BEAM INSTABILITIES IN CIRCULAR PARTICLE ACCELERATORS

E. Métral

Abstract

The theory of impedance-induced bunched-beam coherent instabilities is reviewed following Laclare's formalism, adding the effect of an electronic damper in the transverse plane. Both single-bunch and coupled-bunch instabilities are discussed, both low-intensity and high-intensity regimes are analyzed, both longitudinal and transverse planes are studied, and both short-bunch and long-bunch regimes are considered. Observables and mitigation measures are also examined.
BEAM INSTABILITIES
IN CIRCULAR PARTICLE ACCELERATORS

E. Métral (CERN, BE/ABP)

4 seminars of 2 hours (from Tuesday 23/05/17 to Friday 26/05/17)

◆ Introduction
◆ Longitudinal low-intensity
◆ Transverse low-intensity
◆ Transverse high-intensity
◆ Longitudinal high-intensity
◆ Conclusion
INTRODUCTION
Single-particle trajectory

In the middle of the vacuum chamber

One particle

Circular design orbit

\[ \vec{e}_x(s) \]

\[ \vec{e}_y \]

\[ \vec{e}_z(s) \]

\[ s = vt \]

\[ s = 0 \]
**SPACE CHARGE**

- Coulomb repulsion
- Magnetic attraction

**WAKE FIELD / IMPEDANCE**

- Force vs. Beta
- Magnetic vs. Total

**BEAM-BEAM**

- Long-range
- Head-on
- 285 µrad

**ELECTRON CLOUD**

- LOST or REFLECTED
- Secondary electron
- Photoelectron
50-page article for a special edition of IEEE Transactions on Nuclear Science for the 50th anniversary of the PAC conference (originally launched by IEEE in 1965)

And all the references therein
FOCUS OF THIS COURSE:
IMPEDANCE-INDUCED BEAM INSTABILITIES

- Limits performance of ALL machines
  - Beam instabilities => Increased beam size, beam losses
  - Excessive heating => Deformed / melted components, beam dumps

- Each equipment of each accelerator has an impedance => To be characterized and minimized!
EXAMPLES OF MEASURED BEAM INSTABILITIES

LHC, single bunch, horizontal

<table>
<thead>
<tr>
<th>Time [s]</th>
<th>&lt; x &gt; [a.u.]</th>
</tr>
</thead>
<tbody>
<tr>
<td>22:44:00</td>
<td>120 s</td>
</tr>
</tbody>
</table>

Measured instability rise-time = 9.8 s

In frequency domain

- Amplitude [dB]
- Qh

Elias Métral, La Sapienza University, Rome, Italy, May 23-26, 2017
EXAMPLES OF MEASURED BEAM INSTABILITIES

PS, single bunch, horizontal

PS, single bunch, vertical

Σ, ΔR, ΔV signals

~ 700 MHz

Head stable

Tail unstable

Time (10 ns/div)

SPS, single bunch, vertical

SPS, 72 bunches, horizontal

Elias Métral, La Sapienza University, Rome, Italy, May 23-26, 2017
EXAMPLES OF MEASURED BEAM INSTABILITIES

SPS, single bunch, longitudinal

stable bunch
dipole osc.
quadrapole osc.
SIMULATION (TRACKING) CODES ARE OFTEN USED:
=> HEADTAIL / PyHEADTAIL code at CERN

1. Divide ring into segments separated by interaction points (IP)
2. Initialise bunch
   - Typically $10^6$ macroparticles
   - Various distributions possible: Gaussian, waterbag, matched to rf bucket (longitudinal), ...
3. Linear periodic maps for transverse tracking from one IP to the next
4. Interaction point
   - Wake field kicks
   - Chromaticity
   - Octupoles, RF quadrupole
   - Electron cloud (PyECLOUD)
   - ...
5. Data acquisition
6. Once per turn:
   Apply (non-)linear synchrotron motion
7. 100'000s turns ...

Wake field
Wake kicks

Courtesy K. Li

x
y
z

z [m]

Courtesy of K. Li et al.
PURPOSE OF THIS COURSE

=> Discuss the theory of Bunched-Beam Coherent Instabilities (following Laclare’s formalism from 1987)

- Low-intensity and high-intensity regimes
- Longitudinal and transverse
- Single-bunch and coupled-bunch
- Observables and mitigation measures
LONGITUDINAL: LOW-INTENSITY
PROCEDURE: BOTH LONGITUDINAL (L) & TRANSVERSE (T)

◆ Start with the single particle motion => Harmonic oscillator + beam-induced electromagnetic force (L or T)
PROCEDURE: BOTH LONGITUDINAL (L) & TRANSVERSE (T)

- Start with the single particle motion => Harmonic oscillator + beam-induced electromagnetic force (L or T)
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◆ The beam-induced electromagnetic force can be expressed through the impedance (complex function of frequency) for both L and T
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◆ Apply the Vlasov equation to first order => One ends up with an eigenvalue system to solve
◆ The result is an infinite number of modes of oscillation $mq$
SINGLE PARTICLE LONGITUDINAL MOTION (1/2)

\[ \ddot{\tau} + \omega_{s0}^2 \tau = 0 \]

\[ \tau = \hat{\tau} \cos(\omega_{s0} t + \psi_0) \]

\[ \omega_{s0} = \Omega_0 \left( -\frac{e \hat{V}_{RF} h \eta \cos \phi_{s0}}{2 \pi \beta^2 E_{total}} \right)^{1/2} \]

Time interval between the passage of the synchronous particle and the test particle, for a fixed observer at azimuthal position \( \vartheta \).

e = elementary charge

\( R \) = average machine radius

\( p_0 \) = momentum of the synch. particle

\( \hat{V}_{RF} \) = peak RF voltage

\( h \) = RF harmonic number

\( \phi_{s0} \) = RF phase of the synch. particle

\( R \Omega_0 = v = \beta c \)

\( c = \) speed of light

\( p_0 c = \beta E_{total} \)

\( \eta = \alpha_p - \frac{1}{\gamma^2} = -\frac{\Delta f / f_0}{\Delta p / p_0} = \) slip factor

\( \alpha_p = \frac{1}{\gamma_t^2} = \) mom. comp. factor
SINGLE PARTICLE LONGITUDINAL MOTION (2/2)

- Canonical conjugate variables
  \[
  \left( \tau, \dot{\tau} = \frac{d\tau}{dt} \right) \quad \ddot{\tau} = \frac{d\tau}{dt} = -\frac{df}{f_0} = \eta \frac{dp}{p_0}
  \]

- Linear matching condition
  \[
  \omega_{s0} = \frac{p_0}{\tau_b} \quad \tau_b = 2 \hat{\tau}_{\text{max}}
  \]

- Effect of the (beam-induced) electromagnetic fields
  \[
  \dot{\tau} = \eta \frac{p - p_0}{p_0} \quad \Rightarrow \\
  \ddot{\tau} + \omega_{s0}^2 \tau = \frac{\eta}{p_0} \frac{dp}{dt} = \frac{\eta e}{p_0} \left[ \vec{E} + \vec{v} \times \vec{B} \right] \left( t, \vartheta = \Omega_0 \left( t - \tau \right) \right)
  \]

When following the particle along its trajectory
SINGLE PARTICLE LONGITUDINAL SIGNAL (1/3)

◆ At time $t = 0$, the synchronous particle starts from $\vartheta = 0$ and reaches the Pick-Up (PU) electrode (assuming infinite bandwidth) at times $t_k^0$

$$\Omega_0 t_k^0 = \vartheta + 2k\pi, \quad -\infty \leq k \leq +\infty$$

◆ The test particle is delayed by $\tau$. It goes through the electrode at times $t_k$

$$t_k = t_k^0 + \tau$$

◆ The current signal induced by the test particle is a series of impulses delivered on each passage

$$s_z(t, \vartheta) = e^{\sum_{k=\pm\infty}} \delta \left( t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0} \right)$$

Dirac function
SINGLE PARTICLE LONGITUDINAL SIGNAL (2/3)

Using the relations

\[ \sum_{k = -\infty}^{k = +\infty} \delta\left( u - \frac{2k \pi}{\Omega_0} \right) = \frac{\Omega_0}{2\pi} \sum_{p = -\infty}^{p = +\infty} e^{jp\Omega_0 u} \]

\[ e^{-ju\hat{\tau}\cos(\omega s_0 t + \psi_0)} = \sum_{m = -\infty}^{m = +\infty} j^{-m} J_m(u\hat{\tau}) e^{jm(\omega s_0 t + \psi_0)} \]

Bessel function of mth order

\[ \Rightarrow s_z(t, \vartheta) = \frac{e^{\Omega_0}}{2\pi} \sum_{p, m = -\infty}^{p, m = +\infty} j^{-m} J_m(p\Omega_0 \hat{\tau}) e^{j(\omega pm t - p\vartheta + m\psi_0)} \]

\[ \omega_{pm} = p\Omega_0 + m\omega_{s0} \]

Fourier transform

\[ s_z(\omega, \vartheta) = \frac{e^{\Omega_0}}{2\pi} \sum_{p, m = -\infty}^{p, m = +\infty} j^{-m} J_m(p\Omega_0 \hat{\tau}) e^{-j(p\vartheta - m\psi_0)} \delta(\omega - \omega_{pm}) \]
SINGLE PARTICLE LONGITUDINAL SIGNAL (3/3)

- The single particle spectrum is a line spectrum at frequencies

\[ \omega_{pm} = p \Omega_0 + m \omega_{s0} \]

- Around every harmonic of the revolution frequency \( p \Omega_0 \), there is an infinite number of synchrotron satellites \( m \)

- The spectral amplitude of the mth satellite is given by \( J_m \left( p \Omega_0 \hat{\tau} \right) \)

- The spectrum is centered at the origin

- Because the argument of the Bessel functions is proportional to \( \hat{\tau} \), the width of the spectrum behaves like \( \hat{\tau}^{-1} \)
DISTRIBUTION OF PARTICLES (1/2)

\[ \Psi(\hat{\tau}, \psi_0, t) = \text{particle density in longitudinal phase space} \]

- Signal induced (at the PU electrode) by the whole beam

\[ S_z(t, \vartheta) = N_b \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} \int_{\psi_0=0}^{\psi_0=2\pi} \Psi(\hat{\tau}, \psi_0, t) s_z(t, \vartheta) \hat{\tau} \, d\hat{\tau} \, d\psi_0 \]

Number of particles per bunch

- Canonically conjugated variables derive from a Hamiltonian \( H(q, p, t) \) by the canonical equations

\[ \dot{q} = \frac{\partial H(q, p, t)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q, p, t)}{\partial q} \]
DISTRIBUTION OF PARTICLES (2/2)

◆ According to the Liouville’s theorem, the particles, in a non-dissipative system of forces, move like an incompressible fluid in phase space. The constancy of the phase space density $\Psi(q, p, t)$ is expressed by the equation

$$\frac{d\Psi(q, p, t)}{dt} = 0$$

where the total differentiation indicates that one follows the particle while measuring the density of its immediate neighborhood. This equation, sometimes referred to as the Liouville’s theorem, states that the local particle density does not vary with time when following the motion in canonical variables.

