LAPLACE TRANSFORM

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1. **DEFINITION AND GENERAL PROPERTIES OF THE LAPLACE TRANSFORM**

1.1 **Definition**

Integral transforms have been known in mathematics for a long time. These transforms relate a certain function \( F(x) \) to another function \( f(y) \) by means of a definite integral

\[
\int_{A}^{B} K(x, y) \, F(x) \, dx.
\]

(1.1)

\( F(x) \) and \( f(y) \) are defined in certain regions, \( A \) and \( B \) are constants, often 0 or \( \pm \infty \). Depending on the choice of \( K(x, y) \), these transforms are known, amongst others, as Fourier transforms, Hankel transforms, Hilbert transforms, Mellin transforms, or Laplace transforms.

It can be shown that there exists under certain assumptions another connection between the two functions \( F(x) \) and \( f(y) \) related by Eq. (1.1), namely

\[
\int_{a}^{b} K(x, y) \, f(y) \, dy.
\]

(1.2)

which is called the inverse of the integral transform.

Some of the transforms mentioned are connected to each other in certain ways, and relations may be established between them. We use this fact for the introduction of the Laplace transform.

We start with the widely known expansion of a function into a Fourier series. Let \( F(x) \) be a real function defined in the finite interval \(-\pi \leq x \leq +\pi\). It can be proved that this function can be expressed by a convergent infinite series, the Fourier series

\[
F(x) = \sum_{n=-\infty}^{\infty} \left( a_n \cos nx + b_n \sin nx \right),
\]

(1.3)

provided \( F(x) \) satisfies certain very general conditions, e.g. if \( F(x) \) is composed of a finite number of continuous and monotonous pieces. Since
there are simple relations between the trigonometric functions sine and cosine and the complex exponential function, Eq. (1.3) may be written in complex form

\[ F(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{jn x} \]  

(1.4)

with

\[ c_n = \overline{a_n} + j \overline{b_n}, \quad c_n = \overline{a_n} - j \overline{b_n} \]

and \( j = \sqrt{-1} \).

The factor \( 1/(2\pi) \) in Eq. (1.4) is arbitrary and is introduced for consistency with formulae derived later, where this factor has historical reasons. It is then known from the theory of Fourier series that the Fourier coefficients \( c_n \) are given by

\[ c_n = \overline{a_n} + j \overline{b_n} \]

(1.5)

Since \( c_n \) is generally a complex quantity, we may write

\[ c_n = r_n e^{-jn \varphi_n} \quad (\varphi_n \geq 0) \]  

(1.6)

and obtain from Eq. (1.4)

\[ F(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r_n e^{jn(x - \varphi_n)} \]  

(1.7)

We can explain the last formula in the following way: the right-hand side is a decomposition of \( F(x) \) into harmonic oscillations, with amplitudes \( r_n \) and phases \( \varphi_n \), where the latter two quantities are given by the Fourier coefficient \( c_n \). These coefficients \( c_n \) define the spectrum of \( F(x) \); this means that they indicate which of the harmonic oscillations appear and what is their amplitude and phase. This gives the following result: from a given \( F(x) \) we obtain the spectral coefficients \( c_n \) by means of Eq. (1.5). The knowledge of these coefficients can replace completely the knowledge of \( F(x) \).
We now try to generalize our problem as follows. We consider a variable, e.g. the time $t$, which is not only defined in a finite interval like $x$, but in the infinite interval $-\infty < t < +\infty$. In this case the summation in Eq. (1.4) needs to be replaced by an integration, and one obtains

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{jyt} \, dy.$$  \hspace{1cm} (1.8)

$y$, which replaces $n$, is now a continuous variable, whereas $n$ was a discrete variable. Therefore, the coefficients $c_n$ have been replaced by a function $f(y)$.

Generalizing Eq. (1.5) we obtain for $f(y)$ the relation

$$f(y) = \int_{-\infty}^{\infty} F(t) e^{-jyt} \, dt.$$  \hspace{1cm} (1.9)

The equations (1.8) and (1.9) are of the type (1.2) and (1.1), defining an integral transform and its inverse. In fact, the equations (1.8) and (1.9) define the Fourier transform.*

Formula (1.8) can be interpreted as follows: $F(t)$ cannot be constructed in the infinite interval $-\infty < t < \infty$ from harmonic oscillations only. On the contrary, oscillations of all frequencies are necessary. The spectrum of $F(t)$ is now given by a function of a continuous variable $f(y)$, defined for $-\infty < y < +\infty$. This function may replace $F(t)$ completely, since $F(t)$ can be reconstructed by Eq. (1.8).

The formal similarity of Eqs. (1.8) and (1.9) for infinite intervals with Eqs. (1.4) and (1.5) for finite intervals is very satisfactory. There arises nevertheless an important difficulty in connection with the infinite interval. The integral in Eq. (1.5) always exists [for integrable functions $F(x)$], whereas the integral of Eq. (1.9) becomes meaningless for functions $F(t)$ such that the integral diverges. This is unfortunately already the case for simple functions such as $F(t) = \text{const}$ or $F(t) = e^{j\omega t}$.

*) Some authors write $F(t) = 1/\sqrt{2\pi} \ldots$, $f(y) = 1/\sqrt{2\pi} \ldots$. 
We now try to overcome this difficulty. In a wide range of applications, especially when \( t \) is interpreted as time, negative values of \( t \) are meaningless. We can then restrict our interval to \( 0 \leq t < \infty \), and we set \( F(t) = 0 \) for \( t < 0 \). We obtain from Eq. (1.9):

\[
f(y) = \int_0^\infty e^{-\frac{y}{y^t}} F(t) \, dt,
\]

and from Eq. (1.8)

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy} \, f(y) \, dy = \begin{cases} F(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}
\]

We now introduce a function

\[
F_x(t) = e^{-x} F(t)
\]

with a parameter \( x > 0 \), and we obtain for the spectral function from Eq. (1.10):

\[
f_x(y) = \int_0^\infty e^{-\frac{y}{y^t}} F_x(t) \, dt = \int_0^\infty e^{-\frac{y}{y^t}} \left[ e^{-x} F(t) \right] \, dt.
\]

The right-hand integral now converges for a very much larger set of functions \( F(t) \) than does the original one [Eq. (1.10)]; namely, at least for all those which for \( t \to \infty \), do not grow faster than \( e^{\alpha t} (\alpha > 0) \), if \( x > \alpha \) is chosen.

For formula (1.11) we obtain with \( f_x(y) \)

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy} \, f_x(y) \, dy = \begin{cases} e^{-x} F(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}
\]

Rewriting Eqs. (1.13) and (1.14) as

\[
\int_0^\infty e^{-(x + iy) t} F(t) \, dt = f_x(y)
\]
and

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(x+jy)t} f_x(y) \, dy = \begin{cases} F(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}
\]  \hspace{1cm} (1.16)

shows that the parameter \( x \) and the variable \( y \) appear only in the combination \( x + jy \). We have therefore obtained a complex variable, which we shall call

\[ \Delta = x + jy \text{ (*)).} \]

It is then practical to write

\[ f_x(y) = f(x + jy) = f(\Delta) \hspace{1cm} (1.17) \]

When introducing this variable \( s \) into Eq. (1.16), we have to replace the limits of the integral by \( x - j\infty \) and \( x + j\infty \) and \( dy \) by \( ds/j \) (Fig. 1).

---

*) Some authors call it \( p \) or different.
We then obtain finally

$$f(t) = \int_0^\infty e^{-st} F(t) \, dt$$

(1.18)

$$\frac{1}{2\pi j} \int_{x-j\infty}^{x+j\infty} f(s) \, ds = \begin{cases} F(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

(1.19)

Formula (1.18) is now the definition of the Laplace transform. The integral at the right-hand side is called the Laplace integral. Formula (1.19) is the inverse transform. The function $f(s) = f(x + iy)$ can be interpreted as the spectral function with the frequency variable $y$ of the damped time function $e^{-xt} F(t)$.

1.2 Convergence of the Laplace transform

Since $f(s)$ is a function of a complex variable $s$, we try to find in which region of the complex $s$-plane $f(s)$ is defined; in other words, where the integral (1.18) converges. It can be shown that this is the case at least in a right-half plane $\text{Re} \, s > \beta$ if such a $\beta$ exists; that means, if the integral is not divergent at all. In addition, it can be shown that if the Laplace integral converges absolutely for a point $s = s_0$, that means if

$$\int_0^\infty |e^{-s_0 t} F(t)| \, dt$$

exists and is finite, then it converges absolutely for the right-half plane $\text{Re} \, s \geq \text{Re} \, s_0 = \alpha$. Since
\[ |f(s)| = \left| \int_0^\infty e^{-st} F(t) \, dt \right| \leq \int_0^\infty |e^{-st} F(t)| \, dt \]
\[ \leq \int_0^\infty e^{-\Re s \cdot t} |F(t)| \, dt \leq \int_0^\infty e^{-\Re s \cdot t} |F(t)| \, dt \]
\[ = \int_0^\infty e^{-st} |F(t)| \, dt \leq M, \]

it follows that \( f(s) \) is bounded in the half-plane of absolute convergence. As an example we show that for functions with the property
\[ |F(t)| < M e^{\gamma t}, \]
where \( M \) and \( \gamma \) are constants, the Laplace integral converges absolutely for \( \Re s > \gamma \). We have
\[ \int_0^\infty |e^{-st} F(t)| \, dt < M \int_0^\infty |e^{-st} e^{\gamma t}| \, dt \]
\[ \leq M \int_0^\infty e^{-(\Re s - \gamma) t} \, dt = \frac{M}{\Re s - \gamma}. \]

This is a positive finite quantity for \( \Re s > \gamma \).

It is possible that the half plane of convergence and the half plane of absolute convergence are not identical, \( \alpha \neq \beta \). It is then clear that \( \beta < \alpha \). It may happen that the Laplace integral converges for functions \( F(t) \) for which the modulus \(|F(t)|\) grows very much faster than the exponential function. There are, of course, functions where \( f(s) \) does not exist anywhere, e.g. for \( F(t) = e^{t^2} \). It is now very important that in the half plane \( \Re s > \beta \), \( f(s) \) is an analytic function of \( s \) [this means that \( f(s) \) is differentiable to arbitrary high order]. Therefore the very powerful methods of the theory of complex functions can be applied to \( f(s) \).
This is not the case for $F(t)$, which is a real and nearly arbitrary function. No generally applicable methods exist for such functions.

1.3 The Laplace transform as a mapping relation

We return to Eqs. (1.18) and (1.19) and consider the Laplace transform in a more formal way, namely as a mapping. This transform maps one element $F(t)$ of a set of functions (the original space) into one element $f(s)$ of another set (the image space). $F(t)$ is called the original function, $f(s)$ the image function.

We write this relation symbolically

$$f(s) = \mathcal{L}\{F(t)\},$$

which is similar to well-known, more simple mappings, e.g. to a function of one variable, $w = \varphi(z)$.

It is practical to write elements of the original space with capitals, and elements of the image space with lower case letters.

For the inverse transform [Eq. (1.19)] we may write

$$F(t) = \mathcal{L}^{-1}\{f(s)\}.$$

To a given $F(t)$ belongs exactly one $f(s)$. With a certain restriction (which is not important for practical purposes) we can also say that for a given $f(s)$ we have exactly one $F(t)$.

1.4 Some examples

We now try to construct the image functions $f(s)$ from some given original functions $F(t)$.

Let the original function be

$$U(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \quad (1.20)$$
We leave the definition for $t = 0$ open; we can neglect this point for the integration. $U(t)$ is called Heaviside's (unit) step function. Here the calculation of the image function $f(s)$ is very simple. We have

$$
\mathcal{L}\left\{\frac{1}{s}\right\} = \mathcal{F}\left\{e^{-st}\right\} = \left[\frac{e^{-st}}{-s}\right]_{0}^{\infty} = \frac{1}{s}
$$

if, and only if, $e^{-st} \to 0$ for $t \to \infty$, hence $\text{if } \text{Re } s > 0$.

Therefore we write

$$
\mathcal{L}\left\{U(t)\right\} = \frac{1}{s}, \quad \text{for } \text{Re } s > 0
$$

The step does not happen for $t = 0$ but for $t = a > 0$, which means $U(t-a)$. Hence

$$
\mathcal{L}\left\{U(t-a)\right\} = \mathcal{F}\left\{e^{-st}U(t-a)\right\} = \int_{0}^{\infty} e^{-st} \cdot 0 \cdot dt = \frac{e^{-as}}{s}
$$

for $\text{Re } s > 0$.

Let $F(t) = e^{at}$ (a complex), or more exactly $F(t) = e^{at}$ for $t > 0$, $F(t) = 0$ for $t < 0$, or $F(t) = U(t)e^{at}$). We get

$$
\mathcal{L}\left\{e^{at}\right\} = \int_{0}^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}
$$

for $\text{Re } s > \text{Re } a$.

The original function is $\cos kt$. We can write

$$
\mathcal{L}\left\{\cos kt\right\} = \int_{0}^{\infty} e^{-st} \cos kt \, dt
$$

and solve this integral by partial integration. It is simpler, however, to remember that

*) We know that we always have $F(t) = 0$ for $t < 0$.  


\[ \cos k e^{t} = \frac{1}{2} \left( e^{i k e^{t}} + e^{-i k e^{t}} \right) . \]

Therefore
\[ \mathcal{L} \left\{ \cos k e^{t} \right\} = \mathcal{L} \left\{ \frac{1}{2} \left( e^{i k e^{t}} + e^{-i k e^{t}} \right) \right\} \]
\[ = \frac{1}{2} \left( \frac{1}{s-i k} + \frac{1}{s+i k} \right) = \frac{s}{s^2 + k^2} \]  \hspace{1cm} (1.26)

if \( \text{Re } s > \text{Re } (jk) \) and \( \text{Re } s > \text{Re } (-jk) = \text{Im } k \), hence if \( \text{Re } s > |\text{Im } k| \).

