Multipole-field measurements by sampling oblong apertures of accelerator magnets

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A R T I C L E  I N F O

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A B S T R A C T

The rotating search coil is a commonly used tool to measure magnetic fields of accelerator magnets. The coil intercepts the magnetic flux at a radius given by the dimensions of the measurement shaft that comprises a set of search coils for the analog bucking of the main signal from the dipole field component. For magnets of a rectangular aperture with large aspect ratio (>3:1) the cylindrical domain covers only a portion of the magnet bore. As the field representation is dominated by measurement errors outside that cylindrical domain, a sampling technique is required. The method presented in this paper trades the precision in the measurements against the precision in the shaft positioning and arrives at a multipole representation that covers the entire bore of the magnet.

The magnetic field of accelerator magnets is commonly described by its main field component and the higher-order field errors. In a mathematical sense, this representation is an analytic function with so-called multipole coefficients [1]. These coefficients are determined by a Fourier series expansion of one integrated field component along the design trajectory. This representation is therefore strictly two-dimensional as the magnetic field is integrated along the search coil that covers the magnet extremities and the fringe-field region.

Being the most precise tool to measure the field errors, the rotating search coil [3] nevertheless lacks in versatility, because for a highest precision, the shaft must be as large as possible. This is a problem for magnets with rectangular apertures of large aspect ratio. For rectangular apertures large, stationary fluxmeters are often used for fast ramped magnets. For static operation a combination of field measurements at different transverse positions has found attention in the measurement community.

In synchrotrons the large aspect ratio aperture is usually filled by the non-symmetric transverse beam profile which is generated by the multi-turn injection process in the horizontal plane [2]. The field representation therefore has to cover the full domain.

A field representation in elliptical coordinates is an approach that requires the field distribution on an elliptical boundary from which elliptical multipole coefficients can be extracted [4]. However, the noise spectrum on the measurement is not respected because all multipole orders must be used to express the magnetic field distribution [3]. Experience has shown that the main field component is about two orders of magnitude less precise [6] than the higher-order multipoles.

The novelty of the proposed method lies in the exclusive use of the higher-order multipoles, which allows not only to preserve but even to increase the precision of the combined result with respect to the single measurement. The computed results are valid in the entire sampled domain.

The method was first presented for magnet with round apertures in [7]. In this paper, we present the combination of three rotating-coil measurements on the mid-plane of a normal-conducting dipole magnet.

2. Fundamentals

Consider three multipole measurements acquired by a rotating search coil, as shown in (Fig. 1). Each measurement yields a set of Fourier coefficients/multipoles, which completely determine the field...
inside the coil radius $R_c$. Outside that radius, the measurement errors strongly affect the accuracy of the field representation. In order to obtain a single field representation for the entire domain (aperture of the magnet), the three measurements must be combined in a post-processing step.

2.1. Magnetic field representation

The magnetic field representation reads in complex notation [1]

$$B(z) = B_r + iB_i = \sum_{n=1}^{\infty} C_n \left( \frac{z}{R_c} \right)^{n-1},$$

where $z = x + iy$ and the complex multipoles are given by $C_n = B_r + iB_i$. The radius $R_c$ is the search-coil radius or may be regarded as the reference radius to which the multipoles are scaled. The scaling law $C_n(r_0) = C_n(R_c)(r_0/R_c)^{n-1}$ allows to express the multipoles at a different radius $r_0$. The reference radius $r_0$ is often noted as $R_0$ or $R_{ref}$; in this work the reference radius equals the coil radius and will consequently be denoted $R_c$. In measurement practice the field harmonics are determined by a rotating coil measurement, which yields the radial component of the magnetic field on a circle of the radius of the coil. Owing to the regularity of the magnetic field in an aperture free of magnetic material and current sources, the governing Laplace equation and the eigensolutions of a boundary value problem, yield a field representation inside the measurement domain. In other words, the field harmonics $C_n$ can be determined from the boundary data established by the search-coil measurement.

