a multiplier \(-i\xi_p\). Finally one arrives at the desired result, namely an equation for the characteristic function (2.2), that is
\[
\frac{\partial \Phi(\xi_r, \xi_p; \theta)}{\partial \theta} = \left[ -\frac{BD}{2} \xi_r^2 + \chi \xi_r \frac{\partial}{\partial \xi_r} - \left( \dot{\chi} + \frac{BG}{R} \right) \xi_p \frac{\partial}{\partial \xi_r} \right] \Phi(\xi_r, \xi_p; \theta) + \\
+ \delta_\phi(\theta) \left\{ -\Phi(\xi_r, \xi_p; \theta) + \frac{e^{-\text{sgn}(\xi_p)\pi^2}}{2\sqrt{\pi \kappa_n |\xi_p|}} \int d\xi_r \Phi(\xi_r, \xi_p; \theta) \exp \left[ i \frac{(\xi_r - \xi)^2}{4 \kappa_0 |\xi_p|} \right] \right\}, \tag{2.3}
\]

where \(\text{sgn}(x)\) is the sign function defined as
\[
\begin{align*}
\text{sgn}(x) &= 1, & \text{for } x \geq 0; \\
\text{sgn}(x) &= -1, & \text{for } x < 0.
\end{align*}
\]

In deriving eq. (2.3) use has been made of the formula
\[
\int_{-\infty}^{\infty} dx \exp \left[ \pm i(ax^2 + 2bx) \right] = \frac{\pi}{\sqrt{|a|}} \exp \left[ \mp \frac{b^2}{a} - \frac{\pi}{4} \text{sgn}(a) \right]. \tag{2.4}
\]

The next step is to find the solution of our basic equation for "integer times", that is for integer number of turns. For the time period between the \(n\)-th and \((n+1)\)-th kick produced by the nonlinearity eq. (2.3) reduces to
\[
\frac{\partial \Phi}{\partial \theta} - \chi \xi_r \frac{\partial \Phi}{\partial \xi_r} + \left( \dot{\chi} + \frac{BG}{R} \right) \xi_p \frac{\partial \Phi}{\partial \xi_r} = -\frac{BD}{2} \xi_r^2 \Phi \tag{2.5a}
\]

with the initial condition
\[
\Phi(\xi_r, \xi_p; 2\pi n) = \Phi(\xi_r, \xi_p), \tag{2.5b}
\]

The general solution of eq. (2.5a) given the initial condition (2.5b) can be written as
\[
\Phi = \Phi_a \left[ \left[ \hat{R}(\Delta \psi) \xi_r \right] \Phi \right] \exp \left( -\frac{1}{4} \xi_s^T \hat{\Sigma} \xi_s \right), \tag{2.6}
\]

where \(\xi\) is a 2D-vector with components \(\xi_r\) and \(\xi_p\), \(\hat{R}(\Delta \psi)\) is a rotation \(2 \times 2\) matrix
\[
\hat{R}(\Delta \psi) = \begin{pmatrix} \cos \Delta \psi & -\sin \Delta \psi \\ \sin \Delta \psi & \cos \Delta \psi \end{pmatrix} \tag{2.7}
\]

with
\[
\Delta \psi = \chi(\theta) - \chi(2\pi n) + \frac{1}{2R} \int_{2\pi n}^{\theta} d\tau \beta(\tau) G(\tau) \tag{2.8}
\]

and \(\hat{\Sigma}\) is a symmetric \(2 \times 2\) matrix with components
\[ A_{xx} = \langle \beta D \rangle_{2\pi n}^\theta \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right)_{2\pi n}^\theta \sin 2\Delta \psi , \tag{2.9a} \]

\[ A_{xp} = A_{px} = \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right)_{2\pi n}^\theta (1 - \cos 2\Delta \psi ) , \tag{2.9b} \]

\[ A_{pp} = \langle \beta D \rangle_{2\pi n}^\theta + \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right)_{2\pi n}^\theta \sin 2\Delta \psi . \tag{2.9c} \]

The notation \( \langle \ldots \rangle_{2\pi n}^\theta \) adopted in the expressions (2.9) means an average over the interval indicated.