Here we have used the obvious relation
\[ \mathcal{L} \left\{ \lambda_1 F_1 (t) + \lambda_2 F_2 (t) \right\} = \lambda_1 \mathcal{L} \left\{ F_1 (t) \right\} + \lambda_2 \mathcal{L} \left\{ F_2 (t) \right\} \]  \hspace{1cm} (1.27)

Similarly we obtain
\[ \mathcal{L} \left\{ \sin k e^{t} \right\} = \frac{ke}{s^2 + k^2} \]  \hspace{1cm} (1.28)

for \( \text{Re } s > |\text{Im } k| \).

Since
\[ \cosh k e^{t} = \cos i k e^{t} \]
and
\[ \sinh k e^{t} = -i \sin i k e^{t} , \]
we obtain from Eqs. (1.26) and (1.28) for the Laplace transform of the hyperbolic functions
\[ \mathcal{L} \left\{ \cosh k e^{t} \right\} = \frac{1}{s^2 - k^2} \]  \hspace{1cm} (1.29)

and
\[ \mathcal{L} \left\{ \sinh k e^{t} \right\} = \frac{ke}{s^2 - k^2} \]  \hspace{1cm} (1.30)

for \( \text{Re } s > |\text{Re } k| \).

For \( F(t) = t^\alpha \) with real \( \alpha \) we have
\[ \mathcal{L} \left\{ t^\alpha \right\} = \int_0^\infty e^{-st} t^\alpha \ dt \]  \hspace{1cm} (1.31)

*) We take \( t^\alpha \) positive for \( t > 0 \).
Since the integrand is not integrable at the origin for $\alpha \leq -1$, we have to assume $\alpha > -1$. We make the substitution $st = \tau$, $dt = d\tau/s$, and get for real $s > 0$

$$\mathcal{L}\left\{ t^\alpha \right\} = \int_0^\infty -\tau^{\alpha/s} d\tau/s = \frac{1}{s^{\alpha+1}} \int_0^\infty -\tau^{\alpha} d\tau \quad (1.32)$$

The right-hand integral cannot be expressed by a finite number of elementary functions. It is known as the gamma function

$$\Gamma(z) = \int_0^\infty e^{-\tau} \tau^{z-1} d\tau \quad (1.33)$$

With this function we have

$$\mathcal{L}\left\{ t^\alpha \right\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}} \quad (1.34)$$

A detailed discussion shows that this formula is true for

$$\text{Re}\ s > 0 \quad \text{if } \alpha > 0; \quad \text{Re}\ s > 0, \ s \neq 0 \quad \text{if } -1 < \alpha < 0.$$  

The gamma function is a generalization of the well-known factorial function

$$n! = 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n$$

defined for entire positive $n$. In particular, we have

$$\Gamma(n+1) = n!,$$

and from $n! = n(n-1)!$ the generalization

$$\Gamma(z+1) = z \Gamma(z) \quad (1.35)$$

So we can write for $\alpha = n \geq 0$

$$\mathcal{L}\left\{ t^n \right\} = \frac{n!}{s^{n+1}} \quad (1.36)$$
The straightforward integration of the Laplace integral for a given function $F(t)$ is very difficult in most cases. Fortunately, there exist several very good tables where pairs of functions $F(t)$ and $f(s)$ are tabulated, as in a dictionary for two languages: the origin language and the image language. We give here an example of such a very short table, containing the functions discussed above and the Bessel function $J_0(t)$. The regions of convergence are nearly always omitted in the published tables.

<table>
<thead>
<tr>
<th>$F(t)$</th>
<th>$f(s)$</th>
<th>Convergence region</th>
</tr>
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<tbody>
<tr>
<td>$U(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\text{Re } s &gt; 0$</td>
</tr>
<tr>
<td>$U(t-a)$ ($a &gt; 0$)</td>
<td>$\frac{e^{-as}}{s}$</td>
<td>$\text{Re } s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
<td>$\text{Re } s &gt; \text{Re } a$</td>
</tr>
<tr>
<td>$\cosh kt$</td>
<td>$\frac{s}{s^2-k^2}$</td>
<td>$\text{Re } s &gt;</td>
</tr>
<tr>
<td>$\sinh kt$</td>
<td>$\frac{k}{s^2-k^2}$</td>
<td>$\text{Re } s &gt;</td>
</tr>
<tr>
<td>$\cos kt$</td>
<td>$\frac{s}{s^2+k^2}$</td>
<td>$\text{Re } s &gt;</td>
</tr>
<tr>
<td>$\sin kt$</td>
<td>$\frac{k}{s^2+k^2}$</td>
<td>$\text{Re } s &gt;</td>
</tr>
</tbody>
</table>
| $t^\alpha$ ($\alpha > -1$) | $\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ | $\text{Re } s > 0 \text{ for } \alpha \geq 0$
|               |                 | $\text{Re } s \geq 0, s \neq 0 \text{ for } -1 < \alpha < 0$ |
| $J_0(t)$     | $\frac{1}{\sqrt{s^2+1}}$ | $\text{Re } s > 0$                     |

It is rather surprising to see from this table that the Laplace integral sometimes transforms a transcendent function, such as $e^{at}$, $\sin kt$, $J_0(t)$, etc., into a very elementary function such as $1/(s-a)$, $k/(s^2+k^2)$, $1/\sqrt{s^2+1}$, etc.
2. THE LAPLACE TRANSFORM OF OPERATIONS

2.1 Rules for the transformation calculus

We have seen in the last paragraph that tables can be constructed, in the same way as dictionaries, for \( F(t) \) and \( f(s) \). As in any language, such a dictionary on its own would be very unsatisfactory when we try to handle a problem successfully and properly. Obviously, a number of working rules (such as grammar rules) are necessary for the practical use of the tables. A very elementary and obvious rule was already given in formula (1.27) of the previous section.

In this section we derive rules for the Laplace transform of operations, and start as follows:

2.1.1 Linear substitution

Rule I (theorem of similarity)

We multiply the variable \( t \) of \( F(t) \) with the factor \( a > 0 \) and calculate

\[
\mathcal{L}\{F(at)\} = \int_{0}^{\infty} e^{-st} F(at) \, dt = \int_{0}^{\infty} e^{-\frac{s}{a} \tau} F(\tau) \, d\tau / a
\]

\[
= \frac{1}{a} \int_{0}^{\infty} e^{-\left(\frac{s}{a}\right) \tau} F(\tau) \, d\tau = \frac{1}{a} \mathcal{L}\{F(\frac{\tau}{a})\}
\]

When we replace \( a \) by \( 1/b \), we get a similar relation

\[
\mathcal{L}\{F(at)\} = \frac{1}{a} \mathcal{L}\{F(\frac{\tau}{a})\} \quad (a > 0)
\]

\[
\mathcal{L}^{-1}\{F(\frac{t}{b})\} = \frac{1}{b} F\left(\frac{t}{b}\right) \quad (b > 0)
\]

We apply this theorem to an example. If, for instance, we knew only that [(1.26) for \( k = 1 \)]

\[
\mathcal{L}\{e^{\cos t}\} = \frac{s}{s^2 + 1}
\]
then for real \( a > 0 \) we could use (I) to write

\[
\mathcal{L}\left\{ \cos at \right\} = \frac{1}{a} \frac{s/a}{(s/a)^2 + 1} = \frac{s}{s^2 + a^2}.
\]

This is true, as we know from Eq. (1.26), for all complex \( a \). The restriction \( a > 0 \) in (I) is nevertheless generally necessary. We have, for instance, for the Bessel function \( J_0(t) \)

\[
\mathcal{L}\left\{ J_0(t) \right\} = \frac{1}{\sqrt{s^2 + 1}}
\]

(see Table 1). Applying (I) with \( a = -1 \) would result in

\[
\mathcal{L}\left\{ J_0(-t) \right\} = -1/\sqrt{s^2 + 1},
\]

which is wrong, because one knows that \( J_0(t) \) is an even function and the Laplace transform is unique.

**Rule II** (first shifting theorem)

We calculate the Laplace transform for \( F(t - a) \) with \( a > 0 \). Here it is very important to remember that \( F(t') \) must vanish for \( t' < 0 \), hence \( F(t-a) = 0 \) for \( t < a \). We have (Fig. 3a)

\[
\mathcal{L}\left\{ F(t-a) \right\} = \int_a^\infty e^{-st} F(t-a) \, dt = \int_a^\infty e^{-st} F(t-a) \, dt
\]

\[
= \int_0^\infty e^{-s(t+a)} F(t) \, dt = e^{-sa} \mathcal{L}\{F(t)\}
\]

or

\[
\mathcal{L}\{F(t-a)\} = e^{-as} \mathcal{L}\{F(t)\} \quad (a > 0)
\]

This rule applies especially in the so-called "dead-time" problem; that is, if a procedure starts with a certain delay. When we use this
formula from right to left, we must again take into account the fact that 
$P(t') = 0$ for $t' < 0$, e.g. the image function

$$f_1(n) = e^{-\sqrt{2 \pi} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2 \pi}}}$$

(2.1)

gives the original function

$$F_1(t) = \begin{cases} 
0 & \text{for } 0 < t \leq 1 \\
\sin(t-1) & \text{for } t > 1 
\end{cases}$$

(2.2)

and not $\sin(t-1)$ alone for $t > 0$.

As an example for the first shifting theorem, we calculate the Laplace transform of

$$F(t) = \begin{cases} 
\sin t & \text{for } 0 \leq t \leq 2\pi \\
0 & \text{for } t > 2\pi 
\end{cases}$$

(2.3)

This could be done directly by writing

$$f(s) = \int_0^{2\pi} e^{-st} \sin t \, dt$$

and integrating it by parts. We go another way and observe that $F(t)$
can be written as a difference $F_1(t) - F_2(t)$ of the functions (Fig. 2)

![Fig. 2](image-url)
Thus

\[
\mathcal{L}\{\dot{F}(t)\} = \mathcal{L}\{F_1(t)\} - \mathcal{L}\{F_2(t)\}
\]

\[
= \mathcal{L}\{\sin t\} - \mathcal{L}\{\sin(t-2\pi)\}
\]

\[
= \mathcal{L}\{\sin t\}(1 - e^{-2\pi s})
\]

\[
= \frac{(1 - e^{-2\pi s})}{s^2 + 1}
\]

This procedure may be explained in general as follows. If the Laplace transform \(p(s)\) of a periodic function \(P(t)\) is known, then the Laplace transform \(p_\omega(s)\) of the first period [this means of the function \(P_\omega(t) = P(t)\) for \(0 \leq t < \omega\) and 0 elsewhere] is given by

\[
p_\omega(s) = p(s) \left(1 - e^{-\omega s}\right)
\]

This can also be written as

\[
\mathcal{L}\{P(t)\} = \frac{A}{1 - e^{-\omega s}} \int_0^\omega e^{-st} P(t) dt
\]

The Laplace transform of a periodic function is therefore determined by the "finite" Laplace transform of \(P(t)\) over the first interval.

**Rule III (second shifting theorem)**

Here we try to find the Laplace transform of \(F(t+a)\). This is slightly more complicated than in the previous cases. We have (Fig. 3b)
\[ \mathcal{L}\{\mathcal{F}(t+a)\} = \int_0^{\infty} e^{-st} \mathcal{F}(t+a) \, dt = \int_0^{\infty} e^{-s(\tau-a)} \mathcal{F}(\tau) \, d\tau \]

\[ = \int_0^{\infty} e^{-s(\tau-a)} \mathcal{F}(\tau) \, d\tau - \int_0^{a} e^{-s(\tau-a)} \mathcal{F}(\tau) \, d\tau \]

hence

\[ \mathcal{L}\{\mathcal{F}(t+a)\} = e^{as} \left( \mathcal{F}(a) - \int_0^{a} e^{-st} \mathcal{F}(t) \, dt \right) \quad (a > 0) \]

(III)

Fig. 3
It is plausible from Fig. 3b that the Laplace transform of $F(t+a)$
cannot depend only on $F(t+a)$ for $t > 0$. Therefore one has, in addition,
a "finite" Laplace integral on the right-hand side. This theorem is
especially important for difference equations, where values $F(t)$, $F(t+a)$,
$F(t+2a)$, etc., are given.

Rules I, II, III can be combined and written in the following way

$$\mathcal{L}\{F(a+t-b)\} = \frac{1}{a} e^{-b/a} \mathcal{L}\{f(\frac{t-b}{a})\} \quad (2.8)$$

$$\mathcal{L}\{F(at+b)\} = \frac{1}{a} e^{b/a} \mathcal{L}\{f(\frac{t-b}{a}) - \int_0^t e^{-\frac{t}{a}} F(t) \, dt\} \quad (a > 0, b > 0)$$

Rule IV (damping theorem)

We transform $e^{-\mu t} F(t)$ and get

$$\mathcal{L}\{e^{-\mu t} F(t)\} = \int_0^\infty e^{-(\sigma+\mu) t} F(t) \, dt = \mathcal{L}\{f(\sigma+\mu)\}$$

IV)

$$\mathcal{L}\{e^{-\mu t} F(t)\} = \mathcal{L}\{f(\sigma+\mu)\}$$

$\mu$ may be an arbitrary complex number. However, a real damping of $F(t)$
occurs only for $\mu > 0$.

2.1.2 Differentiation

We first discuss the differentiation in the original space. It can
be shown that $F(t)$ has an image function if $F'(t)$ has one. The contrary
is not always true; for example, $\mathcal{L}[\log t]$ exists but $\mathcal{L}[1/t]$ does not.
Therefore we suppose in the following that the highest appearing

derivative has an image function.