◆ As seen by a stationary observer (like a PU electrode) which does not follow the particle => Vlasov equation

$$\frac{\partial \Psi(q, p, t)}{\partial t} + \dot{q} \frac{\partial \Psi(q, p, t)}{\partial q} + \dot{p} \frac{\partial \Psi(q, p, t)}{\partial p} = 0$$
In the case of a harmonic oscillator

\[ H = \omega \frac{q^2 + p^2}{2} \]

\[ \dot{q} = \frac{\partial H}{\partial p} = p \omega \]

\[ \dot{p} = -\frac{\partial H}{\partial q} = -q \omega \]

Going to polar coordinates

\[ q = r \cos \phi \]
\[ p = -r \sin \phi \]

\[ \Rightarrow \frac{\partial \Psi}{\partial t} + r \frac{\partial \Psi}{\partial r} + \phi \frac{\partial \Psi}{\partial \phi} = 0 \]
As $r$ is a constant of motion $\implies \dot{r} = 0$

\[
\frac{\partial \Psi}{\partial t} + \omega \frac{\partial \Psi}{\partial \phi} = 0 \quad \text{with} \quad \phi = \omega t
\]

\[
\frac{\partial \Psi}{\partial t} = -\omega \frac{\partial \Psi}{\partial \phi} = -\frac{\partial \Psi}{\partial t} \implies \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \phi} = 0
\]

\[
\implies \Psi(r)
\]

A stationary distribution is any function of $r$, or equivalently any function of the Hamiltonian $H$. 
In our case

\[ q = \tau \quad r = \hat{\tau} \quad p = \dot{\tau} \quad \phi = \psi_0 \]

\[ \Psi_0(\hat{\tau}, \psi_0, t) = g_0(\hat{\tau}) \]

\[ S_{z0}(\omega, \vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0(p) \delta(\omega - p \Omega_0) e^{-jp\vartheta} \]

\[ I_b = N_b e \Omega_0 / 2\pi \]

\[ \sigma_0(p) = \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} J_0\left(p \Omega_0 \hat{\tau}\right) g_0(\hat{\tau}) \hat{\tau} d\hat{\tau} \]
Let’s assume a parabolic amplitude density
\[ \hat{z} \equiv \frac{\hat{\tau}}{\left( \frac{\tau_b}{2} \right)} \]

The line density \( \lambda(\tau) \) is the projection of the distribution \( g_0(\hat{\tau}) \) on the \( \tau \) axis
\[
\lambda(\tau) = \int g_0(\hat{\tau}) \frac{d\hat{\tau}}{\omega_{s0}} \\
\int \lambda(\tau) d\tau = 1 \quad \Rightarrow \\
\lambda(z) = \frac{8}{3\pi \left( \frac{\tau_b}{2} \right)^2 \left( 1 - z^2 \right)^{3/2}}
\]

\[ \tau \equiv \frac{\tau}{\left( \frac{\tau_b}{2} \right)} \]
STATIONARY DISTRIBUTION (5/6)

- Parabolic amplitude density
- Parabolic line density
- Gaussian amplitude density
- Water - bag bunch

$\tau_b = 4 \sigma$

$\tau_{b} / 2 \leq \tau$
Using the relations

\[ \int_{u'=0} \int J_0(u') \, u' \, du' = u \, J_1(u) \]

\[ J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \]

\[ \int x^3 J_0(x) \, dx = x^2 \left[ 2J_2(x) - xJ_3(x) \right] \]

\[ \sigma_0(p) = \frac{4}{\pi (p \pi B)^2} J_2(p \pi B) \quad B = \tau_b \Omega_0 / 2\pi \]

and

\[ S_{z0}(\omega, \Theta) = 8 I_b \sum_{p=-\infty}^{p=+\infty} \delta(\omega - p \Omega_0) e^{-jp\Theta} \frac{J_2(p \pi B)}{(p \pi B)^2} \]
LONGITUDINAL IMPEDANCE

\[ 2 \pi R \left[ \vec{E} + \vec{v} \times \vec{B} \right]_z (t, \vartheta) = - \int_{\omega = -\infty}^{\omega = +\infty} Z_l(\omega) S_z(\omega, \vartheta) e^{j\omega t} d\omega \]

All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the impedance (complex function => in \( \Omega \))
EFFECT OF THE STATIONARY DISTRIBUTION (1/9)

\[
\ddot{\tau} + \omega_{s0}^2 \tau = F_0 = \frac{\eta e}{p_0} \left[ \vec{E} + \vec{v} \times \vec{B} \right]_{z0}^t (t, \vartheta = \Omega_0 (t - \tau))
\]

\[
\left[ \vec{E} + \vec{v} \times \vec{B} \right]_{z0}^t (t, \vartheta = \Omega_0 (t - \tau)) = -\frac{1}{2 \pi R} \int_{\omega = -\infty}^{\omega = +\infty} Z_l (\omega) S_{z0} (\omega, \vartheta = \Omega_0 (t - \tau)) e^{j \omega t} \, d\omega
\]

\[
\Rightarrow \quad \ddot{\tau} + \omega_{s0}^2 \tau = F_0 = \frac{2 \pi I_b \omega_{s0}^2}{\Omega_0 \hat{V}_{RF} h \cos \phi_{s0}} \sum_{p = -\infty}^{p = +\infty} Z_l (p) \sigma_0 (p) e^{j p \Omega_0 \tau}
\]
EFFECT OF THE STATIONARY DISTRIBUTION (2/9)

- Expanding the exponential in series (for small amplitudes)

\[
\ddot{\tau} + \omega_{s0}^2 \tau = \frac{2 \pi I_b}{\Omega_0} \frac{\omega_{s0}^2}{h \cos \phi_{s0}} \sum_{p = -\infty}^{+\infty} Z_l(p) \sigma_0(p) \left[ 1 + j p \Omega_0 \tau - \frac{(p \Omega_0 \tau)^2}{2} + \ldots \right]
\]

- Synchronous phase shift
- Incoherent frequency shift (potential-well distortion)
- Nonlinear terms introducing some synchrotron frequency spread
EFFECT OF THE STATIONARY DISTRIBUTION (3/9)

◆ Synchronous phase shift

\[ \ddot{\tau} + \omega_{s0}^2 \tau = \frac{2 \pi I_b}{\Omega_0 \hat{V}_{RF} h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \text{Re}[Z_l(p)] \sigma_0(p) \]

\[ \tau = t_p - t_{s0} \]

Test particle \hspace{1cm} Synchronous particle

\[ \ddot{t}_{s0} = 0 \]

\[ \Rightarrow \ddot{t}_p + \omega_{s0}^2 t_p = \omega_{s0}^2 t_{s0} + \frac{2 \pi I_b}{\Omega_0 \hat{V}_{RF} h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \text{Re}[Z_l(p)] \sigma_0(p) \]

\[ \Rightarrow \ddot{t}_p + \omega_{s0}^2 t_p = \omega_{s0}^2 t_s \]

with \[ \Delta t_s = t_s - t_{s0} = \frac{2 \pi I_b}{\Omega_0 \hat{V}_{RF} h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \text{Re}[Z_l(p)] \sigma_0(p) \]
EFFECT OF THE STATIONARY DISTRIBUTION (4/9)

\[ \phi = \omega_{\text{RF}} \, t \quad \quad \omega_{\text{RF}} = h \, \Omega_0 \]

\[ \phi_s = \omega_{\text{RF}} \, t_s \]

\[ \Delta \phi_s = \phi_s - \phi_{s0} = \omega_{\text{RF}} \, \Delta t_s \]

\[ \Rightarrow \quad \Delta \phi_s = \phi_s - \phi_{s0} = \frac{2 \pi I_b}{\hat{V}_{\text{RF}} \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} \text{Re} \left[ Z_l(p) \right] \sigma_0(p) \]

Only for the small amplitudes. For the power loss of the whole bunch an averaging is needed!

Can be used to probe the resistive part of the longitudinal impedance
Incoherent synchrotron frequency shift (potential-well distortion)

\[ \ddot{\tau} + \omega_s^2 \tau = \frac{2 \pi I_b \omega_{s0}^2}{\Omega_0 \hat{V}_{RF} h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} Z_l(p) \sigma_0(p) j p \Omega_0 \tau \]

\[ \Rightarrow \ddot{\tau} + \omega_s^2 \tau = 0 \]

with \[ \omega_s^2 = \omega_{s0}^2 \left[ 1 - \frac{2 \pi I_b}{\hat{V}_{RF} h \cos \phi_{s0}} \sum_{p=-\infty}^{p=+\infty} j Z_l(p) p \sigma_0(p) \right] \]

- If the impedance is constant (in the frequency range of interest)

\[ \omega_s^2 = \omega_{s0}^2 \left\{ 1 - \frac{2 \pi I_b}{\hat{V}_{RF} h \cos \phi_{s0}} \left[ j \frac{Z_l(p)}{p} \right]_{\text{const}} \sum_{p=-\infty}^{p=+\infty} p^2 \sigma_0(p) \right\} \]
EFFECT OF THE STATIONARY DISTRIBUTION (6/9)

Using the relation

\[ \sum_{p=-\infty}^{p=+\infty} J_2(p \cdot x) = \frac{2}{x} \Rightarrow \sum_{p=-\infty}^{p=+\infty} p^2 \sigma_0(p) = \frac{8}{\pi^4 B^3} \]

For the parabolic amplitude density

\[ \Delta = \frac{\omega_s^2 - \omega_{s0}^2}{\omega_{s0}^2} = -\frac{16 I_b}{\pi^3 B^3 \hat{V}_{RF} h \cos \phi_{s0}} \left[ j \frac{Z_1(p)}{p} \right]_{const} \]

The change in the RF slope corresponds to the effective (total) voltage

\[ \hat{V}_T = \hat{V}_{RF} \left( \frac{\omega_s}{\omega_{s0}} \right)^2 \]
EFFECT OF THE STATIONARY DISTRIBUTION (7/9)

- **Bunch lengthening / shortening** (as a consequence of the shifts of the synchronous phase and incoherent frequency)