2.2. Analytic continuation

Taking the complex representation of the magnetic field in Eq. (1), we can calculate the effect on the multipole-field errors by translating the reference frame into the positions of the measurement coil, $z \rightarrow z', z' = z - z_i$. As this displacement stays within the bore of the magnet, free of magnetic material and current sources, the path between $z$ and $z'$ remains zero-homotopic as required by the method of analytic continuation. For the magnetic flux density being invariant with respect to the frame change we obtain

$$\sum_{n=1}^{\infty} C_n(z_i) \left( \frac{z}{R_c} \right)^{n-1} = B(z) = \sum_{n=1}^{\infty} C_n(z) \left( \frac{z'}{R_c} \right)^{n-1}.$$  

(2)

$z_i$ are the position of the displaced measurements. Using the binomial series expansion for the term $(z' + z_i)^{n-1}$, the left-hand side of Eq. (2) can be transformed as follows:

$$\sum_{n=1}^{\infty} C_n(z_i) \left( \frac{z}{R_c} \right)^{n-1} = \sum_{n=1}^{\infty} C_n(z_i) \left( \frac{z + z_i}{R_c} \right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} C_n(z_i) \left( \frac{n - 1}{k - 1} \right) \left( \frac{z}{R_c} \right)^{k-1} \left( \frac{z_i}{R_c} \right)^{n-k}$$

(3)

Rearranging the double sum [8] according to $\sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{nk}$ yields:

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} C_n(z_i) \left( \frac{z}{R_c} \right)^{k-1} \left( \frac{z_i}{R_c} \right)^{n-k} = \sum_{n=1}^{\infty} C_n(z) \left( \frac{z}{R_c} \right)^{n-1}.$$  

(4)

Comparing the coefficients and using the identity $(\gamma_n^m) = (\delta_{n-k})$ finally results in

$$C_n(z) = \sum_{k=n}^{\infty} C_k(z_i) \left( \frac{z}{R_c} \right)^{k-1} \left( \frac{z_i}{R_c} \right)^{n-k}.$$  

(5)

For measurement practice the series is truncated at an index $k=K$, usually at $K=15$ because of the limited signal-to-noise ratio:

$$C_n(z) \approx \sum_{k=n}^{K} C_k(z_i) \left( \frac{z}{R_c} \right)^{k-1} \left( \frac{z_i}{R_c} \right)^{n-k}.$$  

(6)

Every multipole measured with the displaced coil is coupled to every higher-order multipole in the reference frame. This effect is known as feed-down in the magnet-design community.

2.3. Synthesis of the field measurements

A synthesis of multiple magnetic measurements, presented in [7], relies on the link between the multipoles $C_n$ at the displaced positions and the multipoles $C_n$ at the center. The feed-down formula from Eq. (6) is noted for the multipole orders $n = 2, 3, 4, ..., N$ for all displaced positions $z_i$. The multipoles to compute are the $C_n$ and are the unknowns in an over-determined equation system. The equation system for the three measurements can be written as:

$$C_2(z) = C_2(z_0) \left( \frac{z}{R_c} \right)^2 + C_2(z_1) \left( \frac{z}{R_c} \right)^2 + C_2(z_2) \left( \frac{z}{R_c} \right)^2 + \cdots + C_2(z_N) \left( \frac{z}{R_c} \right)^2$$

$$C_3(z) = C_3(z_0) \left( \frac{z}{R_c} \right)^3 + C_3(z_1) \left( \frac{z}{R_c} \right)^3 + C_3(z_2) \left( \frac{z}{R_c} \right)^3 + \cdots + C_3(z_N) \left( \frac{z}{R_c} \right)^3$$

$$\vdots$$

$$C_N(z) = C_N(z_0) \left( \frac{z}{R_c} \right)^N + C_N(z_1) \left( \frac{z}{R_c} \right)^N + C_N(z_2) \left( \frac{z}{R_c} \right)^N + \cdots + C_N(z_N) \left( \frac{z}{R_c} \right)^N$$

The vertical dots indicate equations up to $C_N$, where $N$ is the highest multipole order extracted from each of the measurements. The central measurement at $z_0$ defines the position of the reference frame in which the reconstructed multipoles are computed. Here, the feed-down formula does not need to be applied so that the measured and reconstructed multipoles are equal and the equations become trivial.

2.3.1. Matrix notation

The equation system can be written in matrix notation as

$$[C] = [M][C],$$  

(7)

where the elements in $[M] \in \mathbb{C}^{3(N-1) \times (K-1)}$ are functions of the search coil radii, the shaft positions, and the binomial coefficients stemming from the analytic continuation. The vector $[C] \in \mathbb{C}^{K-1}$ contains the measured field harmonics; $[C] \in \mathbb{C}^{K-1}$ contains the multipoles in the reference frame, which are the unknown values to be computed. The column vectors are given by

$$[C] = [C_2(z_0), C_3(z_0), ..., C_N(z_0), C_2(z_1), ..., C_N(z_N), ...]^T.$$  

(8)

$$[C] = [C_2, C_3, ..., C_K]^T.$$  

(9)