The Laplace transform of $F'(t) = dF(t)/dt$ can be calculated by
partial integration

$$\mathcal{L}\{F'(t)\} = \mathcal{L}\{dF(t)/dt\} = \mathcal{L}\{F(t)\} - \mathcal{L}\{F(0)\}$$

(2.9)
\[ \mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t) \, dt = \left[e^{-st} F(t)\right]_0^\infty + \delta \int_0^\infty e^{-st} F(t) \, dt \tag{2.9} \]

\[ = \delta \varphi(\delta) - F(+0) \]

This formula shows two very important facts. a) The differentiation, a complicated infinitesimal process in the original space, is replaced by very elementary operations, a multiplication and an addition in the image space. b) The constant "initial" value \( F(+0) \), that is, the limit

\[ \lim_{t \to 0} F(t) = F(+0) \]

appears in the image function. This looks odd and clumsy at first glance, but it will come out as one of the most important advantages when using the Laplace transform for solving differential equations.

In a way similar to the one above, we can derive the image functions for \( F''(t) \), \( F'''(t) \), ..., and obtain the following:

**Rule V**

\[ \mathcal{L}\{F'(t)\} = \delta \varphi(\delta) - F(+0) \]

\[ \mathcal{L}\{F''(t)\} = \delta^2 \varphi(\delta) - F(+0) \delta - F'(+0) \]

\[ \mathcal{L}\{F'''(t)\} = \delta^3 \varphi(\delta) - F(+0) \delta^2 - F''(+0) \delta - F'(+0) \]

\[ \vdots \]

\[ \mathcal{L}\{F^{(m)}(t)\} = \delta^m \varphi(\delta) - F(+0) \delta^{m-1} - F''(+0) \delta^{m-2} - \cdots - F^{(m-2)}(+0) \delta - F^{(m-1)}(+0) \]
This rule is one of the most important rules for practical applications. It should be noted that the values $F(+0)$, $F'(+0)$, ... are limit values. The functions $F(t)$, $F'(t)$, ... may not even be declared for $t = 0$. The unit step function $U(t)$, for instance, is not defined for $t = 0$ and can take any value between 0 and 1 \(^*\)), but we have for $t > 0$

$$\mathcal{L}\{U(t)\} = \frac{1}{s} - \mathcal{L}\{U(+0)\} = \mathcal{L}\{A\} = 0$$

(2.10)

With a value $U(0) \neq 1$, this equation would be wrong.

Rule V may give wrong results if $F(t)$ is not differentiable for $t > 0$. It is not necessary that the derivative of $F(t)$ exists for $t = 0$, e.g. $F(t) = t^{1/2}$ has no derivative for $t = 0$, but (V) comes out right. With Eq. (1.34) we have

$$\mathcal{L}\{F(t)\} = \frac{1}{\sqrt{s}} - \frac{1}{s} = \mathcal{L}\{F'(-0)\} = \frac{1}{s} \frac{\Gamma(1/2)}{\sqrt{s}}$$

$$F(+0) = 0$$

therefore

$$\frac{1}{2} \frac{\Gamma(1/2)}{\sqrt{s}} = \mathcal{L}\{F(t)\} = \frac{\Gamma(3/2)}{\sqrt{s}}$$

(2.11)

This is correct because \(\Gamma(1/2) = 1/2 \Gamma(1/2)\).

We now establish the rule for differentiation in the image space. We have already noted in Section 1.2 that $f(s)$ is differentiable to arbitrary high order inside its half-plane of convergence. The derivative $f'(s) = df(s)/ds$ can be obtained by differentiation under the integral sign

$$f'(s) = \int_0^\infty \frac{d}{ds} e^{-s^t} F(t) \, dt = -\int_0^\infty e^{-s^t} t F(t) \, dt$$

$$= \mathcal{L}\{t F(t)\}$$

\(^*\) It is reasonable to define $U(0) = 1/2$. 

This can easily be generalized to:

**Rule VI**

\[
\mathcal{L}\{ -t F(t) \} = \frac{d}{ds} F(s)
\]

\[
\mathcal{L}\{ t F(t) \} = F'(s)
\]

\[
\mathcal{L}\{ t^2 F(t) \} = F''(s)
\]

\[
\mathcal{L}\{ (-t)^m F(t) \} = \frac{d^m}{ds^m} F(s)
\]

Here again, a complicated operation in one space is reduced to an elementary operation in the other.

2.1.3 **Integration**

The integration from 0 to t of an original function F(t) corresponds to the division by s of the image function f(s). We have by partial integration

\[
\mathcal{L}\{ \int_0^t F(\tau) d\tau \} = \int_0^\infty \frac{1}{s} e^{-st} \left( \int_0^t F(\tau) d\tau \right) \, dt =
\]

\[
= \frac{1}{s} \left[ -\frac{1}{h} e^{-st} \int_0^t F(\tau) d\tau \right]_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} F(t) \, dt
\]

Therefore,

**Rule VII**

\[
\mathcal{L}\{ \int_0^t F(\tau) d\tau \} = \frac{1}{s} \frac{d}{ds} F(s)
\]

under the assumption that F(t) has an image function.
Integration in the image space can be treated by following Rule VIII, which is without great practical importance. We give it for the reason of completeness.

Rule VIII

\[
\mathcal{L}\left\{ \frac{F(t)}{t} \right\} = \int_0^\infty f(\sigma)\,d\sigma
\]

(VIII)

Again the integration corresponds to a division.

2.1.4 Multiplication and convolution

We now treat a subject that is very important with respect to the applications. We have already seen that a very simple formula (1.27) holds for the Laplace transform of a sum of functions. The other familiar combination, the product, is much more complicated to handle. We first discuss the product in the image space, which is more important for the applications than is the product in the original space.

A certain combination of the two functions \( F_1(t) \) and \( F_2(t) \) under the integration sign is well known in physics and other applications. This combination

\[
F_1(t) \ast F_2(t) = \int_0^t F_1(\tau)F_2(t-\tau)\,d\tau \quad (2.12)
\]

is known as "convolution" or "faltung". We now try to calculate its Laplace transform

\[
\mathcal{L}\left\{ F_1 \ast F_2 \right\} = \int_0^\infty e^{-st} \left( \int_0^t F_1(\tau)F_2(t-\tau)\,d\tau \right)\,dt \quad (2.13)
\]

Under certain assumptions we can exchange the order of integration and write (Fig. 4)
\[L \{ f_1(t) \ast f_2(t) \} =\]
\[= \int_{t=0}^{\infty} \int_{\tau=0}^{t} e^{-st} F_1(\tau) F_2(t-\tau) \, d\tau \, dt\]
\[= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} F_1(\tau) F_2(t-\tau) \, dt \, d\tau\]
\[= \int_{\tau=0}^{\infty} F_1(\tau) \int_{t=\tau}^{\infty} e^{-st} F_2(t-\tau) \, dt \, d\tau\]
We use Rule II and obtain
\[
\mathcal{L}\{F_1 \ast F_2\} = \int_0^\infty F_1(\tau) \mathcal{L}\{F_2(t-\tau)\}_T d\tau
\]
\[
= \int_0^\infty e^{-s\tau} F_1(\tau) \mathcal{L}\{F_2(t)\}_T d\tau
\]
\[
= \int_0^\infty e^{-s\tau} F_1(\tau) d\tau \mathcal{L}\{F_2(t)\}_T
\]
\[
= \mathcal{L}\{F_1(t)\}_T \mathcal{L}\{F_2(t)\}_T
\]
\[
= \mathcal{F}_1(\lambda) \cdot \mathcal{F}_2(\lambda)
\]

This remarkable result gives

**Rule IX**

\[
\mathcal{L}\{F_1(t) \ast F_2(t)\} = \mathcal{F}_1(\lambda) \cdot \mathcal{F}_2(\lambda)
\]

The convolution in the original space results in the simple product in the image space. It should be noted that Rule IX is only true if we assume that the Laplace integrals of \(F_1(t)\) and \(F_2(t)\) exist, and at least one of them converges absolutely.

The convolution itself has two very practical properties, which certify the use of a symbol \(\ast\) as in a product. It can be shown that this operation is commutative, that is

\[
F_1 \ast F_2 = \int_0^T F_1(\tau) F_2(t-\tau) d\tau = \int_0^T F_1(t-u) F_2(u) du = F_2 \ast F_1
\]

and associative, that is

\[
(F_1 \ast F_2) \ast F_3 = F_1 \ast (F_2 \ast F_3)
\]
Therefore, a convolution of several functions
\[ F_1 \ast F_2 \ast F_3 \ast \cdots \ast F_n \]
is independent of the order in which the convolutions are carried out between them.

We apply Rule IX to a few mathematical problems. Application to physical problems will be discussed later.

We start with the n-times iterated integral
\[
\Phi_n(t) = \int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} F(\tau_1) \, d\tau_1 \, d\tau_2 \cdots d\tau_{n-1} \quad (2.16)
\]
This can be written as a convolution product
\[
\Phi_n(t) = F(t) \ast 1 \ast 1 \ast \cdots \ast 1 = F \ast 1^* \ast n \quad (2.17)
\]
We have then, by Rule IX and with Eq. (1.21)
\[
\mathcal{L}\{\Phi_n(t)\} = \mathcal{L}\{F\} \ast \mathcal{L}\{1\} \ast n
\]
\[
= \mathcal{L}\{F\} \cdot \left(\frac{1}{\Delta}\right)^n = \mathcal{L}\{F\} \cdot \left(\frac{1}{\Delta}\right)^n \quad (2.18)
\]
Using Eq. (1.36), we obtain
\[
\mathcal{L}\left\{\frac{F}{\Delta} \cdot \left(\frac{1}{\Delta}\right)^n\right\} = \mathcal{L}\left\{\frac{F}{\Delta}\right\} \cdot \mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\right\},
\]
and again with Rule IX
\[
\mathcal{L}\{\Phi_n(t)\} = \mathcal{L}\left\{F \ast \frac{t^{n-1}}{(n-1)!}\right\}
\]
or
\[
\Phi_n(t) = F(t) \ast \frac{t^{n-1}}{(n-1)!},
\]
and finally

\[ \Phi_n(t) = \frac{1}{(n-1)!} \int_0^t \mathcal{F}(\tau) (t-\tau)^{n-1} d\tau \]  
\[ = \frac{1}{(n-1)!} \int_0^t \tau^{n-1} \mathcal{F}(t-\tau) d\tau . \]

As a second example we derive two formulae for integrals with Bessel functions. We know that

\[ \mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{\lambda^2 + 1}} \]  
(2.19)

Therefore, with Rule IX and Eq. (1.28)

\[ \mathcal{L}\{J_0(t) \ast J_0(t)\} = \frac{1}{\lambda^2 + 1} = \mathcal{L}\{\sin t\} \]
or

\[ \int_0^t J_0(\tau) J_0(t-\tau) d\tau = \sin t \]  
(2.20)

We use Eq. (2.19) again and write

\[ \mathcal{L}\{J_0(t)^2\} = \frac{1}{(\lambda^2 + 1)^{1/2}} = \frac{1}{(\lambda + i)^{1/2}} \frac{1}{(\lambda - i)^{1/2}} \]  
(2.21)

With Rule IV and formula (1.34), we obtain

\[ \mathcal{L}\{-i\frac{1}{(\lambda + i)^{1/2}}\} = \frac{1}{\Gamma(1/2)} t^{-1/2} e^{-\alpha t} \]  
(2.22)

Therefore, with Rule IX and \( \Gamma(1/2) = \sqrt{\pi} \):
\[ J_0(t) = \frac{1}{\pi} \left( t^{-\frac{1}{2}} e^{-\frac{i}{2}t} \right) * \left( t^{-\frac{1}{2}} e^{\frac{i}{2}t} \right) \]

\[ = \frac{1}{\pi} \int_0^t t^{-\frac{1}{2}} e^{-\frac{i}{2}t} (t-\tau)^{-\frac{1}{2}} e^{\frac{i}{2}(t-\tau)} d\tau \]

\[ = \frac{1}{\pi} e^{\frac{i}{2}t} \int_0^t t^{-\frac{1}{2}} (t-\tau)^{-\frac{1}{2}} e^{-\frac{i}{2}\tau} d\tau \]

\[ = \frac{1}{\pi} e^{\frac{i}{2}t} \int_0^1 u^{-\frac{1}{2}} (1-u)^{-\frac{1}{2}} e^{-2i\pi u} du \]

\[ = \frac{1}{\pi} \int_0^1 e^{\frac{i}{2}t} (1-2u) \left[ u (1-u) \right]^{-\frac{1}{2}} du \]

We now substitute

\[ 1-2u = \nu, \quad u = \frac{1-\nu}{2}, \quad 1-u = \frac{1+\nu}{2} \]

and obtain

\[ J_0(t) = \frac{1}{\pi} \int_{-1}^1 e^{\frac{i}{2}t\nu} (1-\nu^2)^{-\frac{1}{2}} d\nu, \]

which gives, with

\[ \nu = \cos \phi, \quad 1-\nu^2 = \sin^2 \phi, \]

the Poisson integral for \( J_0(t) \)

\[ J_0(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{it\cos \phi} d\phi = \frac{1}{\pi} \int_{0}^{\pi} \cos (t \cos \phi) d\phi \]  \hspace{1cm} (2.23)
In the theory of heat transfer, the following function is very important

\[ \psi(x, t) = \frac{x}{2\sqrt{\pi}t^{3/2}} e^{-x^2/4t} \]  \hspace{1cm} (2.24)

It can be shown that (\(x\) is a parameter)

\[ \mathcal{L}\{\psi(x, t)\} = e^{-x\sqrt{s}} \]  \hspace{1cm} (x > 0)  \hspace{1cm} (2.25)

Since

\[ e^{-x_1\sqrt{s}} \cdot e^{-x_2\sqrt{s}} = e^{-(x_1 + x_2)\sqrt{s}} \]  \hspace{1cm} (x_1 > 0, x_2 > 0)  \hspace{1cm} (2.26)

we may write with Rule IX

\[ \psi(x_1, t) \ast \psi(x_2, t) = \psi(x_1 + x_2, t) \]  \hspace{1cm} (2.27)

To prove this addition theorem of the function \(\psi(x, t)\) directly is both difficult and lengthy. With Rule IX, it can be shown in one line.