  - **Electrons**

    The equilibrium momentum spread is imposed by synchrotron radiation

    \[
    \frac{\Delta p}{p_0} = \left( \frac{\Delta p}{p_0} \right)_0 \quad \Rightarrow \quad \frac{B}{B_0} = \frac{\omega s_0}{\omega_s} \sqrt{\frac{\cos \phi_{s0}}{\cos \phi_s}}
    \]

    Neglecting the (usually small) synchronous phase shift

    \[
    \Rightarrow \quad \frac{B}{B_0} = \left( \frac{B}{B_0} \right)^3 + \Delta_0 \quad \text{with} \quad \Delta_0 = \Delta_{B=B_0}
    \]
EFFECT OF THE STATIONARY DISTRIBUTION (8/9)

- Protons

The longitudinal emittance is invariant

\[
\tau_b \left( \frac{\Delta p}{p_0} \right) = \tau_{b0} \left( \frac{\Delta p}{p_0} \right)_0 \Rightarrow \left( \frac{B}{B_0} \right)^2 = \frac{\omega_{s0}}{\omega_s} \sqrt{\frac{\cos \phi_{s0}}{\cos \phi_s}}
\]

Again, neglecting the (usually small) synchronous phase shift

\[
\Rightarrow \left( \frac{B}{B_0} \right)^{-1} = \left( \frac{B}{B_0} \right)^3 + \Delta_0
\]
EFFECT OF THE STATIONARY DISTRIBUTION (9/9)

- **General formula**

  \[
  \left( \frac{B}{B_0} \right)^{\pm 1} = \left( \frac{B}{B_0} \right)^3 + \Delta_0
  \]

  + for electrons and – for protons

- **Conclusion of the effect of the stationary distribution:** New fixed point

  - Synchronous phase shift
    \[ \phi_{s0} \Rightarrow \phi_s (I_b) \]
  - Potential-well distortion
    \[ \hat{V}_{RF} \Rightarrow \hat{V}_T (I_b) \]
    \[ \omega_{s0} \Rightarrow \omega_s (I_b) \]
    \[ B_0 \Rightarrow B (I_b) \]
The form is suggested by the single-particle signal

\[
s_z(t, \theta) = \frac{e \Omega_0}{2\pi} \sum_{p,m=-\infty}^{p,m=+\infty} j^{-m} J_m(p \Omega_0 \hat{\tau}) e^{j(\omega_p m t - p \theta + m \psi_0)}
\]

- **Low-intensity**

\[
\Delta \Psi(\hat{\tau}, \psi_0, t) = g_m(\hat{\tau}) e^{-j m \psi_0} e^{j \Delta \omega_{cm} t}
\]

\[
\Delta \omega_{cm} = \omega_c - m \omega_s << \omega_{s0}
\]

Therefore, the spectral amplitude is maximum for satellite number \(m\) and null for the other satellites.

Around the new fixed point

Coherent synchrotron frequency shift to be determined
\[ \Delta S_{zm}(\omega, \vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_m(p) \delta[\omega - (p \Omega_0 + m \omega_s + \Delta \omega_{cm})] e^{-j p \vartheta} \]

\[ \sigma_m(p) = j^{-m} \int_{\hat{\tau} = 0}^{\hat{\tau} = +\infty} J_m(p \Omega_0 \hat{\tau}) g_m(\hat{\tau}) \hat{\tau} d\hat{\tau} \]

\[ \omega_s = \Omega_0 \left( -\frac{e \hat{V}_T h \eta \cos \phi_s}{2\pi \beta^2 E_{total}} \right)^{1/2} \]

- **High-intensity**

\[ \Delta \Psi(\hat{\tau}, \psi_0, t) = \sum_m g_m(\hat{\tau}) e^{-j m \psi_0} e^{j \Delta \omega_{cm} t} \]
EFFECT OF THE PERTURBATION (1/10)

\[ \Psi(\hat{\tau}, \psi_0, t) = \Psi_0 + \Delta \Psi = g_0(\hat{\tau}) + \sum_m g_m(\hat{\tau}) e^{-j m \psi_0} e^{j \Delta \omega_{cm} t} \]

- Vlasov equation with variables \( (\hat{\tau}, \psi_0) \)

\[ \frac{\partial \Psi}{\partial t} + \left( \frac{dg_0}{d\hat{\tau}} + \frac{\partial \Delta \Psi}{\partial \hat{\tau}} \right) \frac{d\hat{\tau}}{dt} + \frac{\partial \Delta \Psi}{\partial \psi_0} \frac{d\psi_0}{dt} = 0 \]

\[ \Rightarrow \quad \text{Linearized Vlasov equation} \]

\[ \frac{\partial \Psi}{\partial t} = - \frac{dg_0}{d\hat{\tau}} \frac{d\hat{\tau}}{dt} \]

\[ \Rightarrow \quad j \sum_m g_m(\hat{\tau}) e^{-j m \psi_0} \Delta \omega_{cm} e^{j \Delta \omega_{cm} t} = - \frac{dg_0}{d\hat{\tau}} \frac{d\hat{\tau}}{dt} \]
EFFECT OF THE PERTURBATION (2/10)

\[ \frac{d\tilde{\tau}}{dt} = \frac{d}{dt} \left( \sqrt{\tau^2 + \frac{\dot{\tau}^2}{\omega_s^2}} \right) = -\frac{F_c}{\omega_s} \sin \left( \omega_s t + \psi_0 \right) \]

with \[ \ddot{\tau} + \omega_s^2 \tau = F_c = \frac{\eta e}{p_0} \left[ \vec{E} + \vec{v} \times \vec{B} \right]_{zc} (t, \vartheta = \Omega_0 (t - \tau)) \]

\[ \Rightarrow F_c = \frac{2\pi I_b \omega_s^2}{\Omega_0 \hat{V}_T h \cos \phi_s} e^{j\omega_c t} \sum_{p=-\infty}^{p=+\infty} Z_l(p) e^{j p \Omega_0 \tau} \sigma(p) \]

with \[ \sigma(p) = \sum_m \sigma_m(p) \] 

Spectrum amplitude at frequency \[ p \Omega_0 + \omega_c \]
EFFECT OF THE PERTURBATION (3/10)

◆ Expanding the product
relations

\[
\sin \psi e^{j p \Omega_0 \tau} \quad \text{(using previously given relations)}
\]

\[
\sin \psi e^{j p \Omega_0 \tau} = \sum_{m=-\infty}^{m=+\infty} j^m e^{-j m \psi} \frac{m}{p \Omega_0 \hat{\tau}} J_m \left( p \Omega_0 \hat{\tau} \right)
\]

\[\Rightarrow \quad \text{Final form of the equation of coherent motion of a single bunch:}\]

\[
j \Delta \omega_{cm} = \omega_c - m \omega_s
\]

\[
j \Delta \omega_{cm} j^{-m} g_m \left( \hat{\tau} \right) \hat{\tau} = \frac{2 \pi I_b}{\Omega_0^2 \hat{V}_T} \frac{m \omega_s}{h \cos \phi_s} \frac{d g_0}{d \hat{\tau}} \sum_{p=-\infty}^{p=+\infty} \frac{Z_l( p )}{p} J_m \left( p \Omega_0 \hat{\tau} \right) \sigma \left( p \right)
\]
EFFECT OF THE PERTURBATION (4/10)

◆ Coherent modes of oscillation at low intensity (i.e. considering only a single mode $m$)

\[ j \Delta \omega_{cm} j^{-m} g_m(\hat{\tau}) \hat{\tau} = \frac{2\pi I_b}{\Omega_0^2 \hat{V}_T} \frac{m \omega_s}{h \cos \phi_s} \frac{dg_0}{d\hat{\tau}} \sum_{p=-\infty}^{+\infty} \frac{Z_l(p)}{p} J_m(p \Omega_0 \hat{\tau}) \sigma_m(p) \]

Multiplying both sides by $J_m(l \Omega_0 \hat{\tau})$ and integrating over $\hat{\tau}$

\[ \Rightarrow \Delta \omega_{cm} \sigma_m(l) = \sum_{p=-\infty}^{+\infty} K_{lp}^m \sigma_m(p) \]

\[ K_{lp}^m = -\frac{2\pi I_b}{\Omega_0^2 \hat{V}_T} \frac{m \omega_s}{h \cos \phi_s} j \int_0^{+\infty} \frac{dg_0}{d\hat{\tau}} J_m(p \Omega_0 \hat{\tau}) J_m(l \Omega_0 \hat{\tau}) d\hat{\tau} \]
EFFECT OF THE PERTURBATION (5/10)

◆ The procedure to obtain first order exact solutions, with realistic modes and a general interaction, thus consists of finding the eigenvalues and eigenvectors of the infinite complex matrix whose elements are $K_{lp}^m$

◆ The result is an infinite number of modes $mq$ ( $-\infty < m, q < +\infty$ ) of oscillation (as there are 2 degrees of freedom $(\hat{\tau}, \psi_0)$)

◆ To each mode, one can associate:
  ▪ a coherent frequency shift $\Delta \omega_{cmq} = \omega_{cmq} - m \omega_s$ (qth eigenvalue)
  ▪ a coherent spectrum $\sigma_{mq}(p)$ (qth eigenvector)
  ▪ a perturbation distribution $g_{mq}(\hat{\tau})$

◆ For numerical reasons, the matrix needs to be truncated, and thus only a finite frequency domain is explored
EFFECT OF THE PERTURBATION (6/10)

◆ The longitudinal signal at the PU electrode is given by

\[
S_{mq}(t, \vartheta) = S_{z0}(t, \vartheta) + \Delta S_{zmq}(t, \vartheta)
\]

\[
S_{z0}(t, \vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_0(p) e^{j p \Omega_0 t} e^{-j p \vartheta}
\]

\[
\Delta S_{zmq}(t, \vartheta) = 2\pi I_b \sum_{p=-\infty}^{p=+\infty} \sigma_{mq}(p) e^{j (p \Omega_0 + m \omega_s + \Delta \omega_{cmq}) t} e^{-j p \vartheta}
\]

◆ For the case of the parabolic amplitude distribution

\[
g_0(\hat{z}) = \frac{2}{\pi \left(\frac{\tau_b}{2}\right)^2} \left(1 - \hat{z}^2\right) \quad S_{z0}(t, \vartheta) = 8 I_b \sum_{p=-\infty}^{p=+\infty} e^{j p \Omega_0 t} e^{-j p \vartheta} \frac{J_2\left(p \pi B\right)}{(p \pi B)^2}
\]
EFFECT OF THE PERTURBATION (7/10)

\[ K_{lp}^m = \frac{128 I_b m \omega_s}{\Omega_0^2 \hat{V}_T \ h \cos \phi_s \ \tau_b^4} \ j \ Z_l(p) \ \hat{t} = +\infty \ \int_{\hat{t}=0}^{\hat{t}=+\infty} J_m \left( p \ \Omega_0 \ \hat{\tau} \right) J_m \left( l \ \Omega_0 \ \hat{\tau} \right) \hat{\tau} \ d\hat{\tau} \]