We are ready to find the solution of eq. (2.3) for one turn, i.e. to express the characteristic function after the \((n+1)\)-th kick as a functional of the characteristic function after the \(n\)-th kick. First we integrate equation (2.3) in \( \theta \) in the interval \([2\pi(n+1) - \varepsilon, 2\pi(n+1) + \varepsilon]\) and then let \( \varepsilon \to 0 \). The result is

\[ \Phi_{\ast n}(\xi_x, \xi_p) = \frac{e^{-i\psi_n(\xi_x, \xi_p)}}{2\sqrt{\pi K_0|\xi_p|}} \int d \xi_1 \exp \left[ \frac{(\xi_1 - \xi_x)^2}{4 K_0 \xi_p} - \frac{1}{4} \left( A_{xx}^{(n)} \xi_1^2 + 2 A_{xp}^{(n)} \xi_1 \xi_p + A_{pp}^{(n)} \xi_p^2 \right) \right] * \Phi_n(\xi_1 \cos \psi_n - \xi_p \sin \psi_n, \xi_1 \sin \psi_n + \xi_p \cos \psi_n) , \tag{2.10} \]

where now

\[ \psi_n = 2\pi n + \frac{1}{2 R} \int_{2\pi n}^{2\pi(n+1)} d\tau \beta (\tau) G_r(\tau) , \tag{2.11a} \]

\[ A_{xx}^{(n)} = 2\pi \langle \beta D \rangle - \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right) \sin 2\psi_n , \tag{2.11b} \]

\[ A_{xp}^{(n)} = A_{px}^{(n)} = \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right)(1 - \cos 2\psi_n) , \tag{2.11c} \]

\[ A_{pp}^{(n)} = 2\pi \langle \beta D \rangle + \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G_r} \right) \sin 2\psi_n , \tag{2.11d} \]

and the notation \( \langle \ldots \rangle \) means an average over \( 2\pi \), namely
\( \langle \ldots \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ldots \). \hspace{1cm} (2.11e)

Our initial task was to find the solution of the original master equation (2.1), which means that we must perform now the inverse Fourier transform in (2.10). To do so we multiply both sides of eq. (2.10) by \((2\pi)^{-2}\exp[-i(X\xi_x + P\xi_p)]\) and integrate in \(\xi_x\) and \(\xi_p\). The integration in \(\xi_x\) can be easily performed with the help of formula (2.4) to give

\[
W_{n+1}(X, P) = \frac{1}{(2\pi)^2} \int d\xi_x d\xi_p \exp\left[-i\xi_x X - i\xi_p (P + K_0 X^2) - \frac{1}{4} \left( A_{11}^{(n)} \xi_x^2 + 2 A_{1p}^{(n)} \xi_x \xi_p + A_{pp}^{(n)} \xi_p^2 \right) \right] 
\]

\* \[ \Phi_n (\xi_1 \cos \psi_n - \xi_p \sin \psi_n, \xi_1 \sin \psi_n + \xi_p \cos \psi_n) . \]

In the last integral we change the integration variables according to

\[
\begin{pmatrix} \xi_1 \\ \xi_p \end{pmatrix} = \hat{R}_n \begin{pmatrix} \xi_{1l} \\ \xi_{pl} \end{pmatrix}, \quad \text{with} \quad \hat{R}_n = \begin{pmatrix} \cos \psi_n & \sin \psi_n \\ -\sin \psi_n & \cos \psi_n \end{pmatrix}, \hspace{1cm} (2.12)
\]

which is in fact a simple rotation by an angle \(\psi_n\). This yields

\[
W_{n+1}(X, P) = \frac{1}{(2\pi)^2} \int d\xi_x d\xi_p \exp\left[-i\xi_x T_n \circ X - i\xi_p T_n \circ P - \frac{1}{4} \left( A_{11}^{(n)} \xi_x^2 - 2 A_{1p}^{(n)} \xi_x \xi_p + A_{pp}^{(n)} \xi_p^2 \right) \right] 
\]

\* \[ \int \int dX \circ dP W_n (X_t, P_t) \exp \left[ i(X_t \xi_x + P_t \xi_p) \right] , \hspace{1cm} (2.13)
\]

where

\[
T_n \circ \begin{pmatrix} X \\ P \end{pmatrix} = \hat{R}_n^T \begin{pmatrix} X \\ P + K_0 X^2 \end{pmatrix} \hspace{1cm} (2.14)
\]

is the time-dependent Henon map (the Henon map with a modulation of the unperturbed frequency). Using the standard formula

\[
\int_{-\infty}^{\infty} dx \exp(-px^2 - qx) = \frac{\pi}{\sqrt{p}} \exp \left( \frac{q^2}{4p} \right)
\]