One can also see in this connection that the same addition theorem holds for all original functions which have image functions \(e^{-s\gamma}(0 < \gamma < 1)\).

Furthermore, very complicated addition theorems, for original functions having image functions of the kind \(s^{-\beta} e^{-\alpha s}(\text{e.g., Bessel functions})\), can be proved in a similar way.

In some applications, the derivative of the convolution \(F_1 \ast F_2\) is needed. It can be shown that under certain conditions

\[ \frac{d}{dt} \left[ F_1(t) \ast F_2(t) \right] = F_1'(t) \ast F_2(t) + F_1(t) \ast F_2'(t) \]  \hspace{1cm} (F_1(+0) means the limit of \(F_1(t)\) if \(t\) goes through positive values to zero.

*) Here we use exceptionally a lower case letter for the original function.
For completeness we also give a Rule X, which expresses the product of two functions from the original space by complex "convolution" integrals in the image space:

$$\mathcal{L}\left\{ F_1(t) F_2(t) \right\} = \left\{ \begin{array}{ll} \frac{1}{2\pi i} \int_{x_1-j\infty}^{x_1+j\infty} f_1(\sigma) \hat{f}_2(s-\sigma) \, d\sigma \\ \frac{1}{2\pi i} \int_{x_2-j\infty}^{x_2+j\infty} f_1(\sigma) \hat{f}_2(s-\sigma) \, d\sigma \end{array} \right.$$

$s$, $x_1$, and $x_2$ must be suitably chosen to ensure convergence. This rule has only a restricted application, and we do not discuss further details here.
3. ORDINARY DIFFERENTIAL EQUATIONS

3.1 The differential equation of first order with constant coefficients

We discuss the application of the Laplace transform to the first-order differential equation with constant coefficients

\[ y'(t) + cy(t) = f(t) \]  \hspace{1cm} (3.1)

\( f(t) \) is a given function, which is often called perturbation function.

In the first section we have always applied the Laplace transform to functions. The important new idea is now to transform an equation directly from the original space to the image space

\[ \mathcal{L}\{y'(t)\} + c \mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\} \]  \hspace{1cm} (3.2)

According to Rule V, we obtain

\[ s \cdot y(s) - y(0) + cy(s) = f(s) \]  \hspace{1cm} (3.3)

This equation is called the image equation. We note two very important facts:

a) the image equation is not a differential equation, but an algebraic equation for \( y(s) \), something very much simpler;

b) this equation contains the value \( y(0) \).

The second point is very useful. The differential equation (3.1) has infinitely many solutions. In order to specify a certain solution, we have to prescribe the value of \( Y(t_0) \) for a special \( t_0 \). This \( t_0 \) can be chosen as \( t_0 = 0 \). \( Y(t_0) \) is called the "initial value". Since our solution \( Y(t) \) is defined only for \( t > 0 \), it is reasonable to define the initial value for \( Y \) as

\[ \lim_{t \to 0^+} Y(t) = Y(0) = y_0 \]

\[ t > 0 \]

*) We assume here that \( f(t) \) and \( y'(t) \) have an image function. This assumption is a preliminary one and it can be shown that it is not necessary for the final result.
The appearance of \( Y(+0) \) in Eq. (3.3) is a great advantage of the Laplace transform method compared with the classical method. There one constructs a general solution, which depends on an arbitrary constant and afterwards tries to determine the appropriate value of this constant.

We solve Eq. (3.3) and find

\[
\mathcal{L}^{-1}\{ \mathcal{L}(Y) \} = \mathcal{L}^{-1}\{ \mathcal{L}(Y(+0)) \} \frac{1}{\lambda + c},
\]

which is the image function of the solution \( Y(t) \). We now try to determine \( Y(t) \) from Eq. (3.4). One way would be to calculate the complex integral (1.19). This is rather tedious, and we will avoid this procedure whenever possible. It is more advisable to search for a suitable correspondence of two functions in the tables, perhaps after using some of the "grammar rules" discussed in Section 2.1. In the case (3.4) this procedure is extremely simple. We have for \( 1/(s + c) \) the original function \( e^{-ct} \), according to Table 1. The first term in Eq. (3.4) is a product of two image functions, hence with Rule IX we obtain

\[
Y(t) = F(t) \ast e^{-ct} + Y(+0) e^{-ct}
\]

\[
= \int_0^t F(\tau) e^{-(t-\tau)} d\tau + Y(+0) e^{-ct}
\]

\[
= e^{-ct} \left( \int_0^t F(t) e^{c\tau} d\tau + Y(+0) \right).
\]

We have obtained here directly the solution of the inhomogeneous equation (3.1). In the classical procedure, one first obtains the solution of the homogeneous equation \( Y' + cY = 0 \), and then, by "variation of the constant", the solution of the inhomogeneous one.

The procedure discussed above can be written in a brief form as follows: If the perturbation function \( F(t) \) is continuous for \( t > 0 \), perhaps with the exception of isolated points \( t = a \), where \( F(t) \) may have steps, that means where \( F(a-0) \) and \( F(a+0) \) exist and \( F(a-0) \neq F(a+0) \),
and if $|F(t)|$ is at least improperly integrable for $t = 0$, then we write after the original equation

$$\gamma'(t) + c \gamma(t) = F(t)$$

the image equation

$$s \gamma(s) - \gamma(0) + c \gamma(s) = F(s),$$

and for its solution

$$\gamma(s) = \frac{F(s)}{s + c} + \gamma(0) \frac{1}{s + c},$$

we search for the original function

$$\gamma(t) = F(t) \times e^{-c t} + \gamma(0) e^{-c t}.$$  
This is then the solution of the differential equation with the initial value $\gamma(0)$. In addition, $\gamma(t)$ is continuous for $t > 0$ and the differential equation is satisfied in each point $t > 0$ at least for $t + 0$ and $t - 0$, that is

$$\gamma'(t) + c \gamma(t) = F(t - 0), \quad \gamma'(t) + c \gamma(t) = F(t + 0).$$

We have therefore

$$\gamma'_+(t) - \gamma'_-(t) = F(t + 0) - F(t - 0),$$

in other words, $\gamma'(t)$ has the same step as $F(t)$, hence $\gamma(t)$ has a corner.

We may write the above procedure in the following way:

**Scheme**

Original space: Differential equation  
+ initial condition  
$\mathcal{L}$-Transform  
$\mathcal{L}^{-1}$-Transform  
Solution

Image space: Algebraic equation  
Solution
We note from Eq. (3.5) that the explicit knowledge of \( f(s) \) is by no means necessary. It may even happen that \( f(s) \) does not exist. It is, however, sometimes practical to determine this function (if it exists) and to translate \( f(s)/(s+c) \) directly, without using the convolution theorem.

We discuss this by means of an example. The initial value problem

\[
Y' + cy = U(t-a), \quad Y(a) = Y_0
\]

may be given. \( U(t-a) \) is the step function for \( t = a \). We have

\[
sy - Y_0 + cy = \frac{e^{-as}}{s},
\]

\[
Y = \frac{e^{-as}}{s(s+c)} + \frac{Y_0}{s+c}.
\]

Because of

\[
\frac{1}{s(s+c)} = \frac{1}{c} \left( \frac{1}{s} - \frac{1}{s+c} \right) = \mathcal{L}\left\{ \frac{1}{c} (1-e^{-ct}) \right\}
\]

we obtain, with Rule II

\[
\mathcal{L}^{-1}\left\{ \frac{e^{-as}}{s(s+c)} \right\} = \begin{cases} 
0 & (0 \leq t < a) \\
\frac{1}{c} (1-e^{-c(t-a)}) & (t \geq a), 
\end{cases}
\]

hence for the solution

\[
Y(t) = \frac{1}{c} (1-e^{-c(t-a)}) U(t-a) + Y_0 e^{-ct},
\]

which can be written explicitly as

\[
Y(t) = \begin{cases} 
Y_0 e^{-ct} & (0 \leq t < a) \\
\frac{1}{c} - \left( \frac{e^{-ac}}{c} - Y_0 \right) e^{-ct} & (t \geq a),
\end{cases}
\]

(3.7)

This function is plotted in Fig. 5 for several values of \( Y_0 \) and \( a = 1, \ c = 3/4. \)
3.2 Expansion of a rational function into partial fractions

In the next paragraph, rational functions \( g(s)/p(s) \) (\( g \) and \( p \) are polynomials) will appear as image functions. We assume that the degree of \( g(s) \) is less than the degree \( n \) of \( p(s) \). It can be shown that if \( \alpha_\mu \) is a zero of \( p(s) \) of order \( k_\mu \), then \( g(s)/p(s) \) can be written near this zero as

\[
\frac{d_\mu}{s - \alpha_\mu} + \frac{d_{\mu+1}}{(s - \alpha_\mu)^2} + \cdots + \frac{d_{\mu+k_\mu}}{(s - \alpha_\mu)^{k_\mu}}
\]  

(3.8)

This holds for all zeros \( \alpha_\mu \) and one can show that

\[
\frac{g(s)}{p(s)} = \sum_{\mu=1}^{m} \left( \frac{d_\mu}{s - \alpha_\mu} + \cdots + \frac{d_{\mu+k_\mu}}{(s - \alpha_\mu)^{k_\mu}} \right),
\]  

(3.9)

where \( m \) is the number of different zeros, hence \( n = \sum_{\mu=1}^{m} k_\mu \).

The sum on the right-hand side is called the expansion of \( g(s)/p(s) \) into partial fractions. This expansion is unique. Two important problems arise from Eq. (3.9).

a) the determination of \( \alpha_\mu \);

b) the calculation of \( d_\mu \).

The first one, the calculation of the zeros of a polynomial, is by no means trivial and may become very complicated for large \( n \). It should be noted, however, that the problem arises not only when using the Laplace transform, but also when solving the \( n \)th order differential equation by the classical method. Fortunately, several methods are available to calculate the zeros \( \alpha_\mu \) of a polynomial on a computer.

The second problem is simpler. We discuss at first the case of simple zeros \( \alpha_\mu \). Formula (3.9) reduces to

\[
\frac{g(s)}{p(s)} = \sum_{\mu=1}^{n} \frac{d_\mu}{s - \alpha_\mu},
\]  

(3.10)

where all the zeros \( \alpha_\mu \) are different from each other. Let \( \alpha_\nu \) be one of these zeros. By multiplication with \( s - \alpha_\nu \) \((s \neq \alpha_\nu)\) we obtain
\[
\frac{g(s)(s-\alpha_v)}{\rho(s)} = dv + \sum_{\mu=1}^{n} \frac{d_{\mu}(s-\alpha_v)}{s-\alpha_{\mu}}.
\]

Because of \(p(\alpha_v) = 0\), we can write
\[
\frac{g(s)(s-\alpha_v)}{\rho(s)} = \frac{g(s)}{\rho(s) - \rho(\alpha_v)}.
\]

For \(s \to \alpha_v\), we have
\[
d_v = \lim_{s \to \alpha_v} \frac{g(s)(s-\alpha_v)}{\rho(s)} = \frac{g(\alpha_v)}{\rho'(\alpha_v)},
\]

hence for the sum of partial fractions
\[
\frac{g(s)}{\rho(s)} = \sum_{\mu=1}^{n} \frac{g(\alpha_{\mu})}{\rho'(\alpha_{\mu})} \frac{1}{s-\alpha_{\mu}}
\]

Especially for \(g(s) = 1\):
\[
\frac{1}{\rho(s)} = \sum_{\mu=1}^{n} \frac{1}{\rho'(\alpha_{\mu})} \frac{1}{s-\alpha_{\mu}}
\]

For the case of multiple zeros \(\alpha_{\mu}\) we give here only the final formula for the coefficients \(d_{v\lambda}\)
\[
d_{v\lambda} = \frac{1}{(k_{\nu v} - \lambda)!} \left\{ \left[ \frac{g(s)}{\tau_{\nu v}(s)} \right]^{(k_{\nu v} - \lambda)} \right\}_{s = \alpha_{\nu v}},
\]

with
\[
\tau_{\nu v}(s) = \frac{\rho(s)}{(s - \alpha_{\nu v}) k_{\nu v}}
\]

and \(\lambda = 1, 2, \ldots, k_{\nu v}\).