◆ Low order eigenvalues and eigenvectors of the matrix can be found quickly by computation, using the relations

\[
\int_0^X J_m^2(a \ x) \ x \ dx = \frac{X^2}{2} \left[ J'_m(a \ X) \right]^2 + \frac{1}{2} \left[ X^2 - \frac{m^2}{a^2} \right] J_m^2(a \ X)
\]

\[
\int_0^X x J_m(a \ x) J_m(b \ x) \ dx = \frac{X}{a^2 - b^2} \left[ a J_m(b \ X) J_{m+1}(a \ X) - b J_m(a \ X) J_{m+1}(b \ X) \right]
\]

◆ The case of a constant inductive impedance is solved in the next slides, and the signal at the PU shown for several superimposed turns
EFFECT OF THE PERTURBATION (8/10)

Signal observed at the PU electrode

DIPOLE mode

QUADRUPOLE mode
EFFECT OF THE PERTURBATION (9/10)

- The spectrum of mode $mq$ is peaked at
  \[ f_q \approx \frac{q + 1}{2 \tau_b} \]
  and extends
  \[ \sim \pm \tau_b^{-1} \]
  \[ q \equiv m + 2k \quad 0 \leq k < +\infty \]

- There are $q$ nodes on these “standing-wave” patterns

SEXTUPOLE mode
EFFECT OF THE PERTURBATION (10/10)

Observations in the CERN SPS in 2007

(Laclare’s) theory
TRANSVERSE:
LOW-INTENSITY
SINGLE PARTICLE TRANSVERSE MOTION (1/3)

◆ A purely linear synchrotron oscillation around the synchronous particle is assumed (with no coherent oscillations)

\[ \ddot{\tau} + \omega_s^2 \tau = 0 \quad \tau = \hat{\tau} \cos(\omega_s t + \psi_0) \]

◆ For the transverse betatron oscillation, the equation of unperturbed motion, e.g. in the horizontal plane, is written as

\[ x = \hat{x} \cos[\varphi_x(t)] \]

\[ x^2 + \frac{\dot{x}^2}{\dot{\varphi}_x^2} = \hat{x}^2 \]

◆ The horizontal betatron frequency is given by

\[ \dot{\varphi}_x = Q_x \Omega \]

Chromaticity

with

\[ \xi_x = \frac{\Delta Q_x / Q_{x0}}{\Delta p / p_0} = \frac{Q'_x}{Q_{x0}} \]

\[ \eta = -\frac{\Delta \Omega / \Omega_0}{\Delta p / p_0} = \frac{\dot{\tau}}{\Delta p / p_0} \]
SINGLE PARTICLE TRANSVERSE MOTION (2/3)

$$\Rightarrow Q_x(p) = Q_{x0} \left(1 + \bar{\xi}_x \frac{\Delta p}{p_0}\right) \quad \Omega(p) = \Omega_0 \left(1 - \eta \frac{\Delta p}{p_0}\right)$$

$$\Rightarrow \dot{\varphi}_x = Q_x \Omega \approx Q_{x0} \Omega_0 \left[1 - \dot{\tau} \left(1 - \frac{\bar{\xi}_x}{\eta}\right)\right]$$

and

$$\varphi_x = Q_{x0} \Omega_0 \left(t - \tau\right) + \omega_{\bar{\xi}_x} \tau + \varphi_{x0}$$

$$\omega_{\bar{\xi}_x} = Q_{x0} \Omega_0 \frac{\bar{\xi}_x}{\eta}$$

Horiz. chromatic frequency
SINGLE PARTICLE TRANSVERSE MOTION (3/3)

◆ In the absence of perturbation, the horizontal coordinate satisfies

\[
\dddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x} \dot{x} + \dot{\varphi}_x^2 x = 0
\]

◆ In the presence of electromagnetic fields induced by the beam, the equation of motion writes

\[
\dddot{x} - \frac{\ddot{\varphi}_x}{\dot{\varphi}_x} \dot{x} + \dot{\varphi}_x^2 x = F_x = \frac{e}{\gamma m_0} \left[ E + \vec{v} \times \vec{B} \right]_x \left( t, \vartheta = \Omega_0 (t - \tau) \right)
\]

When following the particle along its trajectory
SINGLE PARTICLE TRANSVERSE SIGNAL (1/2)

◆ The horizontal signal induced at a perfect PU electrode (infinite bandwidth) at angular position $\vartheta$ in the ring by the off-centered test particle is given by

$$s_x(t, \vartheta) = s_z(t, \vartheta) x(t) = s_z(t, \vartheta) \hat{x} \cos(\varphi_x)$$

$$\Rightarrow s_x(t, \vartheta) = e^{\hat{x}} \cos(\varphi_x) \sum_{k=\infty}^{k=-\infty} \delta \left( t - \tau - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0} \right)$$

◆ Developing $\cos(\varphi_x)$ into exponential functions and using relations given in the longitudinal course, yields

$$s_x(t, \vartheta) = \frac{e^{\Omega_0}}{4\pi} \hat{x} e^{i(Q_{x0} \Omega_0 t + \varphi_{x0})} \sum_{p, m=\infty}^{p, m=-\infty} j^{-m} J_{m, x}(p, \tau) e^{j[\omega_{pm} t + m\psi_0 - p \vartheta]}$$

+ C.C. Complex conjugate

$$\omega_{pm} = p\Omega_0 + m\omega_s$$
SINGLE PARTICLE TRANSVERSE SIGNAL (2/2)

with

\[ J_{m,x}(p, \hat{\tau}) = J_m \left\{ \left[ \left( p + Q_{x0} \right) \Omega_0 - \omega_{\xi_x} \right] \hat{\tau} \right\} \]

\[ s_x(\omega, \vartheta) = \frac{e\Omega_0}{4\pi} \hat{x} e^{j\varphi_{x0}} \]

\[ \sum_{p,m=\pm\infty} j^{-m} J_{m,x}(p, \hat{\tau}) \delta\left\{ \omega - \left[ \left( p + Q_{x0} \right) \Omega_0 + m\omega_s \right] \right\} e^{j(m\psi_0 - p\vartheta)} + c.c. \]

- The spectrum is a line spectrum at frequencies \( \left( p + Q_{x0} \right) \Omega_0 + m\omega_s \)

- Around every betatron line \( \left( p + Q_{x0} \right) \Omega_0 \), there is an infinite number of synchrotron satellites \( m \)

- The spectral amplitude of the mth satellite is given by \( J_{m,x}(p, \hat{\tau}) \)

- The spectrum is centered at the chromatic frequency \( \omega_{\xi_x} = Q_{x0}\Omega_0 \frac{\xi_x}{\eta} \)
STATIONARY DISTRIBUTION (1/2)

◆ In the absence of perturbation, \( \hat{\chi} \) and \( \hat{\tau} \) are constants of the motion.

◆ Therefore, the stationary distribution is a function of the peak amplitudes only:

\[
\Psi_{x0}(\hat{\chi}, \hat{\tau})
\]

◆ No correlation between horizontal and longitudinal planes is assumed and the stationary part is thus written as the product of 2 stationary distributions, one for the longitudinal phase space and one for the horizontal one:

\[
\Psi_{x0}(\hat{\chi}, \hat{\tau}) = f_0(\hat{\chi}) g_0(\hat{\tau})
\]

\[
\int_{\hat{x}=0}^{\hat{x}=+\infty} f_0(\hat{\chi}) \hat{x} \, d\hat{x} = \frac{1}{2\pi}
\]

\[
\int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} g_0(\hat{\tau}) \hat{\tau} \, d\hat{\tau} = \frac{1}{2\pi}
\]
Since on average, the beam center of mass is on axis, the horizontal signal induced by the stationary distribution is null

\[ S_{x0}(t, \vartheta) = N_b \int_{\hat{x}=0}^{+\infty} \int_{\vartheta_{x0}=0}^{+\infty} \int_{\hat{\vartheta}=0}^{+\infty} \int_{\psi_{0}=0}^{+\infty} f_0(\hat{x}) g_0(\hat{\vartheta}) s_x(t, \vartheta) \hat{x} \hat{\vartheta} d\hat{x} d\hat{\vartheta} d\vartheta_{x0} d\psi_0 = 0 \]
In order to get some dipolar fields, density perturbations $\Delta \Psi_x$ that describe beam center-of-mass displacements along the bunch are assumed.

The mathematical form of the perturbations is suggested by the single-particle signal

$$s_x(t, \vartheta) = \frac{e \Omega_0}{4 \pi} \hat{x} \sum_{p, m = -\infty}^{p, m = +\infty} j^{-m} J_{m, x}(p, \hat{\tau}) e^{j(\varphi_{x0} + m\psi_0)} e^{-j p \vartheta} e^{j[(p + Q_{x0})\Omega_0 + m\omega_s] t}$$

+ c.c.