The block matrix $[M]$ is composed of three inner matrices $[W]$, one for
each measurement position $i \in [0, 1, 2]$:

$$[M] = \begin{bmatrix} [W_1] \\ [W_2] \\ [W_3] \end{bmatrix}$$

These inner matrices are upper trapezoidal matrices of the type

$$[W_i] = \begin{bmatrix} w_{i1}^{(0)} & \cdots & w_{i1\,N}^{(0)} \\ 0 & \cdots & w_{i2\,N}^{(0)} \\ \vdots & \ddots & \vdots \\ 0 & 0 & w_{iN\,N}^{(0)} \end{bmatrix}$$

where the central matrix $[W_0]$ is the identity matrix (for $z_0 = 0$). The matrix elements $w_{ik}^{(0)}$ of $[W]$ depend on all indices $n, k$, and $i$:

$$w_{ik}^{(0)} = \left( \frac{k - 1}{k - n} \right) \left( \frac{z_i}{R_i} \right)^{k-n}.$$  \hspace{1cm} (11)

The equation system described by the matrix $[M]$ must be over-determined such that $3(N - 1)$ is larger than $K - 1$.

### 2.3.2. Method of least squares

The method of least squares is applied to find a solution of the over-determined equation system, at which the sum of the square errors takes its minimum. The implementation on the computer is straightforward for different programming environments and platforms. The computations in the following analysis is done by a Python program using the numpy library [9].

The underlying assumption of the method is that the field is sufficiently well approximated by a finite number of coefficients with a limit for $K$ set to 15. Multipoles of higher order are not considered because they are prone to instabilities when they are extracted from magnetic measurements.

Besides alignment errors and additional terms in curved magnets, the odd-numbered multipoles are usually dominant in dipole magnets and are thus called allowed field errors. The multipoles of a single measurement should therefore be considered up to an odd order, which holds more significant information relative to the noise floor. In the following analysis the corresponding parameter $N$ is set to 9. The used indices are summarized in Table 1.

### 2.4. Measurement uncertainties

A single magnetic measurement is subject to several sources of errors, such as mechanical vibrations, coil-calibration errors, and noise in the readout electronics. Due to the signal compensation from two search coils within one common shaft (bucking scheme), these errors can be reduced for the higher-order multipoles but not for the main field component, which is thus about two orders of magnitude less precise than the higher-order multipoles [6]. Therefore, the algorithm for the field synthesis must avoid the main component for highest precision. Moreover, the signal-to-noise ratio decreases with the multipole order as the coefficients become smaller, but the noise floor remains about constant. Fig. 2 shows the higher-order multipoles, which are indistinguishable from the noise floor for the multipole order larger than 9.

As an example for the distribution of the uncertainty on the multipoles, Fig. 3 shows the sample standard deviation for an exemplary measurement by a tangential coil. The uncertainty is highest for the main field component and for the multipoles around the order $n=12$, where the sensitivity of the coil is very low. In between, for the multipoles of order $n \in [2, 9]$, the uncertainty is about constant.

The noise affects the field description inside and outside the coil radius. Outside, the multipoles are scaled with a large number $(z/R)^{n-1}$, $|z| > R$, so that the effect of the higher-order multipoles is visible in Fig. 4.

### 2.5. Positioning uncertainties

Depending on the positioning system and the environment, systematic and random errors are introduced. The positioning and alignment of the magnet and search coil is usually precise (20–100 μm) for measurements at room temperature. In this case the position can be controlled mechanically with precision stages or optically with a laser tracker. However, larger positioning errors are expected when the search coil is not accessible. This may be the case if the coil is immersed in the bath of a vertical cryostat or in the cold-bore of a magnet on a horizontal test bench. Additional flanges and cold-warm transitions do not allow the tracking of the coil motion by means of an optical system. Thus, the accuracy of the position varies from some micrometers to the range of millimeters.

### Table 1

<table>
<thead>
<tr>
<th>Used variables and indices.</th>
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<tbody>
<tr>
<td>$N$</td>
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![Fig. 2. The absolute value of the higher-order multipoles of (Table 2) without noise (red, bold) and with random noise of $\sigma_n = 0.1$ added (blue). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)](image)

![Fig. 3. The sample standard deviation of a measurement by a tangential coil.](image)
3. Application

The method is applied to oblong aperture magnets such as the calibration dipole magnet of the magnetic measurement section at CERN. The magnet has an aperture of 100×80 mm² and a magnetic length of 2500 mm. Although a tight control of the position is possible in this case, a higher uncertainty on the positioning of 0.1 mm to 1 mm is assumed for the following study. As the effect of the positioning error depends on the multipole field errors present in the magnet, different field homogeneities are assumed for the central part of the magnet and its fringe-field regions. The assumed noise levels cover the range of what is typically achieved in magnetic measurements using rotating search coils.