\* There exist several ways of computing the \(d_{\mu}\). This is the most elegant one.
In particular for \( g(s) = 1 \)

\[
\frac{1}{(k\nu - \lambda)!} \left\{ \left( \frac{1}{\tau_\nu(\lambda)} \right)^{k\nu - \lambda} \right\} \Delta = \alpha_\nu \tag{3.15}
\]

\[
= \frac{1}{(k\nu - \lambda)!} \left\{ \left( \frac{1}{\rho(\lambda)} \right)^{k\nu} \right\} \Delta = \alpha_\nu
\]

3.3 The differential equation of \( n \)th order with constant coefficients

Our aim is now to integrate the differential equation

\[
\begin{align*}
Y^{(n)} + c_{n-2} Y^{(n-4)} + \cdots + c_1 Y' + c_0 Y &= F(t) \tag{3.16}
\end{align*}
\]

in the interval \( t \geq 0 \). In order to make the solution unique, we have to specify the initial values of the function itself and of its first \( n-1 \) derivatives, this means \( n \) values

\[
Y'(0) = y_0, \quad Y''(0) = y_0', \quad \ldots, \quad Y^{(n-1)}(0) = y_0^{(n-1)} \tag{3.17}
\]

We assume for \( F(t) \) the same properties as in the case of the first order differential equation. Again with the preliminary assumption that \( F(t) \) and \( Y^{(n)}(t) \) had image functions we can write the image equation for Eq. (3.16)

\[
\begin{align*}
&\left[ \lambda^n - y_0 \lambda^{n-1} - y_0' \lambda^{n-2} - \cdots - y_0^{(n-2)} \lambda - y_0^{(n-1)} \right] \Delta \lambda^n - y_0 \lambda^{n-1} - y_0' \lambda^{n-2} - \cdots - y_0^{(n-2)} \lambda - y_0^{(n-1)} \\
+c_{n-1} \left[ \lambda^{n-1} - y_0 \lambda^{n-2} - y_0' \lambda^{n-3} - \cdots - y_0^{(n-2)} \lambda - y_0^{(n-1)} \right] \\
+c_1 \left[ \lambda - y_0 \right] \\
+c_0 \lambda 
\end{align*} = f(\lambda) \tag{3.18}
\]

We introduce here the "characteristic polynomial" of the differential equation

\[
\rho(\lambda) = \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0, \tag{3.19}
\]

and obtain, for the image equation
\[ p(\lambda) y(\lambda) = f(\lambda) + y_0 (A^{n-1} + c_{n-1} A^{n-2} + \ldots + c_2 A + c_1) + y_0^1 (A^{n-2} + c_{n-1} A^{n-3} + \ldots + c_2) + y_0^{(n-2)} (A + c_{n-1}) + y_0^{(n-1)}. \]

The solution of this linear algebraic equation is

\[ y(\lambda) = \frac{f(\lambda)}{p(\lambda)} + y_0 \frac{(A^{n-1} + c_{n-1} A^{n-2} + \ldots + c_2 A + c_1)}{p(\lambda)} + y_0^1 \frac{(A^{n-2} + c_{n-1} A^{n-3} + \ldots + c_2)}{p(\lambda)} + y_0^{(n-2)} \frac{(A + c_{n-1})}{p(\lambda)} + y_0^{(n-1)} \frac{1}{p(\lambda)}. \]

We have to find the original function for \( f(\lambda) \). In order to make the problem clearer, we split it into two separate parts.

3.3.1 The homogeneous equation with non-vanishing initial conditions

We set \( f(\lambda) = 0 \) and therefore \( F(\tau) = 0 \), and consider the homogeneous differential equation with non-vanishing initial conditions. We start with the special case

\[ y_0 = y_0^1 = \ldots = y_0^{(n-2)} = 0, \quad y_0^{(n-1)} = 1. \]

From Eq. (3.21), we obtain easily

\[ y(\lambda) = \frac{1}{p(\lambda)}. \]

We set

\[ \frac{1}{p(\lambda)} = q(\lambda) \]

and call the solution \( Y(t) \) of the differential equation for this special case \( Q(t) \). We then have
\[ L \{ Q(t) \} = \dot{Q}(\lambda) = \frac{1}{\rho(\lambda)} , \]  
\[ Q^{(n)} + c_{n-2} Q^{(n-4)} + \ldots + c_1 Q' + c_0 Q = 0 , \]  
\[ Q(+0) = Q'(+0) = \ldots = Q^{(m-2)}(+0) = 0 , \]
\[ Q^{(n-4)}(+0) = 1 . \]  

In order to find \( Q(t) \) we decompose \( q(s) \) into partial fractions and have

\[ q(\lambda) = \sum_{\mu=1}^{\infty} \frac{1}{p^1(\alpha_\mu)} \frac{1}{1-\alpha_\mu} = L \left\{ \sum_{\mu=1}^{\infty} \frac{1}{p(\alpha_\mu)} e^{\alpha_\mu t} \right\} \]  
\[ Q(t) = \sum_{\mu=1}^{\infty} \left( d_{\mu 1} + \frac{d_{\mu 2}}{1!} t + \ldots + \frac{d_{\mu k_\mu}}{(k_\mu - 1)!} t^{k_\mu - 1} \right) e^{\alpha_\mu t} \]  

if \( \alpha_\mu \) is a zero of the \( k_\mu \) order. The coefficients \( d_{\mu \lambda} \) are given by Eq. (3.15).

The conditions (3.25), together with the explicit solution (3.26) result in the remarkable relations

\[ \sum_{\mu=1}^{\infty} \frac{1}{p^1(\alpha_\mu)} = 0 , \sum_{\mu=1}^{\infty} \frac{\alpha_\mu}{p^1(\alpha_\mu)} = 0 , \ldots , \]  
\[ \sum_{\mu=1}^{\infty} \frac{\alpha_\mu^{n-2}}{p^1(\alpha_\mu)} = 0 , \sum_{\mu=1}^{\infty} \frac{\alpha_\mu^{n-1}}{p^1(\alpha_\mu)} = 1 \]  

(all \( \alpha_\mu \) simple zeros).
For the solution of the homogeneous differential equation with arbitrary initial conditions, we apply Rule V to \( q(s) \) and obtain, with Eq. (3.25)

\[
\frac{1}{\rho(\lambda)} = \mathcal{L}\{Q(t)^2\}, \quad \frac{s}{\rho(\lambda)} = \mathcal{L}\{Q'(t)^2\},
\]

\[
\frac{s^2}{\rho(\lambda)} = \mathcal{L}\{Q''(t)^2\}, \ldots, \quad \frac{s^{n-1}}{\rho(\lambda)} = \mathcal{L}\{Q^{(n-1)}(t)^2\},
\]

\[
\frac{s^n}{\rho(\lambda)} - 1 = \mathcal{L}\{Q^n(t)^2\}. \tag{3.30}
\]

Thus we get from Eq. (3.21) the desired solution

\[
Y_H(t) = Y_0 \left[ Q^{(n-4)}(t) + c_{m-4} Q^{(n-2)}(t) + \cdots + c_2 Q'(t) + c_1 Q(t) \right] \\
+ Y_0^1 \left[ Q^{(n-2)}(t) + c_{m-1} Q^{(n-3)}(t) + \cdots + c_2 Q(t) \right] \\
+ Y_0^{(n-2)} \left[ Q'(t) + c_{m-1} Q(t) \right] \\
+ Y_0^{(n-1)} Q(t). \tag{3.31}
\]

The functions \( Q'(t), Q''(t), \ldots \) have the same structure as \( Q(t) \), that is they are sums of exponential functions, sometimes (for multiple zeros) multiplied with powers of \( t \).

### 3.3.2 The inhomogeneous equation with vanishing initial conditions

It remains to establish the solution of the inhomogeneous equation with vanishing initial conditions. Thus we take from Eq. (3.21) the first term only and have

\[
Y_f(\lambda) = \frac{1}{\rho(\lambda)} F(\lambda) = Q_f(\lambda) F(\lambda),
\]

which gives, according to Rule IX:

\[
Y_f(t) = Q(t) \times F(t) = \int_0^t Q(t - \tau) F(\tau) d\tau. \tag{3.32}
\]
In the special case of simple zeros $\alpha_\mu$ we have, using Eq. (3.26):

$$\gamma(t) = \sum_{\mu=1}^{n} \frac{e^{\alpha_\mu t}}{\rho^1(\alpha_\mu)} \int_0^t e^{-\alpha_\mu \tau} F(\tau) d\tau \quad (3.33)$$

The complete solution of Eq. (3.16) with initial conditions (3.17) is then given by

$$Y(t) = Y_H(t) + Y_I(t)$$

Similarly as for the first-order equation, it can be shown that $Y(t)$, $Y'(t), ..., Y^{(n-1)}(t)$ are continuous for $t > 0$, provided $F(t)$ fulfills the conditions given in Section 3.1. In addition, Eq. (3.16) is satisfied for all $t > 0$ at least for $t+0$ and $t-0$. $Y^{(n)}(t)$ has the same steps as $F(t)$, that is

$$Y^{(n)}_+(t) - Y^{(n)}_-(t) = F(t+0) - F(t-0)$$

The prescribed procedure gives the general way of solving the $n$th order differential equation. In practical cases, however, one always starts from the image equation and tries to simplify as much as possible. In particular, many original functions corresponding to rational functions $1/p(s)$ are given in the tables.

To end this section, we recall the main advantages of the Laplace transform compared with the classical method.

a) In the classical method the "general solution" is established at first, its constants having to be determined according to the initial conditions. This means the solution of a linear system of equations with $n$ unknowns. With the $\mathcal{L}$-transform, the initial values are introduced automatically and from the beginning. In particular, the very common case of vanishing initial conditions, which has no advantages in the classical method, gives here a very simple procedure.

b) While in the classical method the inhomogeneous equation is solved by "variation of the constants" after solving the homogeneous equation, here the solution is found directly.
3.4 The initial value problems with general initial values and the boundary value problems

The application of the Laplace transform to ordinary differential equations requires the values of the function and its \( n-1 \) first derivatives at \( t = 0 \). It is, however, possible that in practical problems values at \( t = 0 \) are given for other derivatives, e.g. for \( n = 3 \) the values \( Y(0), Y'(0), Y'''(0) \). In this case one can proceed as follows.

The initial value problem will be solved in the same way as if \( Y(0), Y'(0), Y'''(0) \) were given. The solution will be differentiated to obtain \( Y''(t), Y'''(t) \). For \( t = 0 \) one has then two linear equations for the two unknowns \( Y'(0), Y'''(0) \). These equations can easily be solved and the solution of the whole problem is obtained.

A similar method is followed if \( n \) values are given not at one point \( t \), but at two or more points. In this case the problem is called a boundary value problem. We discuss in the following a special case, which we will need later.

Let the second order equation

\[
Y'' - \alpha^2 Y = F(t)
\]

be given together with the two boundary conditions \( Y(0) \) and \( Y(t) \).
\( \alpha \neq 0 \) is an arbitrary complex number and \( F(t) \) a continuous function. We write the image equation

\[
\mathcal{L}^2 Y - Y(0) S - Y'(0) - \alpha^2 y = f(\sigma)
\]

and obtain

\[
y(\sigma) = \frac{f(\sigma)}{\mathcal{L}^2 - \alpha^2} + Y(0) \frac{S}{\mathcal{L}^2 - \alpha^2} + Y'(0) \frac{1}{\mathcal{L}^2 - \alpha^2}.
\]

This gives, with Rule IX and Table 1

\[
Y(t) = \frac{1}{\alpha} F(t) \times \sinh \alpha t + Y(0) \cosh \alpha t + \frac{1}{\alpha} Y'(0) \sinh \alpha t.
\]

* In the following we use \( Y(0) \) instead of \( Y(+0) \), etc.
Again we consider two problems separately.

a) \( F(t) = 0, \ Y(0) \text{ and } Y(t) \) arbitrary

We obtain from Eq. (3.35)
\[
Y(l) = Y(0) \cosh \alpha l + \frac{1}{\alpha} Y'(0) \sinh \alpha l
\]
and
\[
\frac{1}{\alpha} Y'(0) = \frac{Y(l) - Y(0) \cosh \alpha l}{\sinh \alpha l},
\]
hence
\[
Y(t) = Y(0) \cosh \alpha t + (Y(l) - Y(0) \cosh \alpha l) \frac{\sinh \alpha t}{\sinh \alpha l}
\]
\[
= Y(0) \frac{\sinh \alpha (l-t)}{\sinh \alpha l} + Y(l) \frac{\sinh \alpha t}{\sinh \alpha l}.
\]  \hspace{1cm} (3.36)

b) \( F(t) \neq 0, \ Y(0) = Y(l) = 0 \)

With \( Y(0) = 0 \) we get from Eq. (3.35)
\[
Y(t) = \frac{1}{\alpha} F(t) \sinh \alpha t + \frac{1}{\alpha} Y'(0) \sinh \alpha t
\]
Therefore
\[
Y(l) = 0 = \frac{1}{\alpha} \int_0^l F(\tau) \sinh \alpha (l-\tau) d\tau + \frac{1}{\alpha} Y'(0) \sinh \alpha l
\]
which gives
\[
\frac{1}{\alpha} Y'(0) = - \frac{1}{\alpha \sinh \alpha l} \int_0^l F(\tau) \sinh \alpha (l-\tau) d\tau
\]
\[
Y(t) = \frac{1}{\alpha} \int_0^t F(\tau) \sinh \alpha (t-\tau) d\tau \\
- \frac{1}{\alpha} \frac{\sinh \alpha t}{\sinh \alpha l} \int_0^l F(\tau) \sinh \alpha (l-\tau) d\tau.
\]  
(3.37)

We can split the integral from 0 to \(l\) into two integrals from 0 to \(t\) and from \(t\) to \(l\). Using the common denominator \(\sinh \alpha t\) for the integrals from 0 to \(t\), together with the identity

\[
\sinh \alpha (t-\tau) \sinh \alpha l - \sinh \alpha (l-\tau) \sinh \alpha t \\
= - \sinh \alpha \tau \sinh \alpha (l-\tau),
\]

we obtain for Eq. (3.37)

\[
Y(t) = - \frac{1}{\alpha} \frac{\sinh \alpha (l-\tau)}{\sinh \alpha l} \int_0^t F(\tau) \sinh \alpha \tau d\tau \\
- \frac{1}{\alpha} \frac{\sinh \alpha t}{\sinh \alpha l} \int_t^l F(\tau) \sinh \alpha (l-\tau) d\tau.
\]  
(3.38)

Here we can introduce the so-called Green's function

\[
\gamma(t, \tau; \alpha) = \begin{cases} 
- \frac{1}{\alpha} \frac{\sinh \alpha \tau \sinh \alpha (l-\tau)}{\sinh \alpha l} & (0 \leq \tau \leq t) \\
- \frac{1}{\alpha} \frac{\sinh \alpha \tau \sinh \alpha (l-\tau)}{\sinh \alpha l} & (t \leq \tau \leq l)
\end{cases}
\]  
(3.39)

and write

\[
Y(t) = \int_0^l \gamma(t, \tau; \alpha) F(\tau) d\tau.
\]  
(3.40)
The general solution of Eq. (3.34) is then given by the sum of the solutions (3.36) and (3.40).