for computing the double integral in \(\xi_x\) and \(\xi_p\) one gets readily from (2.13)
\[ W_{n+1}(X, P) = \frac{1}{\pi \sqrt{\Delta^{(n)}}} \int \int \prod_{k=0}^{n} \Delta^{(k)} dx_k dp_k W_n(X_n, P_n) \exp \left[ -\left( \begin{array}{c} X_n - T_n \circ X \\ P_n - T_n \circ P \end{array} \right)^T \hat{A}_n^{-1} \left( \begin{array}{c} X_n - T_n \circ X \\ P_n - T_n \circ P \end{array} \right) \right] \]

where

\[ \Delta^{(n)} = 4\pi^2 (\beta D)^2 - 2 \frac{R \beta^2 D}{2R^2 + \beta^2 G} (1 - \cos 2\psi) \]  

(2.16a)

is the determinant of the diffusion matrix defined by its components from (2.11), and the matrix

\[ \hat{A}_n^{-1} = \frac{1}{\Delta^{(n)}} \begin{pmatrix} A_{pp}^{(n)} & A_{xp}^{(n)} \\ A_{xp}^{(n)} & A_{xx}^{(n)} \end{pmatrix} \]  

(2.16b)

is the inverse of that entering eq. (2.13). We may now repeat the whole procedure passing over to the initial distribution \( W_0 \). This yields a 2n-dimensional integral called the path integral [7, 8]

\[ W_n(X, P) = \frac{1}{\pi^n \sqrt{\prod_{k=0}^{n-1} \Delta^{(k)}}} \int \int \prod_{k=0}^{n-1} dx_k dp_k W_0(X_0, P_0) \exp \left[ -\text{Path}(0, n) \right], \]  

(2.17)

where

\[ \text{Path}(0, n) = \sum_{i=0}^{n-1} \left( X_k - T_k \circ X_{k+1} \right)^T \hat{A}_k^{-1} \left( X_k - T_k \circ X_{k+1} \right) \]  

(2.18)

and

\[ X_n = X, \quad P_n = P. \]  

(2.19)

Sometimes in the literature the path function, thus defined by (2.18) is encountered as the generalized Onsager-Machlup function.

If we are interested in the probability of finding the final coordinate \( X_n = X \) and momentum \( P_n = P \) at "time" \( \theta = 2\pi n \) irrespective of the special path chosen, we have to integrate over all intermediate coordinates and momenta. This can be done by a simple Monte Carlo method [9], or by its refined variant initially proposed by Metropolis [9, 10].
Two more features are further evident from the expression for the conditional probability (2.17) after the n-th turn. The most probable path of the particle is that for which Path(0, n) has a minimum, i.e.

\[
\begin{pmatrix}
X_k \\
P_k
\end{pmatrix} = T_k \circ \begin{pmatrix}
X_{k+1} \\
P_{k+1}
\end{pmatrix},
\]

which is just the backwards iterated time-dependent Henon map. Secondly, in the low noise limit, that is D \to 0 the exponential of the generalized Onsager-Machlup function combined with the normalizing factor acts as a delta-function. This allows us to handle the 2n-dimensional integral in (2.17) explicitly and we arrive at

\[
W_n(X, P) = W_0[(T_0 \circ T_1 \circ \ldots T_{n-1} \circ X) , (T_0 \circ T_1 \circ \ldots T_{n-1} \circ P)],
\]

which simply means that the initial functional form of the distribution is retained. One only has to iterate backwards the arguments of the initial distribution according to the law provided by the time-dependent Henon map.

It is important to note that the result displayed by (2.21) may be obtained directly, solving the master equation (2.1) for D=0 and as a consequence it should hold for an arbitrary order nonlinear map.

III. STATISTICAL PROPERTIES OF THE HENON MAP.

In the present Section we discuss two specific cases, namely the time independent Henon map and the Henon map with a quadratic term slowly varying in time.