3.1. Measurement configuration

The magnet is measured at three longitudinal positions: The two magnet extremities covering the fringe fields up to a point of longitudinal uniformity within the magnet bore and the central part of the magnet. Choosing two example cases with the good and poor (transverse) field homogeneity yields an understanding of the error propagation in the post processing of the measurement data.

The multipoles in the central part have low values, denoted by $C_n^{(c)}(ℓ)$ and are given in Table 2, left. The multipoles at the magnet extremities have much higher values, are denoted $C_n^{(h)}(ℓ)$ and given in Table 2, right. These multipoles are given for the central position at position $z_0$. The measurement positions are assumed to be $z_1 = 30$ mm, $z_0 = 0$ mm and $z_2 = 30$ mm. The position of the central measurement $z_0$ equals the position of the reconstructed multipoles. The coil radius $R_c$ is 17 mm.

As is common practice in magnetic measurements, the multipoles are normalized to the main (dipole) field component and expressed as units in $10^4$.

4. Error propagation in the field synthesis

A Python code was written to simulate a synthetic field with additional error sources by using the NumPy and SciPy libraries [10]. The simulations were done with different sources of errors: First, measurement errors (two levels) or position errors (two levels) are...
taken into account, individually. Finally the two errors sources are combined.

All imposed errors are assumed to be of Gaussian distribution with zero mean. It is assumed, that the noise amplitude over the multipole order is similar, with the two values $\sigma_n^h = 1.0 \text{m}(\ell)$ and $\sigma_n^p = 0.1 \text{m}(\ell)$. These values correspond to an absolute difference of $10^{-4}$ T and $10^{-5}$ T, respectively, for a main dipole flux density of 1 T. The positioning error is also assumed to be Gaussian with zero mean and $\sigma_p^h = 1 \text{mm}(\ell)$ and $\sigma_p^p = 0.1 \text{mm}(\ell)$ although higher precisions can be obtained with appropriate alignment stages.

As a figure of merit for the field synthesis an error norm is defined for the field on the rectangular path $\gamma$, shown in Fig. 5. The characteristic points are $y \in \{ \pm (2/3)R \}$ and $x \in \{ x_0 + (\pm \sqrt{3}/3)R, x_0 - (\pm \sqrt{3}/3)R \}$, so that the corners of the path lie on the circles. The long sides are sampled 100 times, the short ones 50 times. The error $\xi$ is defined as the sum over the errors along that path $\gamma$ divided by the number of samples and normalized to the central field at $z_0$:

$$\xi = \frac{1}{300} \sum_{i=1}^{300} \left| B_{\text{rec}}(y_i) - B_R(y_i) \right|.$$  

This error norm is computed 250 times for the Gaussian input data so that a mean and standard deviation can be computed.

### 5. Results

A field synthesis without additional noise gives results that are accurate within machine precision, as it should be.

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$\sigma_n^h$ & $\sigma_n^p$ & \\
\hline
$C_n^h$ & $0.27 \pm 0.11$ & $2.79 \pm 0.97$ \\
$C_n^p$ & $0.01 \pm 0.01$ & $0.14 \pm 0.05$ \\
\hline
\end{tabular}
\caption{Error norm multiplied by 10000 over the path $y$ for different configurations: $\xi \times 10^4$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$\sigma_p^h$ & $\sigma_p^p$ & \\
\hline
$C_n^h$ & $2.49 \pm 0.61$ & $2.54 \pm 0.65$ \\
$C_n^p$ & $0.25 \pm 0.06$ & $0.29 \pm 0.07$ \\
\hline
\end{tabular}
\caption{Error $\xi \times 10^4$ for the low multipoles $C_n^p$ in the left table, and the high multipoles in the right table.}
\end{table}

Fig. 7. The relative error of the field affected by noise to the original one: $B_{\text{rec}} = B_{\text{rec}} - B_R(\gamma, z_0)$. The graphs are given for $C_n^h$ and the error sum $\xi$ is $0.27 \times 10^{-4}$ and $2.51 \times 10^{-4}$ for left and right plots, respectively.