It should be noted that this solution has a meaning only if \( \sinh \alpha t \neq 0 \). The values \( \alpha^2 \neq 0 \) for which \( \sinh \alpha t = 0 \), namely

\[
\alpha^2 = -n^2 \left( \frac{\pi}{\ell} \right)^2 \quad (n = 1, 2, \ldots)
\]

are called eigenvalues of the boundary value problem. For these eigenvalues the homogeneous boundary value problem \( f(t) = 0, y(0) = y(\ell) = 0 \) has the eigenfunction solution \( y(t) = \sin n(\pi/\ell)t \). For all other values of \( \alpha \) this problem has the solution \( y(t) = 0 \) only.
\[ Y(t) = \frac{1}{c} \left( 1 - e^{-c(t-a)} \right) U(t-a) + Y_0 e^{ct} \]

\[ a = 1 \quad c = 0.75 \]

**Fig. 5**
3.5 The solution of the differential equation for special perturbation functions

For a physical system which is described by the differential equation (3.16), the solution of the homogeneous equation $Y_H(t)$ [Eq. (3.31)] determines the proper motion of the system. This motion starts from the given initial values [Eq. (3.17)] without the interference of any perturbation function. It can be described by a sum of functions $t^\alpha$.

In the following we disregard this solution, which means we assume

$$Y(0) = Y'(0) = \ldots = Y^{(n-1)}(0) = 0$$

and discuss the solution (3.32) of the inhomogeneous equation only

$$Y(t) = Q(t) \times F(t). \quad (3.41)$$

In electrotechnics, $F(t)$ is often called input function and $Y(t)$ output function. Simpler than the integral relation (3.41) is the formula

$$y(s) = q(s) \times f(s) \quad (3.42)$$

in the image space. It is, therefore, desirable to calculate as much as possible in the latter. The factor $q(s)$, which relates $f(s)$ to $y(s)$, is called the transmission factor. This factor depends only on the coefficients $c_{n-1}, \ldots, c_1, c_0$ of the differential equation and is the mathematical equivalent of the physical system described by Eq. (3.16).

We can demonstrate Eq. (3.42) with a "block diagram"
This is especially useful if several systems are connected together, e.g.

\[ f(s) \rightarrow q_{r_1}(s) \rightarrow y_{d_1}(s) \rightarrow q_{r_2}(s) \rightarrow y_{d_2}(s) \]

Here we have

\[ y_{d_1}(s) = q_{r_1}(s) f(s) \quad ; \quad y_{d_2}(s) = q_{r_2}(s) y_{d_1}(s) \]

hence

\[ y_{d_2}(s) = q_{r_2}(s) y_{d_1}(s) f(s) \]

This means that two systems connected in series are equivalent to one system with \( q(s) = q_1(s) q_2(s) \). A more complicated example is the so-called feedback. Here we have

\[ f(s) \rightarrow \bigcirc_{f(s)-y_{d_2}(s)} \rightarrow q_{r_1}(s) \rightarrow y_{d_1}(s) \]

or

\[ y_{d_1}(s) = q_{r_1}(s) \left[ f(s) - y_{d_2}(s) \right], \quad y_{d_2}(s) = q_{r_2}(s) y_{d_1}(s) \]

Of special interest here is the relation between \( f(s) \) and \( y_1(s) \), namely
\[ y_1(s) = \frac{q_1(s)}{1 + q_1(s) q_2(s)} f(s) \]
\[ = \frac{1}{q_1(s) + q_2(s)} f(s) \] (3.43)

or with the characteristic polynomials

\[ y_1(s) = \frac{1}{p_1(s) + \frac{1}{p_2(s)}} f(s) = \frac{p_2(s)}{p_1(s) p_2(s) + 1} f(s) \] (3.44)

We now discuss the behaviour of the output function \( Y(t) \) for some special input functions \( F(t) \), which are of particular interest in technical and physical problems.

3.5.1 The step function \( U(t) \)

For \( F(t) = U(t) \) we obtain as output function

\[ y_u(t) = Q(t) \ast 1 \] (3.45)

or

\[ y_u(s) = \frac{1}{\Delta p(s)} = \frac{1}{\Delta \rho(s)} \] (3.46)

Instead of calculating the integral in Eq. (3.45), we can expand Eq. (3.46) into partial fractions and obtain for the case of simple zeros \( \alpha_\mu \) of \( p(s) \) and \( \alpha_\mu \neq 0 \) (\( c_0 \neq 0 \)) with

\[ \left[ \Delta p(s) \right]' = p(s) + \Delta p'(s) = \delta \]

\[ \alpha_\mu \rho'(\alpha_\mu) \] for \( s = \alpha_\mu \)

from formula (3.13)

\[ y_u(s) = \frac{1}{\Delta \rho(s)} = \frac{1}{\rho(0)} \frac{1}{\Delta} + \sum_{\mu=1}^{w} \frac{1}{\alpha_\mu \rho'(\alpha_\mu)} \frac{1}{s - \alpha_\mu} \] (3.47)
hence the original function

\[
Y(t) = \frac{1}{\rho(0)} + \sum_{\mu=1}^{\infty} \frac{e^{\alpha_{\mu} t}}{\alpha_{\mu} \rho(\alpha_{\mu})}.
\]  

(3.48)

This can be explained in the following way: an idle system \([Y(0) = Y'(0) = \ldots = 0]\) reacts on a unit step with a step \(U(t)/\rho(0) = U(t)/c_0\). Upon this step, a sum of exponential functions is superposed. Especially for \(\text{Re} \alpha_{\mu} < 0\), these functions die away for \(t \to \infty\). \(Y_U(t)\) is called the time admittance of the physical system. Because of Eq. (3.45), we have

\[
Y_U'(t) = \frac{d}{dt} \int_0^t Q(\tau) \, d\tau = Q(t).
\]

(3.49)

The output function \(Y(t)\) can be given for an arbitrary input function \(F(t)\) if \(Y_U(t)\) is available. We have from Eqs. (3.41) and (3.49)

\[
Y(t) = Y_U'(t) * F(t)
\]

(3.50)

or, using Eq. (2.28) and \(Y_U(0) = 0\):

\[
Y(t) = \frac{d}{dt} \left[ Y_U * F \right].
\]

(3.51)

The relations (3.50) and (3.51) are called Duhamel's formulae.

3.5.2 The harmonic oscillation e^{j\omega t}

It is advantageous to calculate with \(e^{j\omega t}\) instead of \(\cos \omega t\) or \(\sin \omega t\). If one of the latter functions is given as the input function, the real or imaginary part of the final result is taken, respectively. In the case of real coefficients \(c_0, c_1, \ldots\), that means in most technical problems, with each \(a_{\mu}\) the complex conjugate \(\overline{a_{\mu}}\) is also a root of the characteristic equation. Therefore we have in Eq. (3.26) two terms
$$\frac{1}{\rho(\alpha)} e^{\alpha \cdot t} \quad \text{and} \quad \frac{1}{\rho(\overline{\alpha})} e^{\overline{\alpha} \cdot t}$$

which are conjugated to each other. \( Q(t) \) is, therefore, a real function. This is also true for multiple roots.

For \( F(t) = e^{j\omega t} \) we call the solution \( Y_\omega(t) \) and have

$$y_\omega(s) = \frac{q(s)}{s - j\omega} \quad (3.52)$$

and

$$Y_\omega(t) = Q(t) * e^{j\omega t} = e^{j\omega t} \int_0^t e^{-j\omega \tau} Q(t) d\tau \quad (3.53)$$

For a so-called "passive" system (that means a system with no internal energy sources), which dissipates energy (e.g. friction, electrical resistance) one can show that

$$\text{Re } \alpha_\mu < 0$$

for all \( \mu \). For such a system we can easily find the steady-state solution \( \bar{Y}_\omega(t) \), which characterizes \( Y_\omega(t) \) for large \( t \to \infty \).

With

$$0 > \text{Re } \alpha_1 \geq \text{Re } \alpha_2 \geq \ldots \geq \text{Re } \alpha_m$$

we see that \( L[Q] = q(s) \) converges at least for \( \text{Re } s > \text{Re } \alpha_1 \), therefore in any case on the imaginary axis. Thus, we can calculate

$$q(j\omega) = \int_0^\infty e^{-j\omega t} Q(t) d\tau \quad (3.54)$$

and write Eq. (3.53) as
\[ Y_\omega(t) = e^{j\omega t} \left( \int_0^t e^{-j\omega \tau} \mathcal{Q}(\tau) \, d\tau - \int_t^\infty e^{-j\omega \tau} \mathcal{Q}(\tau) \, d\tau \right) \]

\[ = q(j\omega) e^{j\omega t} - e^{j\omega t} \int_t^\infty e^{-j\omega \tau} \mathcal{Q}(\tau) \, d\tau \]

The second term on the right-hand side goes to zero for \( t \to \infty \); therefore we have for sufficiently large \( t \) the steady-state solution

\[ \tilde{Y}_\omega(t) = q(j\omega) e^{j\omega t} \]  

(3.56)

We can obtain \( \tilde{Y}_\omega(t) \) also from the partial fraction expansion

\[ Y_\omega(s) = \frac{1}{(s-j\omega)R(s)} = \frac{q(j\omega)}{s-j\omega} + \sum_{\mu=1}^n \frac{1}{s-(\alpha_\mu-j\omega)p(\alpha_\mu)} \]

(3.57)

which gives

\[ Y_\omega(t) = q(j\omega) e^{j\omega t} + \sum_{\mu=1}^n \frac{e^{\alpha_\mu t}}{(s-(\alpha_\mu-j\omega)p'(\alpha_\mu))} \]

(3.58)

The steady-state solution \( \tilde{Y}_\omega(t) \) remains, by the way, unchanged if \( Y_\omega(t) \) has non-vanishing initial conditions. This is easily verified from Eq. (3.31) for Re \( \alpha_\mu < 0 \).

\( q(j\omega) \) is in general a complex quantity

\[ q(j\omega) = |q(j\omega)| e^{j\Omega} \]

(3.59)

The steady-state solution is therefore

\[ \tilde{Y}_\omega(t) = |q(j\omega)| e^{j(\omega t + \Phi)} \]

(3.60)

in other words it is an oscillation with the same frequency \( \omega \) as the input function, but with another amplitude \( |q(j\omega)| \) and another phase \( \Phi \).

*) Given here for \( \alpha_\mu \) simple and \( \alpha_\mu \neq j\omega \).
The function \( q(j\omega) \) is called frequency admittance of the system. The functions \( q(j\omega) \) and \( \phi(\omega) \) are extremely important for technical purposes. It is especially advantageous that one can obtain the steady-state solution from them without transformation into the original space.

It is also remarkable that \( q(j\omega) \) has a relationship with the time admittance, namely by Eq. (3.46)

\[
q(j\omega) = j\omega Y_u(j\omega) = \left[ \omega \int Y_u \right]_s = j\omega
\]

(3.61)

In addition, we note without further detail that the time admittance can be obtained for \( Re \alpha < 0 \) and \( t > 0 \) from the frequency admittance by

\[
Y_u(t) = \frac{2}{\pi} \int_0^\infty \frac{u(\omega)}{\omega} \sin t\omega \, d\omega
\]

(3.62)

and

\[
Y_v(t) = \frac{2}{\pi} \int_0^\infty \frac{v(\omega)}{\omega} \cos t\omega \, d\omega + u(0).
\]

(3.63)

\( u(\omega) = Re \, q(j\omega) \) and \( v(\omega) = Im \, q(j\omega) \) are called conductance and susceptance, respectively.

### 3.5.3 The impulsive function \( \delta(t) \)

It happens often in physics that a system is subject to a force \( F_\varepsilon(t) \), which is effective during a very short time \( \varepsilon \) only. The total power

\[
\varepsilon \int_0 F_\varepsilon(t) \, dt,
\]

however, is not of the order \( \varepsilon \) but of the order 1

\[
\int_0^t F_\varepsilon(\tau) \, d\tau = 1 \quad \text{for} \quad t \geq \varepsilon.
\]

(3.64)
A very simple function of this type is

$$F_{\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } 0 \leq t \leq \varepsilon \\ 0 & \text{for } t > \varepsilon \end{cases}$$ \hspace{1cm} (3.65)

It is now not unreasonable to introduce a limit $\varepsilon \to 0$, which leads to a force $\delta(t)$. This "function" equals zero for $t > 0$ and is "infinite" for $t = 0$, in such a way that

$$\int_{0}^{t} \delta(\tau) \, d\tau = 1 \quad \text{for } t > 0 .$$ \hspace{1cm} (3.66)

This is very crude and by no means allowed in common analysis. There are, fortunately, several ways to define this process in a proper way\(^*\), which results in all cases in an extension of the common analysis. We do not discuss these details here.

$\delta(t)$ is therefore not a function in the proper sense. We call it a pseudofunction. It is also called impulsive function or Dirac function.

For functions $G(t)$, which are continuous for $t = 0$, we define

$$\int_{a}^{b} G(t) \delta(t) \, dt = \begin{cases} G(0) & \text{for } a < 0 < b \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (3.67)

The integral sign in Eq. (3.67) has only a formal meaning and is not covered by the usual definitions of Riemann or Lebesgue. It is not clear a priori that the rules of integral calculus are applicable to such integrals. There is, however, a way to overcome this difficulty.