In the first situation there is no modulation of the unperturbed tune and equation (2.21) reads as

\[
W_n(X, P) = W_0(T^n \circ X, T^n \circ P),
\]

where \(T\) denotes the usual Henon map. It is important to know the connection between the description in terms of maps and statistical description in terms of probability densities. In particular let us consider in some detail the relations between the distribution function \(W_n\) and the invariant sets of the map \(T\) (fixed points, cycles, invariant manifolds). First of all suppose that

\[
W_0 : \mathbb{R}^2 \xrightarrow{1-1} \mathbb{R} \quad ; \quad W_0 : \mathbb{R} \xrightarrow{|t| \to 0} 0
\]

Under these hypotheses we can translate the concept of fixed point from the map to the distribution function according to:

Definition 1:

\(\tilde{X} \in \mathbb{R}^2\) is a fixed point for \(W_n \Leftrightarrow W_n(\tilde{X}) = W_0(\tilde{X}).\)

As far as the stability of a fixed point is concerned, we can state the following:
Definition 2:

$\tilde{X} \in \mathbb{R}^2$ is a stable fixed point for $W_n \Leftrightarrow \forall U, \exists V$ so that $W_n : U \rightarrow V$ independently of the number of iterations. Here $U$ is a domain in the neighborhood of the fixed point $\tilde{X}$ with radius smaller than every initially given number $\varepsilon > 0$, while $V$ is a subset of $\mathbb{R}$ not containing the origin.

Definition 3:

$\tilde{X} \in \mathbb{R}^2$ is an unstable fixed point for $W_n \Leftrightarrow \forall U, \exists \tilde{X}_1 \in U$ such that $W_n(\tilde{X}_1) \xrightarrow{n \to \infty} 0$.

Using these three definitions it is readily proved that $\tilde{X} \in \mathbb{R}^2$ is a fixed point for $W_n \Rightarrow T^n \circ \tilde{X} = \tilde{X}$. Moreover its stability type (as a fixed point of $W_n$) is completely determined by the map. In other words the concept of fixed point and its stability is automatically transferred from maps to probability distributions.

One can also analyze the behavior of $W_n$ on the hyperbolic invariant sets of the Henon map, namely the invariant manifolds emanating from unstable fixed points of arbitrary periods. It is worth to consider the unstable fixed point of period one $\tilde{X}_h$ not only for the sake of simplicity, but also for it determines the dynamic aperture of $T$ [11].

Each point belonging to the invariant manifolds $W^{u, e}_{\tilde{X}_h}$ tends to the hyperbolic fixed point under the iteration of the map or its inverse. This means that:

$$W_n(\tilde{X}_m) \xrightarrow{n \to \infty} W_0(\tilde{X}_h) \quad \text{if} \quad \tilde{X}_m \in W^{u, e}_{\tilde{X}_h} \quad (3.2)$$

and due to the homoclinic intersections the limit (3.2) is reached in an oscillatory way as can be seen from the figures.

We can apply these concepts to understand the behavior of a time-dependent version of the Henon map given by:

$$\begin{pmatrix} X_{n+1} \\ P_{n+1} \end{pmatrix} = \tilde{R}(v) \begin{pmatrix} X_n \\ P_n + K(n)X_n^2 \end{pmatrix} \quad (3.3)$$

This corresponds to a FODO cell with the strength of the sextupole changing in time. In particular we are interested in the behavior of $W_n$ under adiabatic variation of $K(n)$ in eq. (3.3). In a recent work [12] Dutt et al. showed that: if the nonlinearity is switched on adiabatically and then maintained at a certain level, the stable islands disappear and the distribution function becomes more regular near the border of the dynamic aperture. In order to explain this observation it should be noted that with a time-dependent rescaling of the variables one can always recover the time-independent Henon map, i.e.:

$$\mathcal{N}_n = \Lambda_n \circ \mathcal{K} \circ \Lambda_n^{-1} \quad (3.4)$$

where
\[
\Lambda_s \begin{pmatrix} X_{\text{old}} \\ P_{\text{old}} \end{pmatrix} = \frac{1}{K(n)} \begin{pmatrix} X_{\text{new}} \\ P_{\text{new}} \end{pmatrix}
\] (3.5)

should be performed at each iteration. With formula (3.4) in hand it is easy to compute the fixed point of \( \mathcal{H}_s \), that is:

\[
X_s = -\frac{2}{K(n)} \cdot \tau g \pi v \quad ; \quad P_s = -\frac{2}{K(n)} \cdot \tau g^2 \pi v .
\] (3.6)

This fixed point is exactly \( \tilde{X}_s \) rescaled according to (3.5). In order to deal with the general problem of higher order fixed points let us define:

\[
\mathcal{H}_p^k = \mathcal{H}_{pk-1} \circ \mathcal{H}_{pk-2} \circ \ldots \circ \mathcal{H}_{(p-1)k} .
\] (3.7)