- **Low-intensity**

$$\Delta \Psi_x = h_m(\hat{x}, \hat{\tau}) e^{-j(\varphi_{x0} + m\psi_0)} e^{j\Delta \omega_{cm}^x t}$$

$$\Delta \omega_{cm}^x = \omega_c - m \omega_s << \omega_s$$

Coherent betatron frequency shift to be determined
PERTURBATION DISTRIBUTION (2/3)

- In the time domain, the horizontal signal takes the form (for a single value $m$)

\[ S_x(t, \vartheta) = 2 \pi^2 I_b \sum_{p=-\infty}^{p=+\infty} e^{-jp\vartheta} \sigma_{x,m}(p) e^{j[(p+Q_{x0})\Omega_0 + \omega_c]t} \]

**Fourier transform**

\[ S_x(\omega, \vartheta) = 2 \pi^2 I_b \sum_{p=-\infty}^{p=+\infty} e^{-jp\vartheta} \sigma_{x,m}(p) \delta \left\{ \omega - \left[ (p + Q_{x0})\Omega_0 + \omega_c \right] \right\} \]

with

\[ \sigma_{x,m}(p) = j^{-m} \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{\tau}=0}^{\hat{\tau}=+\infty} h_m(\hat{x}, \hat{\tau}) J_{m,x}(p, \hat{\tau}) \hat{x}^2 d\hat{x} \hat{\tau} d\hat{\tau} \]
High-intensity

\[ \Delta \Psi_x = \sum_m h_m (\hat{x}, \hat{\tau}) e^{-j(\varphi_{x0} + m\psi_0)} e^{j\Delta \omega_{cm}^x t} \]
All the properties of the electromagnetic response of a given machine to a passing particle is gathered into the transverse impedance (complex function $\Rightarrow$ in $\Omega / m$).
EFFECT OF THE PERTURBATION (1/10)

\[ \Psi_x (\hat{x}, \varphi_{x0}, \hat{\tau}, \psi_0, t) = \Psi_{x0} + \Delta \Psi_x \]

\[ \Rightarrow \quad \Psi_x = f_0 (\hat{x}) g_0 (\hat{\tau}) + \sum_m h_m (\hat{x}, \hat{\tau}) e^{-j(\varphi_{x0} + m\psi_0)} e^{j\Delta \omega_{cm} t} \]

- Vlasov equation

\[ \frac{\partial \Psi_x}{\partial t} + \frac{\partial \Psi_x}{\partial \hat{x}} \dot{\hat{x}} + \frac{\partial \Psi_x}{\partial \varphi_{x0}} \dot{\varphi}_{x0} + \frac{\partial \Psi_x}{\partial \hat{\tau}} \dot{\hat{\tau}} + \frac{\partial \Psi_x}{\partial \psi_0} \dot{\psi}_0 = 0 \]

\[ \Rightarrow \quad \text{Linearized Vlasov equation} \]

\[ \frac{\partial \Psi_x}{\partial t} = - \frac{df_0 (\hat{x})}{d\hat{x}} g_0 (\hat{\tau}) \dot{\hat{x}} \]

\[ \Rightarrow \quad j \sum_m h_m (\hat{x}, \hat{\tau}) e^{-j(\varphi_{x0} + m\psi_0)} \Delta \omega_{cm} e^{j\Delta \omega_{cm} t} = - \frac{df_0 (\hat{x})}{d\hat{x}} g_0 (\hat{\tau}) \dot{\hat{x}} \]
EFFECT OF THE PERTURBATION (2/10)

◆ The expression of \( \dot{x} \) can be drawn from the single-particle horizontal equation of motion

\[
\dot{x} = \frac{d}{dt} \left( \dot{x} \right) = \frac{d}{dt} \left[ x^2 + \left( \frac{\dot{x}}{\dot{\varphi}_x} \right)^2 \right]^{1/2} = F_x \frac{\dot{x}}{\dot{x} \dot{\varphi}_x}
\]

\[
\frac{\dot{x}}{\dot{x} \dot{\varphi}_x} = - \sin(\varphi_x)
\]

\[
\Rightarrow \dot{x} = - \frac{\sin(\varphi_x)}{\dot{\varphi}_x} F_x
\]
EFFECT OF THE PERTURBATION (3/10)

- Using the definition of the transverse impedance, the force can be written

\[ F_x = -\frac{j e \beta \pi I_b}{R \gamma m_0} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \sigma_{x,m}(p) e^{-j p \Omega_0(t-\tau)} e^{j [(p + Q_{x0}) \Omega_0 + \omega_c] t} \]

- Developing the \( \sin(\varphi_x) \) into exponential functions, keeping then only the slowly varying term, making the approximation \( \dot{\varphi}_x \approx Q_{x0} \Omega_0 \) and using the relations \( J_{-m}(-x) = J_m(x) \) and one from the longitudinal course, yields

\[ \dot{x} = -\frac{e \pi I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p,m=-\infty}^{p,m=+\infty} Z_x(p) \sigma_{x,m}(p) j^m J_{m,x}(p, \hat{\tau}) e^{-j(\varphi_{x0} + m \psi_0)} e^{j \Delta \omega_{cm} x t} \]
EFFECT OF THE PERTURBATION (4/10)

⇒ For each mode $m$, one has

$$j h_m(\hat{x}, \hat{\tau}) \Delta \omega_{cm}^x = \frac{e \pi I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \sigma_x(p) j^m J_{m,x}(p, \hat{\tau}) \frac{d f_0(\hat{x})}{d\hat{x}} g_0(\hat{\tau})$$

with

$$\sigma_x(p) = \sum_m \sigma_{x,m}(p)$$

Spectrum amplitude at frequency

$$\left( p + Q_{x0} \right) \Omega_0 + \omega_c$$

Multiplying both sides by $\hat{x}^2$ and integrating over $\hat{x}$

$$j \Delta \omega_{cm}^x \int_{\hat{x}=0}^{\hat{x}=+\infty} h_m(\hat{x}, \hat{\tau}) \hat{x}^2 d\hat{x} = -\frac{e I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \sigma_x(p) j^m J_{m,x}(p, \hat{\tau}) g_0(\hat{\tau})$$

using the relation

$$\int_{\hat{x}=0}^{\hat{x}=+\infty} \frac{d f_0(\hat{x})}{d\hat{x}} \hat{x}^2 d\hat{x} = -2 \int_{\hat{x}=0}^{\hat{x}=+\infty} f_0(\hat{x}) \hat{x} d\hat{x} = -\frac{1}{\pi}$$
EFFECT OF THE PERTURBATION (5/10)

Note that the horizontal stationary distribution disappeared and only the longitudinal one remains ⇒ Only the beam center of mass is important (in our case). This should also be valid for the perturbation, which can be written

\[ \hat{x} = +\infty \]
\[ \int_{\hat{x} = 0} h_m \left( \hat{x}, \hat{\tau} \right) \hat{x}^2 \, d\hat{x} = g_0(\hat{\tau}) \hat{x}_m(\hat{\tau}) \]

Averaged peak betatron amplitude

⇒ Final form of the equation of coherent motion of a single bunch:

\[ j \Delta \omega_{cm}^x \hat{x}_m(\hat{\tau}) = - \frac{e I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p = -\infty}^{+\infty} Z_x(p) \sigma_x(p) j^m J_{m,x}(p, \hat{\tau}) \]

\[ \Delta \omega_{cm}^x = \omega_c - m \omega_s \]

Contribution from all the modes \( m \)
EFFECT OF THE PERTURBATION (6/10)

with

\[ \sigma_{x,m}(p) = j^{-m} \int_{\dot{x}=0}^{\dot{x}=+\infty} \int_{\dot{\tau}=0}^{\dot{\tau}=+\infty} h_m(\dot{x}, \dot{\tau}) J_{m,x}(p, \dot{x}) \dot{x}^2 \, d\dot{x} \, \dot{\tau} \, d\dot{\tau} \]

\[ = j^{-m} \int_{\dot{\tau}=0}^{\dot{\tau}=+\infty} J_{m,x}(p, \dot{\tau}) g_0(\dot{\tau}) \dot{x}_m(\dot{\tau}) \dot{\tau} \, d\dot{\tau} \]

◆ Coherent modes of oscillation at low intensity (i.e. considering only a single mode \( m \))

\[ j \Delta \omega_{cm}^x \dot{x}_m(\dot{\tau}) = -\frac{e I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x(p) \sigma_{x,m}(p) j^m J_{m,x}(p, \dot{\tau}) \]

Multiplying both sides by \( j^{-m} J_{m,x}(l, \dot{\tau}) g_0(\dot{\tau}) \dot{\tau} \) and integrating over \( \dot{\tau} \)

Elias Métral, La Sapienza University, Rome, Italy, May 23-26, 2017
EFFECT OF THE PERTURBATION (7/10)

\[
\Delta \omega_{cm}^x \sigma_{x,m}(l) = \sum_{p = +\infty}^{p = -\infty} K_{lp}^{x,m} \sigma_{x,m}(p)
\]

\[
K_{lp}^{x,m} = \frac{jeI_b}{2\gamma m_0 c Q_{x0}} Z_x(p) \int_{\hat{\tau} = 0}^{\hat{\tau} = +\infty} J_{m,x}(l, \hat{\tau}) J_{m,x}(p, \hat{\tau}) g_0(\hat{\tau}) \hat{\tau} d\hat{\tau}
\]

◆ Following the same procedure as for the longitudinal plane, the horizontal coherent oscillations (over several turns) of a “water-bag” bunch interacting with a constant inductive impedance are shown in the next slides for the first head-tail modes

* Note that the index \(x\) has been removed for clarity

\[
g_0(\hat{\tau}) = \frac{4}{\pi \tau_b^2}
\]
EFFECT OF THE PERTURBATION (8/10)

\[ f_{\xi_x} = \frac{\xi_x}{\eta} Q_{x0} f_0 \]

\[ \chi_x = \omega_{\xi_x} \tau_b = 10 \]

\[ \omega_{\xi_x} = 0 \]

\[ Q_{x0} = x.13 \]
EFFECT OF THE PERTURBATION (9/10)
EFFECT OF THE PERTURBATION (10/10)

Observation in the CERN PSB in ~1974 (J. Gareyte and F. Sacherer)

Observation in the CERN PS in 1999

(Laclare’s) theory
REMINDER OF THE PROCEDURE: BOTH L & T

♦ Start with the single particle motion => Harmonic oscillator + beam-induced electromagnetic force (L or T)
♦ Look at the single particle signal => Line spectrum with an infinite number of synchrotron satellites \( m \) (centered at 0 for L and at the chromatic frequency for T)
♦ Consider a distribution of particles (particle density in phase space) and express it as a sum of a stationary distribution + a perturbation
♦ The beam-induced electromagnetic force can be expressed through the impedance (complex function of frequency) for both L and T
♦ Study the effect of the impedance on the stationary distribution (for L) => A new fixed point is obtained, with a dependency on the bunch intensity of the synchronous phase, incoherent frequency, effective (total) voltage and bunch length
♦ Around the new fixed point (for L), write the perturbation => Coherent with respect to the satellite number \( m \)
♦ Apply the Vlasov equation to first order => One ends up with an eigenvalue system to solve
♦ The result is an infinite number of modes of oscillation \( mq \)
Finding the eigenvalues and eigenvectors of a complex matrix by computer can be difficult in some cases, and a simple approximate formula for the eigenvalues is useful in practice to have a rough estimate.