From Eq. (3.67) it follows that for $G(t) = 1$:

$$\int_{-\infty}^{t} \delta(\tau) \, d\tau = \begin{cases} 1 & \text{for } 0 < t \\ 0 & \text{for } 0 \geq t \end{cases}$$

\(^*\) e.g. with so-called distributions.
\[ \int_{-\infty}^{t} \delta(t) \, dt = u(t), \quad (3.68) \]

where \( u(t) \) is the step function. If \( \delta(t) \) were continuous, one could write \( dU(t)/dt = \delta(t) \). A formal replacement of \( \delta(t) \, dt = dU(t) \) in Eq. (3.67) results in

\[ \int_{a}^{b} \alpha(t) \, dU(t) = \begin{cases} \alpha(0) & \text{for } a \leq 0 < b \\ 0 & \text{otherwise} \end{cases} \quad (3.69) \]

This type of integral is known in common analysis as Stieltjes \(^{1} \) integral. We can, therefore, understand

\[ \int_{a}^{b} \alpha(t) \delta(t) \, dt \text{ as } \int_{a}^{b} \alpha(t) \, dU(t) \]

We obtain for \( \delta(t-t_0) \)

\[ \int_{a}^{b} \alpha(t) \delta(t-t_0) \, dt = \int_{a}^{b} \alpha(t) \, dU(t-t_0) = \begin{cases} \alpha(t_0) & \text{for } a \leq t_0 < b \\ 0 & \text{otherwise} \end{cases} \quad (3.70) \]

In particular for the Laplace transform

\[ \mathcal{L}\{\delta(t)\} = \int_{0}^{\infty} e^{-st} \delta(t) \, dt = 1 \quad (3.71) \]

and

\(^{1} \) The Stieltjes integral is defined as

\[ \int_{a}^{b} \alpha(t) \, dH(t) \]

is defined as

\[ \lim_{\max(t_v-t_{v-1}) \to \infty} \sum_{v=1}^{M} \alpha(t_v) [H(t_v) - H(t_{v-1})] \quad (t_{v-1} \leq t_v \leq t_v) \]

The Riemann integral is the special case \( H(t) = t \).
\[
\mathcal{L}^{-1} \delta(t-t_0) = \int_0^\infty e^{-s \tau} \delta(t-t_0) \, d\tau = e^{-t_0 s}
\]  

(3.72)

It should be noted that 1 and \( e^{-t_0 s} \) do not belong to the image space of the L-transform if only functions of the common analysis are allowed in the original space.

In addition, we find that
\[
\mathcal{L} \{ f(t) \} \ast \mathcal{L} \{ \delta(t-t) \} = \int_0^t \mathcal{L} \{ f(t-\tau) \} d\tau = \int_0^t \mathcal{L} \{ f(t-\tau) \delta(t-\tau) \} d\tau = \mathcal{L} \{ f(t) \} \]  

(3.73)

This relation is not possible in common analysis either.

We discuss now a physical system with \( \delta(t) \) as input function,
\[
Y^{(n)} + c_{n-1} Y^{(n-1)} + \ldots + c_0 Y = \delta(t)
\]  

(3.74)

with vanishing initial conditions
\[
Y_0 = \ldots = Y_0^{(n-1)} = 0.
\]

For exact mathematical interpretation, we convolute Eq. (3.74) with 1 and obtain
\[
(Y^{(n)} + c_{n-1} Y^{(n-1)} + \ldots + c_0 Y) \ast 1 = \delta(t) \ast 1
\]  

(3.75)

and with \( \mathcal{L} \)-transform
\[
\mathcal{L} \{ \delta(t) \} \mathcal{L} \{ Y(t) \} \frac{A}{\delta} = \frac{A}{\delta}
\]

or
\[
\mathcal{L} \{ \delta(t) \} \mathcal{L} \{ Y(t) \} = 1
\]  

(3.76)

One obtains the same result, by the way, when transforming Eq. (3.74) directly. We have
\[
Y(t) = \frac{1}{\mathcal{L} \{ \delta(t) \}}
\]  

(3.77)
and

\[ X_f(t) = \mathcal{Q}(t) \]  \hspace{1cm} (3.78)

To the input function \( \delta(t) \) corresponds therefore the output function \( Q(t) \).

It is worth while to look at this result carefully. A comparison of Eq. (3.74) with Eq. (3.24), and their appropriate initial conditions, shows that Eq. (3.78) does not satisfy the problem without further explication. The left-hand side of Eq. (3.74) for \( Y(t) = Q(t) \) is equal to zero and not \( \delta(t) \). In addition, \( Q^{(n-1)}(+0) \neq 1 \) and not zero. We can avoid these difficulties by setting \( Q^{(n-1)}(+0) = 0 \). Then \( Q^{(n-1)}(t) \) has in \( t = 0 \) the same step as \( U(t) \). Therefore, the \( n \)th derivative \( Q^{(n)}(t) \) is not really a function and we have to add to it the impulsive function \( \delta(t) \). The equation (3.74) is then fulfilled, together with the initial conditions.

This procedure is translated into the image space as follows: the genuine function part of \( Q^{(n)}(t) \) gives \( s^n - 1 \), the pseudofunction part \( \delta(t) \) results in 1, therefore

\[ \int \left\{ Q^{(n)}(t) \right\} = s^n - 1 + 1 = s^n, \]

which is what we assumed for the calculation of Eq. (3.77).

As a physical example, it is known that a linear oscillator (e.g. a point of mass at the end of a long thread) can be described by a second order differential equation. If this oscillator is subject to an impulsive force \( \delta(t) \), the velocity (the first derivative) jumps suddenly from zero to a non-zero value, whereas the position of the mass (the function itself) starts moving continuously from zero onwards.

3.5.4 Examples

We illustrate the procedures discussed in the previous paragraphs by simple examples. Let the equation

\[ Y'' + 6Y' + 5Y = F(t) \]

be given. We solve it for \( F(t) = U(t) \) and for \( F(t) = \delta(t) \). The characteristic polynomial and its derivative become
\[ p(s) = s^2 + 6s + 5 \quad ; \quad p'(s) = 2s + 6 \]

The roots \( a_1 \) and \( a_2 \) of \( p(s) \) are simple, namely \( a_1 = -1, a_2 = -5 \), thus we obtain from Eq. (3.26) and with \( p'(-1) = 4, \ p'(-5) = -4 \)

\[
Q(t) = \sum_{\mu=1}^{2} \frac{1}{p'(\alpha_{\mu})} e^{\alpha_{\mu} t} = \frac{1}{4} \left( e^{-t} - e^{-5t} \right)
\]

and by differentiation

\[
Q'(t) = \frac{1}{4} \left( -e^{-t} + 5e^{-5t} \right).
\]

We realize that \( Q(+0) = 0, \ Q'(+0) = 1. \)

For \( F(t) = U(t) \) we obtain

\[
Y_U(t) = Q(t) \times 1 = \int_{0}^{t} Q(\tau) d\tau
\]

\[
= \frac{1}{4} \int_{0}^{t} (e^{-t} - e^{-5t}) \, dt = \frac{1}{5} - \frac{1}{4} e^{-t} + \frac{1}{20} e^{-5t}
\]

and

\[
\lim_{t \to \infty} Y_U(t) = \frac{1}{p(0)} = \frac{1}{5}.
\]

\( Y_U(t) \) can be obtained also from Eq. (3.48).

For \( F(t) = 5(t) \) we obtain directly \( Y_5(t) = Q(t) \). The first derivative jumps from 0 to 1 for \( t = 0 \) (Fig. 6).

As an example for an equation with an oscillatory input function we solve

\[
Y'' + 2Y' + 5Y = e^{j\omega t}
\]

and find
Fig. 6
\[ \rho(\alpha) = s^2 + 2s + 5, \quad \rho'(\alpha) = 2(s+1) \]

\[ \alpha_1 = -1+2i, \quad \alpha_2 = -1-2i \]

\[ \rho'(\alpha_1) = 4i, \quad \rho'(\alpha_2) = -4i \]

Therefore

\[ Q(t) = \frac{1}{4i} \left( e^{(-1+2i)t} - e^{(-1-2i)t} \right) \]

\[ = \frac{1}{2} e^{-t} \sin 2t \]

and \( Q(0^+) = 0 \).

For the solution \( Y_\omega(t) \) we obtain from Eq. (3.53)

\[ Y_\omega(t) = e^{j\omega t} \int_0^t e^{-j\omega \tau} Q(\tau) \, d\tau \]

\[ = \frac{1}{2} e^{j\omega t} \int_0^t e^{-(1+j\omega)\tau} \sin 2\tau \, d\tau \]

Using the formula

\[ \int e^{ax} \sin bx \, dx = \frac{a x}{a^2 + b^2} \left( a \sin bx - b \cos bx \right) \]

we obtain

\[ Y_\omega(t) = \frac{1}{2} \frac{1}{(1+j\omega)^2 + 4} \left\{ 2e^{j\omega t} - e^{-t} (\sin 2t + 2 \cos 2t + j\omega \sin 2t) \right\} \]

or in the case of \( F(t) = \cos \omega t \) the solution (Fig. 7).
$\omega = \frac{1}{2}$

$F(t) = \cos \omega t$

$Q(t)$

$\text{Re } Y_\omega(t)$

Fig. 7
\[ \text{Re} \gamma_\omega (t) = \frac{1}{25 - 6\omega^2 + \omega^4} \times \]
\[ \times \left\{ \begin{array}{l}
(5 - \omega^2) \cos \omega t + 2\omega \sin \omega t \\
- e^{-t} \left[ (5 - \omega^2) \cos 2t + \frac{1}{2} (5 + \omega^2) \sin 2t \right] \end{array} \right\} \]

and for \( F(t) = \sin \omega t \)

\[ \text{Im} \gamma_\omega (t) = \frac{1}{25 - 6\omega^2 + \omega^4} \times \]
\[ \times \left\{ \begin{array}{l}
-2\omega \cos \omega t + (5 - \omega^2) \sin \omega t \\
+ e^{-t} \left[ 2\omega \cos 2t + \frac{1}{2} \cos (\omega^2 - 3) \sin 2t \right] \end{array} \right\} \]

We have then for the steady-state solution

\[ \gamma_\omega (t) = \frac{1}{(1 + j\omega)^2 + 4} e^{j\omega t} = \frac{1}{5 - \omega^2 + 2j\omega} e^{j\omega t} \]

hence (Fig. 3)

\[ q'(j\omega) = \int_0^\infty e^{-j\omega t} Q(t) \, dt = \frac{1}{5 - \omega^2 + 2j\omega} \]

We obtain for the modulus of \( q(j\omega) \)

\[ |q(j\omega)| = \frac{1}{\sqrt{25 - 6\omega^2 + \omega^4}} \]

and for the phase
\[ q(j\omega) = \frac{1}{5 - \omega^2 + 2j\omega} \]
\[ \Psi(\omega) = \arctan \frac{\text{Im} q(j\omega)}{\text{Re} q(j\omega)} \]

\[ = \arctan \frac{2\omega}{\omega^2 - 5} \]

3.6 Ordinary differential equations with polynomial coefficients

In the previous paragraphs we considered ordinary differential equations with constant coefficients. In the case of non-constant coefficients

\[ A_{m}(t) \Psi^{(m)}(t) + \ldots + A_{1}(t) \Psi^{(1)}(t) + A_{0}(t) \Psi(t) = F(t) \quad (3.79) \]

the \( \mathcal{L} \)-transform cannot be applied in general. If, however, the functions \( A_{\nu}(t) \) are polynomials (or can be approximated by polynomials) one can try to find a solution with the help of Rule VI. Applying this rule one obtains in the image space a differential equation for \( \Psi(s) \), which is at most of order \( m \), if the degree of the \( A_{\nu}(t) \) is at most of order \( m \). In the case \( m < n \) the order of the differential equation is therefore reduced, and the procedure seems to be promising. For \( m \geq n \), however, the problem does not in general become simpler and can be solved in exceptional cases only in this way. We discuss two examples for \( m < n \).

a) The Bessel differential equation

\[ t \Psi''(t) + \Psi'(t) + t \Psi(t) = 0. \quad (3.80) \]

By formal application of Rules V and VI we obtain

\[ -(s^2 \Psi(s) - s \Psi'_{s} - \Psi''_{s}) + s \Psi(s) - \Psi'_{s} - \Psi''_{s} = 0, \]

which gives

\[ (s^2 + 1) \Psi'(s) + s \Psi(s) = 0 \]

* This includes obviously the case of rational functions.
a first order equation for \( y(s) \), which is easily integrated to

\[
y'(s) = \frac{C}{\sqrt{s^2 + 1}}
\]

The corresponding original function is \( Y(t) = C J_0(t) \), which satisfies Eq. (3.80). It is therefore a solution of the equation.

b) The Laguerre differential equation

This equation reads

\[
t Y''(t) + (1-t) Y'(t) + n Y(t) = 0 \tag{3.81}
\]

where \( n \) is an entire number. We obtain

\[
-(s^2 y'(s) - sy_0 - y_0') + sy(s) - y_0 +
\]

or

\[
(s y'(s) - y_0') + n y(s) = 0
\]

Again, by separation of the variables, we get from

\[
\frac{dy}{y} = \frac{n}{s-1} ds - \frac{n+1}{s} ds
\]

the image function (except a constant factor):

\[
y'(s) = \frac{1}{s} \left( \frac{s-1}{s} \right)^n \tag{3.82}
\]

With the binomial theorem

\[
\frac{1}{s} \left( 1 - \frac{1}{s} \right)^n = \sum_{v=0}^{n} \binom{n}{v} \left( \frac{-1}{s} \right)^v \frac{1}{s^{v+1}}
\]
we obtain with Table 1 the Laguerre polynomials

\[ Y_n(t) \equiv L_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(t-1)^\nu}{\nu!} t^\nu, \quad (3.83) \]

which are indeed solutions of Eq. (3.81). We can apply the damping theorem (Rule IV) and obtain

\[ \mathcal{L} \left\{ e^{-t} L_n(t) \right\} = \frac{S^n}{(S+1)^{n+1}} \quad (3.84) \]

and with Rule IV

\[ \mathcal{L} \left\{ e^{-t} \frac{t^n}{n!} \right\} = \frac{1}{(S+1)^{n+1}}. \]