The k-th order "time-dependent fixed point" will be a solution of the equation:

\[
\mathcal{H}_p^k \left( \tilde{X}_s^k \right) = \tilde{X}_s^k .
\] (3.8)

With the use of (3.4) we can write expression (3.7) in a more explicit form as

\[
\mathcal{H}_p^k = \Lambda_{pk-1} \circ \mathcal{H} \circ \Lambda_{pk-2}^{-1} \circ \mathcal{H} \circ \Lambda_{pk-2}^{-1} \circ \ldots \circ \mathcal{H} \circ \Lambda_{(p-1)k}^{-1} .
\] (3.9)

The adiabatic variation of \( K(n) \) allows us to further simplify (3.9), provided

\[
\Lambda_{j-1}^{-1} \circ \Lambda_{j-1} = id .
\]

Under these assumptions one finally arrives at

\[
\mathcal{H}_p^k = \Lambda_{pk-1} \circ \mathcal{H} \circ \Lambda_{(p-1)k}^{-1} .
\] (3.10)

Now we are in a position to write down the approximate solution of eq. (3.8) with the help of (3.10):

\[
\tilde{X}_p^k = \Lambda_{(p-1)k} \tilde{X}_s^k , \quad \text{where} \quad \mathcal{H}^k \left( \tilde{X}_s^k \right) = \tilde{X}_s^k .
\] (3.11)

Recall that according to expression (2.21) the evolution in time of the process under consideration, described by the distribution function \( W_s \) is obtained by composing the time-dependent map in reverse order and retaining the functional form of the initial distribution for the tracked variables. Iterating the map for a constant value of the nonlinearity one generates a certain number of fixed points and stable islands with given probability measure on these sets, prescribed by \( W_0 \). Decreasing (because of the reverse composition order) the strength of the nonlinearity according to \( K(n) \to 0 \) and observing that \( \Lambda_s \to \infty \), from (3.11) it is readily seen that \( \tilde{X}_p^k \to \infty \). This means that the fixed points and the stable islands, produced in the course of iteration at a constant value of the nonlinearity are driven to infinity. From (3.1a) it follows that the value of the distribution function will tend to zero, explaining the vanishing of these sets (See figure 5).
IV. THE FOKKER-PLANCK EQUATION WITH THE NUMBER OF TURNS AS A "TIME" VARIABLE.

In this Section we derive an equivalent form of the solution (2.17) of our master equation (2.1) [see eq. (4.8)] which may turn out to be more suitable for practical use. Our starting point is equation (2.15), which we write as

\[
W_{n+1}(X, P) = \frac{1}{\pi \sqrt{\Delta_n}} \int \int dX_1 dP_1 W_n(X_1 + \bar{X}_n, P_1 + \bar{P}_n) \exp \left[ -\left( \begin{array}{c} X_1 \\ P_1 \end{array} \right)^T \hat{A}_n^{-1} \left( \begin{array}{c} X_1 \\ P_1 \end{array} \right) \right], \tag{4.1}
\]

where for brevity the following notation

\[
\bar{X}_n = T_n \circ X \quad ; \quad \bar{P}_n = T_n \circ P \tag{4.2}
\]

has been adopted. Next we develop \(W_n(X_1 + \bar{X}_n, P_1 + \bar{P}_n)\) in Taylor series under the integral (4.1), so that the latter becomes

\[
W_{n+1}(X, P) = \frac{1}{\pi \sqrt{\Delta_n}} \sum_{m=0}^{\infty} \frac{1}{m!} \int \int dX_1 dP_1 \exp \left[ -\left( \begin{array}{c} X_1 \\ P_1 \end{array} \right)^T \hat{A}_n^{-1} \left( \begin{array}{c} X_1 \\ P_1 \end{array} \right) \right] \left( X_1 \frac{\partial}{\partial \bar{X}_n} + P_1 \frac{\partial}{\partial \bar{P}_n} \right)^m W_n(\bar{X}_n, \bar{P}_n) \tag{4.3}
\]

In the last integral (4.3) we first change the integration variables according to a rotation by an angle \(\psi_n / 2\), namely

\[
\left( \begin{array}{c} X_1 \\ P_1 \end{array} \right) = \left( \begin{array}{cc} \cos \frac{\psi_n}{2} & -\sin \frac{\psi_n}{2} \\ \sin \frac{\psi_n}{2} & \cos \frac{\psi_n}{2} \end{array} \right) \left( \begin{array}{c} X_2 \\ P_2 \end{array} \right) \tag{4.4}
\]

and then scale the new variables \(X_2\) and \(P_2\) by the factors \(\hat{\lambda}_s^{(n)}\) and \(\hat{\lambda}_p^{(n)}\) respectively, where

\[
\hat{\lambda}_{s,p}^{(n)} = \left[ 2\pi \langle BD \rangle \mp 2 \left( \frac{R \beta^2 D}{2 R^2 + \beta^2 G} \right) \sin \psi_n \right]^{-1} \tag{4.5}
\]