**Assuming sinusoidal modes**

\[
p_m(t) = \begin{cases} 
\cos\left(\left(\left|m\right| + 1\right)\frac{\pi}{\tau_b} t\right), & m \text{ even} \\
\sin\left(\left(\left|m\right| + 1\right)\frac{\pi}{\tau_b} t\right), & m \text{ odd}
\end{cases}
\]

The difference signal from a beam position monitor has the form

\[
\Delta - \text{signal} \propto p_m(t) e^{j\left(\chi_x t/\tau_b + 2\pi kQ_{x0}\right)}
\]

\[
\chi_x = \omega_{\xi_x} \tau_b
\]

For the kth revolution

**Total phase shift between head and tail**
The function \( h_{m,m}(\omega - \omega_{\xi x}) = \left| p_m(\omega - \omega_{\xi x}) \right|^2 \),

where \( p_m(\omega - \omega_{\xi x}) \) is the Fourier transform of \( p_m(t) e^{j\omega_{\xi x}t} \),

is a good approximation of the power spectrum

\[
h_{m,m}(\omega) \approx \left| \sigma_{mm}(\omega) \right|^2
\]

\[
h_{m,m}(\omega) = \frac{\tau_b^2}{2\pi^4} \left( |m| + 1 \right)^2 \frac{1+(-1)^{|m|}}{\left( \frac{\omega \tau_b}{\pi} \right)^2 - \left( |m| + 1 \right)^2} \cos(\omega \tau_b)
\]
APPROXIMATE FORMULAE: SACHERER FORMULAE (3/4)

Making this approximation, it can be shown that the Sacherer formulae are obtained

- **In longitudinal**

\[
\Delta \omega_{m,m}^l = \left| \frac{m}{m+1} \right| \frac{j I_b \omega_s}{3 B^3 \hat{V}_T h \cos \phi_s} \left[ \frac{Z_l(p)}{p} \right]_{m,m}^{\text{eff}}
\]

\[
\left[ \frac{Z_l(p)}{p} \right]_{m,m}^{\text{eff}} = \sum_{p=-\infty}^{p=+\infty} \frac{Z_l(\omega_p)}{p} \frac{h_{m,m}(\omega_p^l)}{\sum_{p=-\infty}^{p=+\infty} h_{m,m}(\omega_p^l)} \quad \omega_p^l = p \Omega_0 + m \omega_s
\]

- **In transverse**

\[
\Delta \omega_{m,m}^{x,y} = \left( \left| m \right| + 1 \right)^{-1} \frac{j e \beta I_b}{4 \pi R B m_0 \gamma Q_{x0,y0} \Omega_0} \left( Z_{x,y}^{\text{eff}} \right)_{m,m}
\]

\[
\left( Z_{x,y}^{\text{eff}} \right)_{m,m} = \sum_{p=-\infty}^{p=+\infty} Z_{x,y}(\omega_p^{x,y}) h_{m,m}(\omega_p^{x,y} - \omega_{\xi x,y}^{x,y}) \quad \omega_p^{x,y} = \left( p + Q_{x0,y0} \right) \Omega_0 + m \omega_s
\]
APPROXIMATE FORMULAE: SACHERER FORMULAE (4/4)

Power spectrum

\[ h_{m,m}(\omega - \omega_{\xi_x}) \]

\[ m = 0 \]
\[ |m| = 1 \]
\[ |m| = 2 \]

Pick-up (Beam Position Monitor) signal

\[ \Delta R\text{-signal} \]

\[ m = 0 \]
\[ |m| = 1 \]

One particular turn
COUPLED-BUNCH INSTABILITIES: BOTH L & T

◆ In the case of \( M \) equi-populated equi-spaced bunches
  ▪ \( M \) possible coupled-bunch modes \( n \) (from 0 to \( M – 1 \))
  ▪ Mode \( n \) corresponds to a phase shift between 2 adjacent bunches of

\[
2 \pi \frac{n}{M}
\]

▪ The single-bunch eigenvalue is extended to the coupled-bunch regime by making the following modifications

- \( I_b \) => \( M I_b \)
- \( l \) => \( n + l M \)
- \( p \) => \( n + p M \)

◆ As concerns Sacherer formulae => Only change: sum over the coupled-bunch mode spectrum (see above, instead of the single-bunch spectrum)
MITIGATIONS (1/6)

◆ Electronic dampers

- Used for coupled-bunch instabilities (both L & T) and intra-bunch instabilities with long bunches => Work very well

- Not used yet for intra-bunch instabilities with short bunches => Bandwidth issue. Intense studies since several years to develop a transverse wide-band damper in SPS and promising results have been reached
MITIGATIONS (2/6)

- Example of studies in the SPS in 2016

=> All these instabilities will / should be cured in the future with dampers!

Courtesy of J.D. Fox et al.
MITIGATIONS (3/6)

- **Landau damping** => Generate a (controlled) tune spread such that the coherent tune shift remains inside the spread (in fact *inside a stability diagram*)

---

**Landau Damping**

- **Three Coupled Oscillators:**
  - *equal frequencies*
  - *instability*

- **Three Coupled Oscillators:**
  - *different frequencies*
  - *no instability*

- **Limit:**
  - *frequency spread (tune spread)*
  - *single particle resonances*
MITIGATIONS (4/6)

- **In longitudinal**: use the nonlinearity of the RF bucket => It can be shown that

\[ I_m^{-1}(\omega) = \Delta \omega_{m,m}^l \]

\[ I_m(\omega) = \int_0^\infty \frac{\hat{t}^{2m}}{\omega - m\omega_s(\hat{t})} \frac{dg_0(\hat{t})}{d\hat{t}} d\hat{t} \]

\[ \omega = m(\omega_s - S) \]

\[ g_0(\hat{t}) \propto (1 - \hat{t}^2)^2 \]

Full spread between the centre and the edge of the bunch

\[ S = \left(1+\frac{5}{3}\tan^2\phi_s\right)\frac{\pi^2}{16} (hB)^2 \omega_s \]

Sacherer stability criterion

\[ S \geq \frac{4}{\sqrt{|m|}} \left| \Delta \omega_{m,m}^l \right| \]
MITIGATIONS (5/6)

- **In transverse**: use controlled nonlinearities (e.g. Landau octupoles) =>

  It can be shown that (e.g. in the horizontal plane)

  \[
  I^{-1}_m (\omega) = \Delta \omega^x_{m,m},
  \]

  \[
  I_m (\omega) = - \int_{J_x=0}^{+\infty} dJ_x \int_{J_y=0}^{+\infty} dJ_y \frac{J_x \frac{\partial f (J_x, J_y)}{\partial J_x}}{\omega - \omega_x (J_x, J_y) - m \omega_s} \]

  \[
  \omega_x (J_x, J_y) = \omega_0 + a J_x + b J_y
  \]

  Example of the LHC at 7 TeV with nominal transverse emittance and maximum current in the Landau octupoles

Gaussian, \(a < 0\)

Gaussian, \(a > 0\)

Up to 6 \(\sigma\)

Up to 3.2 \(\sigma\)
MITIGATIONS (6/6)

- Linear coupling
  - Can have a beneficial effect if asymmetries between the 2 transverse planes (different impedances, chromaticities, etc.) => Sharing of the instability growth rates and frequency spreads
  - Detrimental effect in case there is nothing to gain from one plane (two identical planes) and coupling is too strong => Loss of Landau damping (coherent tune outside of tune spread)
  - Stabilization of the PS low-energy instability by linear coupling

![Graph showing intensity vs. time with and without linear coupling]
TRANSVERSE: HIGH-INTENSITY
Reminder: general equation of coherent motion considering the contributions from all the modes \( m \)

\[
j \Delta \omega_{cm}^x \hat{x}_m (\hat{\tau}) = - \frac{e I_b}{2 \gamma m_0 c Q_{x0}} \sum_{p=-\infty}^{p=+\infty} Z_x (p) \sigma_x (p) j^m J_{m,x} (p, \hat{\tau})
\]

Multiplying both sides by \( j^{-m} J_{m,x} (l, \hat{\tau}) g_0 (\hat{\tau}) \hat{\tau} \) and integrating over \( \hat{\tau} \)

\[
\Rightarrow \quad \Delta \omega_{cm}^x \sigma_{x,m} (l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^{x,m} \sigma_x (p)
\]

Dividing both sides by \( \Delta \omega_{cm}^x \) and summing over \( m \)

\[
\sigma_x (l) = \varepsilon_x \sum_{p=-\infty}^{p=+\infty} \left[ j Z_x (p) \right] M_{lp}^x \sigma_x (p)
\]
with

\[ M_{lp}^x = 2B \sum_m \frac{1}{\omega_c / \omega_s - m} \int_0^1 J_{m,x} \left( l, \frac{\tau_b}{2} u \right) J_{m,x} \left( p, \frac{\tau_b}{2} u \right) u \, du \]

(assuming a water-bag for the stationary distribution, as before)

and

\[ \varepsilon_x = \frac{eI_b}{4 \pi \gamma m_0 c Q_{x0} B \omega_s} \]

### Method to solve this equation

- Assume a real coherent betatron frequency shift measured in incoherent synchrotron frequency unit \( \omega_c / \omega_s = (\omega - \omega_{x0}) / \omega_s \)
- Look for the eigenvalues of the matrix \( \left[ jZ_x(p) M_{lp}^x \right] \)
- Scale the intensity parameter \( \varepsilon_x \) in order to adjust the eigenvalue to unity
Case of a constant (vertical) inductive impedance

Also noted sometimes $- j Z \varepsilon$ later

$$x = - j Z_y (0) \varepsilon_y$$
Case of a Broad-Band resonator impedance

\[ \tilde{\omega}_r = \frac{\omega_r}{\sqrt{1 - \frac{1}{4Q^2}}} \]

\[ \alpha = \frac{\omega_r}{2Q} \]

\[ Z_y(\omega) = \frac{\omega_r R_y}{\omega} \frac{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \]

\[ G_y(t) = \frac{\omega_r^2 R_y}{Q \tilde{\omega}_r} e^{-\alpha t} \sin(\tilde{\omega}_r t) \]

Elias Métral, La Sapienza University, Rome, Italy, May 23-26, 2017
~ SPS case: \( f_r \tau_b = 2.8 \)

\[ x = -j Z_y(0) \varepsilon_y \]
Another possibility to solve this problem is to use a decomposition on the low-intensity eigenvectors (as proposed by Garnier in 1987).

Using this formalism, the effect of a transverse damper was recently added.

Remark: 2 other codes (Vlasov solvers) including the transverse damper were developed in the recent years:

- A. Burov developed a Nested Head-Tail Vlasov Solver (NHTVS) with transverse damper in 2014.
- N. Mounet solved Sacherer integral equation with transverse damper, using a decomposition over Laguerre polynomials of the radial functions (DELPHI code, 2015).