Fig. 8. The difference of the reconstruction and the original field relative to the field in the center $B_{\text{cen}} = B_{\text{cen}} - B_R(\gamma, z_0)$ for $C_n^h$. For the right figure, the positioning error ($\Delta z = \Delta x + \Delta y$) of the left coil is $\Delta z_1 = (1.1 - 0.05) \text{mm}$ and $\Delta z_2 = -(1.21 + 0.28) \text{mm}$ for the right coil.

Fig. 9. The error norm $\xi$ as a function of the positioning error $\sigma_p$. The green (upper) graph is made for the high multipole values $C_n^h$ and the blue one for the low $C_n^p$. The red rhombus mark $\sigma_p^h$ and $\sigma_p^p$. 
Table 3 shows the error norm $\xi$ for the two noise levels $\sigma_m$. Fig. 6 shows the linear dependency (standard deviation shown as error bars) for the large multipoles $C_n^{(h)}$ in the magnet extremity. For a constant noise floor, the error $\xi$ does not depend on the strength of the multipoles.

Fig. 7 shows the difference of the reference field and the reconstructed one related to the central field. The figures are given for the case with the high multipoles $C_n^{(h)}$.

The effect of a positioning error is first studied by assuming zero noise on the measurements. The positioning error $\sigma_p$ is set to $\sigma_p^{(h)} = 1.0 \text{ mm}$ and $\sigma_p^{(l)} = 0.1 \text{ mm}$. Both values are relatively high and representative for an application in a cryostat or in a difficult measurement environment. Again, both values for the multipoles are used and shown in Table 4.

There is a significant difference for the two levels of multipole field errors. The high multipoles $C_n^{(h)}$ lead to an error norm $\xi$, which is about 20 times higher than for the low multipoles. This can be explained by the fact, that as the field variation gets larger, the sensitivity gets higher for the same displacement error. For both sets of multipoles, the error norm $\xi$ depends linearly on the positioning error $\sigma_p$ as seen in Table 4.

Fig. 8 shows the distribution of the absolute field error for the field synthesis in the magnet extremity (high multipole values). The influence of the positioning error $\sigma_p^{(h)}$ is smaller than $10^{-4}$ relative to the field in the center. However, $\sigma_p^{(h)}$ (1 mm) results in relative field errors of $10 \times 10^{-4}$ (Fig. 9).

5.1. Comparison of the error sources

Considering both error sources together, i.e, the positioning errors and measurement noise reveals the underlying principle of the error propagation: The noise on the measurement manifest itself as an additive term on the multipoles $C_n^{(l)} = C_n + \Delta C_n$, while the positioning error is characterized by the feed-down formula $\tilde{C}_n = \sum_{k=1}^{l-1} C_{1-k}^{(l)} (\Delta/\omega)^{1-k}$, which depends on the strength of the multipole field errors themselves.

The eight possible combinations of noise and positioning errors are given in Table 5. The low multipoles are printed in the left table, where the positioning errors on the top are linked to the measurement errors on the side. The corresponding field deviation in the magnet aperture is shown in Fig. 10.

For the low multipoles the total error is dominated by the noise on the measurements, while for magnets with a larger field error the uncertainty on the field reconstruction is dominated by the positioning error of the individual measurements.

Fig. 11, left, shows the multipoles of 30 combinations affected by low error sources from noise and displacement. In comparison to Fig. 2, the noise floor is not visible and thus the reconstructed multipoles are closer to the reference values. The higher accuracy obtained for the higher-order multipoles allows the extrapolation of the field valid for a larger domain.

The higher precision is shown in Fig. 11, right, where the sample
standard deviation $S$ of the combination and a single measurement is plotted. The single measurement (green), affected by a measurement error of $\sigma_m(\ell)$, shows the noise floor for the higher-order field harmonics. The combination, however, shows a large improvement of the precision for the higher-order multipoles. These simulations were done with $\sigma_m(\ell)$ and $\sigma_p(\ell)$.

6. Conclusion

The combination of the three measurements yields a field synthesis in oblong magnet apertures, which is more accurate than the extrapolation from a single measurement. The accuracy of the field representation by the multipole coefficients is not only better in the outer domain (not covered by the single measurement), but also within the area covered by the central coil.

The proposed field synthesis is affected by the positioning errors of the single measurements. This error scales with the field errors present in the magnet and is therefore largest for measurements in the magnet extremity. For measurements of normal-conducting magnets at room temperatures, it is however, not technically difficult to obtain a precise alignment of the measurement system with respect to the magnet.

References