Finally we come to the expression
\[ W_{n+1}(X, P) = \frac{1}{\pi} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{I_s}{s!} \frac{I_k}{k!} \frac{1}{\lambda_x^{s+1}} \frac{1}{\lambda_p^{k+1}} \partial_{\tilde{X}_n}^{s+1} \partial_{\tilde{P}_n}^{k+1} W_n(\tilde{X}_n, \tilde{P}_n), \quad (4.6) \]

in which \( \tilde{X}_n \) and \( \tilde{P}_n \) are the variables (4.2) transformed through the rotation (4.4) and the coefficients \( I_m \) are given by

\[ I_m = \int_{-\infty}^{\infty} dz z^m \exp(-z^2). \quad (4.7) \]

Substituting their values

\[ I_{2m} = \sqrt{\pi} \frac{(2m-1)!!}{2^m}; \quad I_{2m-1} = 0 \quad (m=1, 2, \ldots) \]

into the expression (4.6) and summing over we find

\[ W_{n+1}(X, P) = \exp \left[ \frac{1}{4} \left( A^{(n)}_{xx} \partial_{\tilde{X}_n}^2 - 2A^{(n)}_{xp} \partial_{\tilde{X}_n} \partial_{\tilde{P}_n} + A^{(n)}_{pp} \partial_{\tilde{P}_n}^2 \right) \right] W_n(\tilde{X}_n, \tilde{P}_n). \quad (4.8) \]

In the low noise approximation we expand the operator \( \exp(...) \) entering the expression (4.8) up to the first order in the noise strength and obtain

\[ W_{n+1}(X, P) - W_n(\tilde{X}_n, \tilde{P}_n) = \frac{1}{4} \left( A^{(n)}_{xx} \partial_{\tilde{X}_n}^2 - 2A^{(n)}_{xp} \partial_{\tilde{X}_n} \partial_{\tilde{P}_n} + A^{(n)}_{pp} \partial_{\tilde{P}_n}^2 \right) W_n(\tilde{X}_n, \tilde{P}_n). \quad (4.9) \]

For sufficiently large number of turns one can take \( n \) as a new continuous "time" variable and then write eq. (4.9) in the limit as a Fokker-Planck equation

\[ \frac{\partial W(\tilde{X}_n, \tilde{P}_n)}{\partial n} = \frac{1}{4} \left[ B_{xx}(n) \partial_{\tilde{X}_n}^2 - 2B_{xp}(n) \partial_{\tilde{X}_n} \partial_{\tilde{P}_n} + B_{pp}(n) \partial_{\tilde{P}_n}^2 \right] W(\tilde{X}_n, \tilde{P}_n), \quad (4.10) \]

where

\[ B_{xx}(n) = 2\pi \beta D(1 - \cos 2\Delta \psi), \quad B_{xp}(n) = 2\pi \beta D(1 + \cos 2\Delta \psi), \quad (4.11a) \]

\[ B_{pp}(n) = 2\pi \beta D \sin 2\Delta \psi. \quad (4.11b) \]

In other words we have enlarged the "time" scale so that it is possible to see the diffusive motion, sponsored by the phenomenological noise simultaneously with the effect of the nonlinear kicks. The contribution of the latter is "hidden" in the change of variables (4.2) whose Jacobian has a unit value, for the Henon map is an area preserving map.