Sacherer integral equation was also solved using a decomposition over Laguerre polynomials of the radial functions by Besnier in 1974 and Y.H. Chin in 1985 in the code MOSES without transverse damper.
Damping time of a transverse damper

- The damper gain $G$ is defined by $G = \frac{\Delta \vartheta}{\beta_x}$, where $\Delta \vartheta$ is the change of the slope produced by a measured displacement $x$, assuming the same $\beta_x$ value at the PU and the kicker. After one turn, the displacement has been corrected by $\Delta x = \beta_x \Delta \vartheta = G x$

  \[ \Rightarrow \quad \frac{dx}{dt} = \frac{G x}{T_0} = G f_0 x \quad \Rightarrow \text{Damping time: } \tau_{\text{damper}} = \frac{1}{G f_0} \]

- Averaging over all the possible betatron phases at the PU position (as the tune cannot be an integer):

\[ \tau_{\text{damper}} = \frac{2}{G f_0} = \frac{n_d}{f_0} \]
◆ Decomposition on the low-intensity modes (following Garnier 1987) + adding a (perfect) transverse damper

\[ \sigma_x(l) = \sum_{i,j} a_{ij} \sigma_{x,ij}(l) \]

\[ \frac{\omega_c}{\omega_s} a_{kl} = H^x a_{ij} \]

\[ H^x_{kl,ij} = k \delta_{ki} \delta_{lj} \]

\[ + \Delta \omega_{ckl} \sum_{p=\pm\infty} \sigma_x^{*}(p) \left[ - j \frac{Z_x(p)}{Z_x(0)} \right] + \delta_{p0} \frac{F_{\text{damper}} f_0}{n_d \omega_s \left( -x 2 \pi \omega_s B \right)} \sigma_{x,ij}(p) \]

- \( F_{\text{damper}} = 0 \Rightarrow \) No damper
- \( F_{\text{damper}} = +j \Rightarrow \) Resistive damper
- \( F_{\text{damper}} = +1 \Rightarrow \) Reactive damper
- \( n_d = \) Damper damping time (in # of turns)

Eigenvalue solution of the low-intensity eigenvalue problem with constant inductive impedance

Eigenvector solution of the low-intensity eigenvalue problem with constant inductive impedance
- Check between the 2 methods => **1) Constant inductive impedance**

Laclare1987: 
Eigenvalue problem without decomposition (without damper)

Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)

\[ x = - j Z_y(0) \epsilon_y \]
2) Broad-Band impedance \[ f_r \tau_b = 2.8 \]

Laclare1987: 
Eigenvalue problem without decomposition (without damper)

Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)

Courtesy of D. Amorim
2) Broad-Band impedance $f_r \tau_b = 2.8$ $Q' = 0$

Laclare1987:
Eigenvalue problem without decomposition (without damper)

Decomposition on the low-intensity Eigenvectors following Garnier1987 formalism (without damper)

$e^{j(\omega_R + j\omega_I)t} = e^{j\omega_R t} e^{\tau}$

Courtesy of D. Amorim
Past comparison between MOSES and HEADTAIL simulations (for a Gaussian longitudinal distribution)
$Q' = +7$

$Q' = -7$

The diagrams illustrate the real and imaginary parts of $(\omega - \omega_y) / \omega_s$ as functions of $-jZ\epsilon$ for two different values of $Q'$. The graphs show a clear trend in the data for both $Q'$ values.
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

With transverse damper (reactive, 50 turns) in red

With transverse damper (-reactive, 50 turns) in red

\[ Q' = 0 \]
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE OF THE (BROAD-BAND) SPS

With transverse damper (reactive) in red: 25, 50 and 100 turns

25 turns

50 turns

100 turns
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: CASE OF THE (BROAD-BAND) SPS

\[ Q' = 0 \]

With transverse damper (resistive, 50 turns) in red

With transverse damper (-resistive, 50 turns) in red
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE OF THE (BROAD-BAND) SPS

Q’ = 0

With transverse damper (resistive) in red: 25, 50 and 100 turns
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE WITH $fr \times taub = 0.8$ (instead of 2.8 before) $Q' = 0$

With transverse damper
(reactive, 50 turns) in red

With transverse damper
(-reactive, 50 turns) in red
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE WITH $fr \times taub = 0.8$ (instead of 2.8 before) $Q' = 0$

With transverse damper (resistive, 50 turns) in red

With transverse damper (-resistive, 50 turns) in red
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE WITH $f_r \times \tau_{aub} = 0.8$ (instead of 2.8 before) $Q' = 0$

With transverse damper (resistive) in red: 25, 50 and 100 turns
RESONATOR IMPEDANCE + TRANSVERSE DAMPER: 
CASE WITH \(fr \times taub = 0.8\) (instead of 2.8 before) & \(Q' = + 7\)

With transverse damper (reactive, 50 turns) in red

With transverse damper (resistive, 50 turns) in red
RESONATOR IMPEDANCE + TRANSVERSE DAMPER:
CASE WITH $fr \times taub = 0.8$ (instead of 2.8 before) & $Q' = -7$

With transverse damper (reactive, 50 turns) in red

With transverse damper (resistive, 50 turns) in red
Destabilising effect of the resistive transverse damper (e.g. with 50 turns) for $Q' = 0$ => Where does the instability come from?
• Mode 0 (1st radial mode) only
  => Stable
- Mode -1 (1st radial mode) only
  => Stable
Instability appears when both modes -1 and 0 (with only 1\textsuperscript{st} radial mode) are considered.

\[ \text{Re} \left( \frac{\omega - \omega_y}{\omega_s} \right) \]

\[ \text{Im} \left( \frac{\omega - \omega_y}{\omega_s} \right) \]

\[ -j Z \epsilon \]

\[ 0.0 \quad 0.5 \quad 1.0 \quad 1.5 \quad 2.0 \]

\[ 0.0 \quad 0.5 \quad 1.0 \quad -0.5 \quad -1.0 \]

=> This is the interaction between modes -1 and 0 through the damper which creates the instability.

The “coupling” between the 2 modes pushes apart the instability growth rates and as the lowest one is 0, it becomes negative.
If one looks at the matrix to be diagonalized, it can be approximated by (with $x = -j Z \varepsilon$)

$$
\begin{pmatrix}
-1 & -0.23 j x \\
-0.55 j x & -0.92 x + 0.48 j
\end{pmatrix}
$$
If one looks at the matrix to be diagonalized, it can be approximated by (with $x = -jZ\varepsilon$)

$$
\begin{pmatrix}
-1 & -0.23jx \\
-0.55jx & -0.92x + 0.48j
\end{pmatrix}
$$

N.B.: Would be $+0.48$ for a + reactive damper
\[ x = -j Z \epsilon \]
\[ x = -j Z \epsilon \]

\[ \frac{\text{Im} (\omega - \omega_y)}{\omega_s} + \frac{0.48}{2} j \]
\[ x = -jZ \epsilon \]

\[ \frac{\Im(\omega - \omega_y)}{\omega_s} = \frac{0.48}{4} j \]
\[ x = -jZ\epsilon + \frac{0.48}{10}j \]
\[ x = -j Z \epsilon \]

\[ + \frac{0.48}{100} j \]

i.e. almost no damper
\[ x = -j Z \epsilon \]

i.e. using a REACTIVE damper
0.0 \times 0.48 \text{ i.e. using a REACTIVE damper 2 times stronger}
◆ **Simple formula for the intensity threshold** in the case of a bunch interacting with a Broad-Band impedance in the long-bunch regime (as for the SPS case before), considering only the mode-coupling between the 2 adjacent modes overlapping the maximum of the resonator impedance.

“Long-bunch” regime:

\[ \tau_b \gg 0.5 / f_r \]

\[ |m| + 1 \approx 2 f_r \tau_b \left( 1 + f_{\xi_y} / f_r \right) \]
\[ \tau_b = 0.5 / f_r \]
Using \( h_{m,n}(\omega - \omega_{\xi_x}) = p_m^*(\omega - \omega_{\xi_x}) p_n(\omega - \omega_{\xi_x}) \)

\[
h_{m,n}(\omega) = \frac{\tau_b^2}{\pi^4} (|m| + 1) \times (|n| + 1) \times F^m_n
\]

\[
= \times \left\{ \left( \frac{\omega \tau_b}{\pi} \right)^2 - (|m| + 1)^2 \right\}^{-1} \times \left\{ \left( \frac{\omega \tau_b}{\pi} \right)^2 - (|n| + 1)^2 \right\}^{-1}
\]

\[
F^m_n\text{ even} = (-1)^{(|m| + |n|)/2} \times \cos^2 \left[ \frac{\omega \tau_b}{2} \right]
\]

\[
F^m_n\text{ odd} = \frac{(-1)^{(|m| + |n| + 3)/2}}{2 j} \times \sin \left[ \frac{\omega \tau_b}{2} \right]
\]

\[
F^m_n\text{ even} = (-1)^{(|m| + |n| + 1)/2} \times \sin \left[ \frac{\omega \tau_b}{2} \right]
\]

\[
F^m_n\text{ odd} = (-1)^{(|m| + |n| + 2)/2} \times \sin^2 \left[ \frac{\omega \tau_b}{2} \right]
\]
In longitudinal

\[ \Delta \omega_{m,n}^l = \frac{|m|}{|m| + 1} \frac{j I_b \omega_s}{3 B^3 \hat{V}_T h \cos \phi_s} \times \left[ \frac{Z_l(p)}{p} \right]_{m,n}^{\text{eff}} \]

\[ \left[ \frac{Z_l(p)}{p} \right]_{m,n}^{\text{eff}} = \sum_{p=-\infty}^{p=+\infty} \frac{Z_l(\omega_p^l)}{p} h_{m,n}(\omega_p^l) \quad \omega_p^l = p \Omega_0 + m \omega_s \]

In transverse

\[ \Delta \omega_{m,n}^{x,y} = (|m| + 1)^{-1} \frac{j e \beta I_b}{4 \pi R B m_0 \gamma Q_{x0,y0} \Omega_0} \left( Z_{x,y}^{\text{eff}} \right)_{m,n} \]

\[ \left( Z_{x,y}^{\text{eff}} \right)_{m,n} = \sum_{p=-\infty}^{p=+\infty} \sum_{p=-\infty}^{p=+\infty} Z_{x,y}(\omega_p^{x,y}) h_{m,n}(\omega_p^{x,y} - \omega_{\xi,x,y}) \quad \omega_p^{x,y} = \left( p + Q_{x0,y0} \right) \Omega_0 + m \omega_s \]
Long-bunch regime

Short-bunch regime

From TMCI theory

\[ N_{b,th}^y \]

0.5 1 1.5 2

\[ 2f_r \tau_b \]
- Increase the chromatic frequency
- Chromaticity jump in case transition has to be crossed

Try to decrease the impedance and/or increase the resonance frequency => Impedance reduction campaign

Change the optics to increase the betatron tune (decrease the beta function at critical impedances) and/or go further away from transition => New optics needed

Increase the beam longitudinal emittance (when possible)

Simple formula for TMCI (with the 2 assumptions):

\[ N_{b,th}^y \propto \frac{f_r}{|Z_y|} |\eta| Q_y \varepsilon_L \left( 1 + \frac{f_{\xi_y}}{f_r} \right) \]
- Increase the chromatic frequency
- Chromaticity jump in case transition has to be crossed
Increase the beam longitudinal emittance (when possible)
Change the optics to increase the betatron tune (decrease the beta function at critical impedances) and/or go further away from transition => New optics needed

* No dependence on $Q_s$!
** It is the same formula as for coasting beams (with peak values)!