Now we already know how to proceed in finding the solution of eq. (4.10) and may write it in a straightforward manner as

\[ W(X, P; n) = \frac{1}{\pi \sqrt{\Delta}} \int dX_n dP_n W_0(X_n + \tilde{T}_n X, P_n + \tilde{T}_n P) \exp \left[ -\frac{1}{\Delta} \left( A_{pp} X_n^2 + 2A_{xp} X_n P_n + A_{xx} P_n^2 \right) \right] \]
Here

$$\hat{T} = T_0 \circ T_1 \circ \ldots \circ T_{n-1},$$  \hspace{1cm} (4.12)$$

$$A_{ss}(n) = \int_0^{2\pi} d\theta \beta D(1 - \cos 2\Delta \psi), \quad A_{pp}(n) = \int_0^{2\pi} d\theta \beta D(1 + \cos 2\Delta \psi),$$  \hspace{1cm} (4.13a)$$

$$A_{sp}(n) = \int_0^{2\pi} d\theta \beta D \sin 2\Delta \psi.$$  \hspace{1cm} (4.13b)$$

The advantage of formula (4.12) in comparison with the path-integral representation (2.17) is obvious. Instead of solving 2n-tuple integral and passing over all intermediate positions of the particle along its trajectory, one has to perform a double integral that bridges the transition from the initial state to the final state after n turns. Let us point out once again that equation (4.10) and therefore its solution (4.12) is valid in the low noise approximation, which is most commonly encountered in practice. Note also, that in the limit $D \to 0$ equations (4.9) and (4.12) reduce to the expression (2.21).

V. NUMERICAL RESULTS.

From experimental data available one can conclude that in real machines the external phenomenological noise is negligible compared to transport phenomena, induced by nonlinearities. This will be probably the case in superconducting machines like LHC and SSC, where nonlinear forces produced by high order multipoles can not be avoided in general. For these reasons we performed numerical simulations of the beam phase space density evolution according to equation (2.21), where the initial particle distribution function has been taken to be a Gaussian one

$$W_0(X, P) = \frac{1}{2\pi \sigma_x \sigma_p} \exp \left\{ - \frac{(X - X_0)^2}{2\sigma_x^2} - \frac{(P - P_0)^2}{2\sigma_p^2} \right\}$$

with $\sigma_x = \sigma_p = 0.5$ and $X_0 = P_0 = 0$.

A simple computer code has been written to simulate the evolution of the probability distribution after a certain number of turns. The algorithm is based on the fact that the probability of finding a particle at a certain point in phase space after n turns is precisely the value of the initial distribution function computed at the tracked point (this is essentially the Liouville theorem). With the aid of this code we have studied three different models:

(i) a plain FODO cell (Figures 1-2),

(ii) a FODO cell with a modulation of the unperturbed betatron tune, due to ripple in the power supply of linear machine elements (Figures 3-4),

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(iii) a FODO cell with adiabatic ramp of sextupolar strength from zero to its nominal value (Figure 5).

The unperturbed betatron tune has been taken to be \( v = 68.255 \) in all of them so that one can easily compare the results. It is worth while to note that a special care has to be taken in case of explicit time dependence [cases (ii) and (iii) above] as the tracking should be performed in reverse order [see eq. (2.21)].

For the models studied, representing systems that are essentially one and a half degree of freedom, one can always define a region in phase space where bounded motion occurs. As a consequence the distribution function asymptotically will differ from zero only on the stable regions. This asymptotic limit is reached in a rather fast way. In Figure 1 the distribution function after ten turns is shown. One can clearly see the speed of the process; few iterations are enough to enhance the tail of the initial probability distribution. This process is sponsored by the invariant manifolds of the Henon map, acting as a channel for particle transport. The subsequent evolution is shown in Figure 2. In Fig. 2a the probability density has reached its zero value on the invariant manifolds, but is still different from zero on the stochastic layer due to the hyperbolic fixed points of period four. Increasing the number of turns this layer is depopulated (Fig. 2b) until the asymptotic state is reached (Fig. 2c).

For the case with small modulation of the unperturbed betatron tune we have obtained qualitatively similar results. The major differences consist of the speed the particle transport takes place and the shrinking of dynamic aperture. For instance in Fig. 3a and 3b the faster depopulation of the central peak compared to case (i) is visible. Figure 4 shows that the regions on which the distribution function is non vanishing are smaller and also the probability of finding a particle trapped in the stable islands is decreased.

In the third model specified above we analyzed the effect of ramping the sextupolar strength according to the adiabatic function:

\[
K(n) = K_{\text{max}} - (K_{\text{max}} - K_{\text{min}}) \exp\left\{\frac{-an^2}{b^2}\right\}
\]

where \( b \) is the number of turns, used to reach the maximum value of the nonlinearity strength. In Fig. 5 we have depicted the case of \( b = 10^3 \) and total number of iterations equal to 12000. As it was explained in Section 3 the islands are no more visible for they moved to infinity.