Simple formula for TMCI (with the 2 assumptions):

$$N_{b,th}^y \propto \frac{f_r}{|Z_y|} |\eta| Q_y \varepsilon_L \left(1 + \frac{f_{\xi_y}}{f_r}\right)$$

Try to decrease the impedance and/or increase the resonance frequency => Impedance reduction campaign

Try to decrease the impendence and/or increase the resonance frequency => Impedance reduction campaign
- **Ex.1 => In the PS**: a fast vertical single-bunch instability is observed (with high-intensity bunches) when transition is crossed and when no longitudinal emittance blow-up is applied before transition.

\[ \Sigma, \Delta R, \Delta V \text{ signals} \]

- Head stable
- Tail unstable
- \( \sim 700 \text{ MHz} \)
- Time (10 ns/div)
=> Instability suppressed by increasing the longitudinal emittance
Ex.2 => In the SPS

Synchrotron period ≈ 7 ms

Instability (initially) suppressed by increasing the chromaticity

\[ \xi_y \approx 0 \]

\[ \xi_y = 0.8 \]
$\Rightarrow$ Travelling-wave pattern along the bunch

$\xi_y = 0.14$

$\langle y \rangle$ [a.u.]

Head

Tail

$1^{st}$ trace (in red) = turn 2

Last trace = turn 150

Every turn shown
$\xi_y = 2.04$

1st trace (in red) = turn 2  Last trace = turn 150  Every turn shown
• \( \gamma_t \) was recently modified in the SPS to increase the TMCI intensity threshold above the foreseen intensities for the future upgrade

• Simple rough estimate of \( \gamma_t \) for machines made of simple FODO cells:
  
  \[ D_x \approx \frac{\rho}{Q_x^2} \]
  
  Inserting this in the definition of \( \alpha_p \) (and then expressing \( \gamma_t \)) yields
  
  \[ \gamma_t \approx Q_x \]

  => If one wants to modify \( \gamma_t \) (increase or decrease its value) one should modify the horizontal tune
• TMCI intensity threshold with the old (Q26) optics at injection: $\sim 1.7 \times 10^{11}$ p/b

• Predictions going from Q26 to the new (Q20) optics:

  ⊕ **Q26:** $\eta | Q_y = 0.62 \times 10^{-3} \times 26.13 \approx 0.0162$ $\gamma_t = 22.8$

  ⊕ **Q20:** $\eta | Q_y = 1.80 \times 10^{-3} \times 20.13 \approx 0.0362$ $\gamma_t = 18$

  => A gain of a factor $0.0362 / 0.0162 \approx 2.2$ in the intensity threshold was expected
• Measurements

=> Good agreement with simple formula

Q26 optics: $\gamma_t = 22.8$

Q20 optics: $\gamma_t = 18$

Gain of a factor

$4.5 / 1.7 \approx 2.6$

Courtesy of B. Salvant et al.

Courtesy of H. Bartosik et al.
- Very good agreement between measurements and simulations

=> Intensity threshold with the new (Q20) optics: $\sim 4.5 \times 10^{11}$ p/b

[Graph showing measurements and simulations with annotations and color scale]

_Courtesy of H. Bartosik et al._
LONGITUDINAL: HIGH-INTENSITY
**Reminder:** general equation of coherent motion considering the contributions from all the modes $m$

\[
j \Delta \omega_{cm} \ j^{-m} \ g_m(\hat{\tau}) \ \hat{\tau} = \frac{2 \pi I_b \ m \ \omega_s}{\Omega_0^2 \ \hat{V}_T \ h \ \cos \phi_s} \ \frac{dg_0}{d\hat{\tau}} \ \sum_{p=-\infty}^{p=+\infty} \ \frac{Z_l(p)}{p} \ J_m(p \ \Omega_0 \ \hat{\tau}) \ \sigma(p)
\]

**Multiplying both sides by** $J_m(l \ \Omega_0 \ \hat{\tau})$ **and integrating over** $\hat{\tau}$

\[
\Rightarrow \ \Delta \omega_{cm} \ \sigma_m(l) = \sum_{p=-\infty}^{p=+\infty} K_{lp}^m \ \sigma(p)
\]

**Dividing both sides by** $\Delta \omega_{cm}^x$ **and summing over** $m$

\[
\sigma(l) = \varepsilon_{long} \ \sum_{p=-\infty}^{p=+\infty} \left[ j \ \frac{Z_l(p)}{p} \right] M_{lp} \ \sigma(p)
\]
with

\[ M_{lp} = 2B \sum_{m} \frac{m}{\omega_c - m} \int_{0}^{1} J_m(p \pi Bu) J_m(l \pi Bu) u \, du \]

and

\[ \varepsilon_{\text{long}} = \frac{4I_b}{\pi^2 B^3 \hat{V}_T h \cos \phi_s} \]
Case of a constant inductive impedance

\[ x = -j \left[ \frac{Z_l(p)}{p} \right]_{p=0} \]
\[
x = -j \left[ \frac{Z_l(p)}{p} \right]_{p=0}^{\varepsilon_{\text{long}}}
\]
Case of a Broad-Band resonator impedance

\[
\begin{align*}
\bar{\omega}_r &= \omega_r \sqrt{1 - \frac{1}{4Q^2}} \\
\alpha &= \frac{\omega_r}{2Q}
\end{align*}
\]

\[
Z_l(\omega) = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)}
\]

\[
G_l(t) = \frac{\omega_r R_s}{Q} e^{-\alpha t} \left[ \cos\left(\frac{\bar{\omega}_r t}{\omega_r}\right) - \frac{\alpha}{\bar{\omega}_r} \sin\left(\frac{\bar{\omega}_r t}{\omega_r}\right) \right]
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plots.png}
\caption{Graphs showing the impedance and current response for different parameters.}
\end{figure}
Case of a Broad-Band resonator impedance $f_r \tau_b = 2.8$

$$x = -j \left[ \frac{Z_l(p)}{p} \right]_{p=0} \varepsilon_{\text{long}}$$
Case of a Broad-Band resonator impedance \( f_r \tau_b = 2.8 \)

\[
x = - j \left[ \frac{Z_l(p)}{p} \right]_{p=0} \epsilon_{long}
\]

LMCI: Longitudinal Mode-Coupling Instability
Case of a Broad-Band resonator impedance $f_r \tau_b = 2.8$

- The threshold (mode-coupling) is reached when
  \[ |\epsilon_{long}^{th} \left| \left( \frac{Z_l(p)}{p} \right) \right|_{p=0} \approx 0.8 \]
  which can be re-written

  \[
  \left( \frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} \geq \frac{10}{3 \pi} \frac{I_{b,peak}}{\beta^2 \left( E_{\text{total}} / e \right) \eta} \left| \frac{Z_l(p)}{p} \right|_0
  \]

  using

  \[
  I_{b,peak} = \frac{3 I_b}{2 B} \quad \left( \frac{\Delta p}{p_0} \right)^2_{\text{FWHH}} = \omega_s^2 \tau_b^2 \frac{2}{2 \eta^2}
  \]

- This is the Keil-Schnell-Boussard criterion (i.e. the Keil-Schnell criterion for coasting beams applied with peak values for bunched beams as proposed by Boussard). Note that PWD leads to different thresholds below and above transition.
◆ Case of a Broad-Band resonator impedance \( f_r \tau_b = 2.8 \)

- The threshold (mode-coupling) is reached when \( \left| \varepsilon_{\text{th}}^{\text{long}} \right| \left| \frac{Z_1(p)}{p} \right|_{p=0} \approx 0.8 \)
  which can be re-written

\[
\left( \frac{\Delta p}{p_0} \right)_{\text{FWHH}}^2 \geq \frac{10}{3 \pi} \frac{I_{b,\text{peak}}}{\beta^2 (E_{\text{total}}/e) \eta} \left| \frac{Z_1(p)}{p} \right|_{p=0}
\]

using \( I_{b,\text{peak}} = \frac{3 I_b}{2 B} \)

- This is the Keil-Schnell-Boussard criterion (i.e. the Keil-Schnell criterion for coasting beams applied with peak values for bunched beams as proposed by Boussard). **Note that PWD leads to different thresholds below and above transition**

* No dependence on \( Q_s \! \)!
** The same formula can also be obtained by considering only the mode-coupling between the 2 adjacent modes overlapping the maximum of the resonator impedance**
CONCLUSION
◆ **Low-intensity** (for both Longitudinal and Transverse)

- **Each mode can be treated individually**
- Eigen-value system to be solved in general
- Solution can be approximated by Sacherer formula
- Landau damping used to stabilize these instabilities
  - From the non-linearity of the RF bucket in L
  - From external (controlled) nonlinearities in T: (Landau) octupoles
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★ **High-intensity** (for both Longitudinal and Transverse)
  - The modes cannot be treated independently => Mode influencing and mode-coupling
  - In the case of a Broad-Band resonator impedance and in the long-bunch regime, the same formulae as for coasting beams are recovered (using the peak values) in both L and T
Good understanding of impedance-induced beam instabilities BUT what is usually missing is a precise model of the machine impedance (e.g.: huge effort in the CERN SPS machine => Now the transverse impedance model can reproduce very well all the observables!)
◆ **Good understanding of impedance-induced beam instabilities BUT** what is usually missing is a precise model of the machine impedance (e.g.: huge effort in the CERN SPS machine => Now the transverse impedance model can reproduce very well all the observables!)

◆ **FURTHERMORE**, in a machine like the LHC, not only all the mechanisms have to be understood separately, but (ALL) the **possible interplays between the different phenomena need to be analyzed in detail** as they can play important roles in the beam stability

- Linear (Q’) and nonlinear (Q”) chromaticity
- Landau octupoles (and other nonlinearities) or RFQs (under study)
- Transverse damper (using realistic models)
- Space charge
- Beam-beam: head-on and long-range
- Electron cloud
- Linear coupling strength
- Tune separation between the transverse planes (bunch by bunch)
- Tune split between the two beams (bunch by bunch)
- Transverse beam separation between the two beams
- Noise, etc.