A relevant information about particle transport can be further extracted from the evolution of the distribution function. The integral of the probability density over a certain domain \( \Delta \) in phase space is proportional to the number of particles contained there:

\[
\mathcal{P}_n = \int_{\Delta} dX dP W_n(X, P).
\]

Its evolution in time provides us the law that rules the escape of particles from certain regions of interest in phase space. It is worth to note that for the first several hundred turns \( \mathcal{P}_n \) will strongly depend on the integration region if it is not chosen properly, especially in the case of tune modulation. This effect is due to the pulsation of the
separatrix. The asymptotic behavior of $\rho_n$ however, is uniquely defined. In Figure 6 we show $\hat{\rho}_n = \frac{\rho_n}{\rho_0}$, where the domain $\Delta$ is specified as a disc of radius 1.5 that is three times the sigmas of the initial distribution. One can readily see the presence of an asymptotic state in both cases (with and without modulation) and the reduction of the stability region in the first case.

It has been pointed out [13, 14] that $\rho_n$ has an algebraic tail, namely it decays according to a power law:

$$\rho_n \sim n^{-\alpha_D}.$$  

We recover this result, obtaining the following values for $\alpha_D$:

- $\alpha_D = 0.333$ for the pure Henon map,
- $\alpha_D = 0.425$ for the modulated Henon map.

In both cases the correlation coefficient for the regression line is about 0.997. Figure 7 in which $\hat{\rho}_n = \frac{\rho_n}{\rho_0}$ is plotted in double logarithmic scale ($\ln \hat{\rho}_n$ versus $\ln n$), shows a good agreement with the results from the fit.

VI. CONCLUDING REMARKS.

As a result of the analysis performed we have obtained an exact solution of the master equation (2.1) in the presence of external noise. In the limit of vanishing noise and using the number of turns as an independent "time" variable, we derived also a Fokker-Planck equation. Unfortunately, we are not in a position to provide an explicit formula for the transport coefficients, as they depend on the canonical variables in a very complicated way. We are also not able to derive a pure Fokker-Planck equation valid for all regions in phase space, when $D=0$. In this case we have studied numerically the beam distribution in phase space and we computed numerically $\alpha_D$.

Interesting subjects for further investigation are the study of the dependence of $\alpha_D$ on the unperturbed betatron tune and on the modulation parameters, as well as the computation of transport coefficients in simple cases modeling FODO cell with uniformly distributed nonlinearities.

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Figure 1: Distribution function for the Henon map without modulation after 10 turns (Fig. 1a). The initial probability distribution has zero means ($X_0 = P_0 = 0$) and $\sigma_x = \sigma_p = 0.5$. The unperturbed betatron tune is $\nu = 68.255$. In Fig. 1b the projection of the surface on X-P plane is shown. It is visible that the unstable manifold acts as a channel for particle transport.
Figure 2: Evolution of the distribution function for the Henon map. The three cases refer to 1000, 3000 and 9000 turns respectively. The stochastic layer surrounding the four stable islands is slowly depopulated until the asymptotic state is reached (Fig. 2c).
Figure 3: Distribution function for the modulated Henon map after 10 turns. The modulation amplitude is equal to $10^{-3}$, while the modulation frequency is $\Omega = 4.1 \times 10^{-3}$. The unperturbed betatron frequency is $\nu = 68.255$. The parameters of the distribution function are: $X_0 = P_0 = 0$ and $\sigma_x = \sigma_p = 0.5$. In Fig. 3b a projection on X-P plane is depicted. The central region is less populated than in Fig. 1b.
Figure 4: Evolution of the distribution function for the modulated Henon map. The three cases refer to 1000, 3000 and 9000 turns respectively. The stable region is considerably smaller than in Figs. 2a-c. The value of the probability distribution on the stable islands has decreased as well.
Figure 5: Distribution function after 12000 turns in the case of adiabatic ramp of the sextupolar strength. The ramp lasts for 1000 turns and the final value of K is one. The tune is taken to be $v = 68.255$ as in the cases specified above. The islands are no more present and the distribution function is smoother on the boundary of the stability domain.
Figure 6: Decay rates for the Henon map with modulation (dashed line) and without modulation (solid line). The integration domain is a disc with radius 1.5 (three times the dynamic aperture) and the other parameters are the same as in previous cases.
Figure 7: Plot of $\ln \tilde{\rho}_x$ versus $\ln n$ for the Henon map. Figure 7a refers to the situation without modulation, while Fig. 7b represents the case with modulation. The dashed lines superimposed correspond to the result from the fit.