Since \( \lim_{t \to 0} t^n e^{-t} = 0 \), we have

\[ \mathcal{L} \left\{ \frac{1}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \right\} = \frac{S^n}{(S+1)^{n+1}}, \]

which gives by comparison with Eq. (3.84)

\[ L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}). \quad (3.85) \]

It should be noted that, in general, only special solutions of the differential equation with polynomial coefficients are found by the \( \mathcal{L} \)-transform. This is nevertheless often progress, and methods from the classical theory can be used to reduce the order of the given equation with the help of the special solution.
3.7 Systems of linear differential equations

The $\mathcal{L}$-transform is very advantageous for solving systems of linear differential equations. In such a system of $N$ equations of order $n$, in every equation, there appears (at least theoretically) a differential expression of $n^{th}$ order for each of the $N$ unknown functions, e.g. in the $\alpha^{th}$ equation for $Y_\beta(t)$ the expression

$$C_n^{\alpha\beta} Y_\beta^{(n)} + C_{n-1}^{\alpha\beta} Y_\beta^{(n-1)} + \cdots + C_1^{\alpha\beta} Y_\beta + C_0^{\alpha\beta} Y_\beta = \mathcal{P}_{\alpha\beta}(t)$$ (3.86)

In practice, some of these coefficients $C$ may be zero, e.g. it may happen that $Y_\beta$ is not present in the $\alpha^{th}$ equation or the differential expression is of an order less than $n$. We introduce as abbreviations

$$C_n^{\alpha\beta} + C_{n-1}^{\alpha\beta} Y_\beta^{(n-1)} + \cdots + C_1^{\alpha\beta} Y_\beta + C_0^{\alpha\beta} = \mathcal{P}_{\alpha\beta}(t)$$ (3.87)

and

$$\frac{d}{dt} \equiv D, \quad \frac{d^2}{dt^2} \equiv D^2, \quad \ldots,$$ (3.88)

and obtain for the differential expression the symbolic relation

$$C_n^{\alpha\beta} Y_\beta^{(m)} + \cdots + C_1^{\alpha\beta} Y_\beta^{(1)} + C_0^{\alpha\beta} Y_\beta = \mathcal{P}_{\alpha\beta}(D) Y_\beta$$ (3.89)

We can then write the system of differential equations as

$$\mathcal{P}_{11}(D) Y_1 + \mathcal{P}_{12}(D) Y_2 + \cdots + \mathcal{P}_{1N}(D) Y_N = F_1(t)$$

$$\mathcal{P}_{N1}(D) Y_1 + \mathcal{P}_{N2}(D) Y_2 + \cdots + \mathcal{P}_{NN}(D) Y_N = F_N(t),$$ (3.90)

or, using matrix calculus, as

$$\begin{bmatrix} \mathcal{P}_{\alpha\beta}(D) \end{bmatrix} \begin{bmatrix} Y_\beta(t) \end{bmatrix} = \begin{bmatrix} F_\alpha(t) \end{bmatrix} \quad (\alpha, \beta = 1, 2, \ldots, N).$$ (3.91)

When we translate this matrix equation into the image space, we have to take care of the initial values

$$\|Y_\beta(0)\|, \|Y_\beta'(0)\|, \ldots, \|Y_\beta^{(n-1)}(0)\|,$$ (3.92)
which are \( n \) values for each of the \( N \) functions \( Y_\beta \). We obtain
\[
\| p_{\alpha \beta}(s) \| \| Y_\beta(s) \| = \| f_\alpha(s) \| + \\
\| c_{\alpha \beta}^n s^{n-1} + \cdots + c_{\alpha \beta}^1 s \| \| Y_\beta(s) \| + \\
\| c_{\alpha \beta}^n s^{n-2} + \cdots + c_{\alpha \beta}^2 s \| \| Y_\beta'(s) \| + \\
\| c_{\alpha \beta}^n s^{n-3} + c_{\alpha \beta}^{n-1} s \| \| Y_\beta''(s) \| + \\
\| c_{\alpha \beta}^n s \| \| Y_\beta'''(s) \| + \] (3.93)

as the image equation. This is a generalization of Eq. (3.20), which can be obtained from Eq. (3.93) for \( N = 1, \alpha = \beta = 1 \) and \( c_n = 1 \).

The behaviour of the determinant \( \Delta(s) = \det[p_{\alpha \beta}(s)] \) is very important for the type of solution of the system (3.93). If and only if the determinant \( \det[c_n^\alpha s^\beta] \) of the coefficients of \( Y_\beta^{(n)} \) in Eq. (3.86) is different from zero, then \( \Delta(s) \) is a polynomial in \( s \) of degree \( nN \). In this case the system is called "normal". On the other hand, if \( \det[c_n^\alpha s^\beta] = 0 \), then \( \det[p_{\alpha \beta}(s)] \) is of a degree \( < nN \) and the system is called "abnormal". We discuss the normal case first and give later an example for the abnormal case.

Similarly, as in the case of one equation, we split the problem into two parts.

3.7.1 **The inhomogeneous system with vanishing initial conditions**

We have from Eq. (3.93)
\[
\| p_{\alpha \beta}(s) \| \| Y_\beta(s) \| = \| f_\alpha(s) \| \] (3.94)

with the solution
\[
\| Y_\beta(s) \| = \| p_{\alpha \beta}(s) \|^{-1} \| f_\alpha(s) \|. \] (3.95)

Denoting by \( \Delta_{\alpha \beta}(s) \) the determinants of those matrices which are built by taking away the \( \alpha \)th row and the \( \beta \)th column of \( \| p_{\alpha \beta}(s) \| \), we have from the theory of matrices with \( \Delta(s) = \det[p_{\alpha \beta}(s)] \):
\[ \left\| p_{\alpha\beta}(s) \right\|^{-1} = \frac{1}{\Delta(s)} \left\| \Delta_{\alpha\beta}(s) \right\| \]  

(3.96)

where T means the transposed matrix. It is clear that \( \Delta_{\alpha\beta}(s) \) is a polynomial in \( s \) of at most degree \( n(N-1) \), so that

\[ \frac{\Delta_{\alpha\beta}(s)}{\Delta(s)} = \mathcal{L}\{ \mathcal{Q}_{\alpha\beta}(t) \} \]  

(3.97)

is an image function, which can be found by partial fraction expansion. Using the convolution theorem (Rule IX) we obtain from Eq. (3.95) together with Eq. (3.96)

\[ \left\| Y_{\beta}(t) \right\| = \left\| \mathcal{Q}_{\alpha\beta}(t) \right\| \left\| F_{\alpha}(t) \right\| \]  

(3.98)

as a solution for Eq. (3.94).

It can be shown that Eq. (3.98) is a solution for Eq. (3.94) even in the case where \( \left\| F_{\alpha}(t) \right\| \) has no image vector \( \left\| f_{\alpha}(s) \right\| \), provided the \( F_{\alpha}(t) \) are continuous for \( t > 0 \).

### 3.7.2 The homogeneous system with arbitrary initial conditions

From Eq. (3.93) we have

\[ \left\| y_{\beta}(s) \right\| = \left\| p_{\alpha\beta}(s) \right\|^{-1} \left\| c_{n}^{\alpha\beta} s^{n-1} + \ldots + c_{1}^{\alpha\beta} Y_{\beta}(s) \right\| + \left\| p_{\alpha\beta}(s) \right\|^{-1} \left\| c_{n-1}^{\alpha\beta} s^{n-2} + \ldots + c_{1}^{\alpha\beta} Y_{\beta}(s) \right\| \]  

\[ + \left\| p_{\alpha\beta}(s) \right\|^{-1} \left\| c_{n-2}^{\alpha\beta} s^{n-3} + \ldots + c_{1}^{\alpha\beta} Y_{\beta}(s) \right\| \]  

\[ + \left\| p_{\alpha\beta}(s) \right\|^{-1} \left\| c_{n-3}^{\alpha\beta} s^{n-4} + \ldots + c_{1}^{\alpha\beta} Y_{\beta}(s) \right\| \]  

(3.99)

Since the degree of \( \Delta_{\alpha\beta}(s) \) is at least lower by \( n \) than the degree of \( \Delta(s) \), we can write

\[ \frac{\Delta_{\alpha\beta}(s)}{\Delta(s)} = \gamma_{1} s^{-n} + \gamma_{2} s^{-n-1} + \ldots \]

and obtain

\[ \mathcal{Q}_{\alpha\beta}(t) = \gamma_{1} t^{-n} + \gamma_{2} t^{-n-1} + \ldots \]
We see here that at least
\[ Q_{\alpha \beta}^{(0)}(\theta) = Q_{\alpha \beta}^{(1)}(\theta) = \ldots = Q_{\alpha \beta}^{(n-2)}(\theta) = 0 \] (3.100)

A multiplication of \( \Delta_{q\beta}(s)/\Delta(s) \) with \( s^\nu \) (\( \nu = 0, 1, \ldots, n-1 \)) corresponds therefore (Rule V) to the \( \nu \)th derivative of \( Q_{q\beta}(t) \). We obtain with Eq. (3.96) and the above considerations
\[
\| Y_\beta^H(t) \| = \left[ \| Q_{\alpha \beta}(t) \| \right] \left[ T \| c_\alpha^\beta d^{n-1} + \ldots + c_2^\beta d + c_1^\alpha \| \right] \| Y_\beta(\theta) \| \\
+ \left[ \| Q_{\alpha \beta}(t) \| \right] \left[ T \| c_\alpha^\beta d^{n-2} + \ldots + c_2^\beta \| \right] \| Y_\beta(\theta) \| \\
+ \left[ \| Q_{\alpha \beta}(t) \| \right] \left[ T \| c_\alpha^\beta \| \right] \| Y_\beta^{(n-4)}(\theta) \|,
\] (3.101)

the desired solution.

The general solution of the system (3.91) with the initial conditions (3.92) is then given by superposition of Eqs. (3.98) and (3.101)
\[
\| Y_\beta(t) \| = \| Y_\beta^I(t) \| + Y_\beta^H(t) \| \] (3.102)

For a "normal" system (3.91) the solution (3.102) always satisfies the nN arbitrarily given initial values (3.92) and the solution \( \| Y_\beta(t) \| \) is continuously connected to these values. This is not the case in general for an "abnormal" system.

Before discussing an example, we note the advantages of the \( \ell \)-transform against the classical method in the case of a system of differential equations.

a) One has to solve only one system of N algebraic equations with N unknowns, the image equations.

b) The special solution of the inhomogeneous system is obtained directly, without going through the general solution of the homogeneous system. One obtains the solution with vanishing initial conditions, which is desired in many practical problems, in a simple way.
c) One gets directly the solution which (in the normal case) satisfies the initial conditions.

d) One can calculate each of the $N$ functions $Y_\beta(t)$ alone, without knowing the other ones.

The matrix calculus described above is very elegant for the general case. In practice, however, many simplifications may be possible and at each step one can decide which is the best procedure for going on, e.g. one can solve the system (3.94) directly, without going explicitly through the inverse $||p_{\alpha\beta}(s)||^{-1}$.

### 3.7.3 Example of the normal case

We discuss a normal system for $n = N = 2$ in detail. This system arises in a mechanical problem. Let

$$
a_1 \frac{d^2 Y_1}{dt^2} + c (Y_1 - Y_2) = -\gamma (1 - \Upsilon(t - T)) \tag{3.103}
$$

$$
a_2 \frac{d^2 Y_2}{dt^2} + c (-Y_1 + Y_2) = 0 \quad (T > 0, a_1 \neq 0, a_2 \neq 0)
$$

be given together with the initial conditions

$$
Y_1(+0) = Y_2(+0) = 0, \quad Y_1(+0) = Y_2(+0) = \infty \tag{3.104}
$$

The derivative of $Y_2$ only, namely $Y_2(t)$, is of interest. The right-hand side is the step function shown in Fig. 9. Here it is unnecessary
to go to the matrix procedure, but for a better understanding of the
previous paragraphs we write Eq. (103) as follows:
\[
\begin{align*}
\alpha_1 y''_1 + \sigma y'_1 + c y'_1 + \sigma y''_2 + \sigma y'_2 - c y_2 &= -\eta (1-u(t-T)) \\
\sigma y''_1 + \sigma y'_1 - c y'_1 + \alpha_2 y''_2 + \sigma y'_2 + c y_2 &= 0.
\end{align*}
\]
(3.105)
We have therefore
\[
\| f_{\alpha\beta}(d) \| = \begin{pmatrix}
\alpha_1 d^2 + c & -c \\
-c & \alpha_2 d^2 + c
\end{pmatrix}
\]
\[
\| F_\alpha(t) \| = \begin{pmatrix}
-\eta (1-U(t-T)) \\
0
\end{pmatrix}
\]
(3.106)
\[
\| Y_\beta(0) \| = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \| Y'_\beta(0) \| = \begin{pmatrix}
\omega \\
\omega
\end{pmatrix}
\]
(\alpha, \beta = 1, 2).
Because of
\[
\det \left[ c_{\alpha\beta} \right] = \begin{vmatrix}
\alpha_1 & 0 \\
0 & \alpha_2
\end{vmatrix} \neq 0,
\]
the system is normal.

From Eqs. (3.103) and (3.104) we obtain the image equations
\[
\alpha_1 (s^2 y_1 - \omega) + c (y_1 - y_2) = -\frac{\eta}{\delta} (1-e^{-T_2})
\]
\[
\alpha_2 (s^2 y_2 - \omega) + c (-y_1 + y_2) = 0
\]
or
\[
(a_1 s^2 + c) y_1 - c y_2 = \alpha_1 \omega - \frac{\eta}{\delta} (1-e^{-T_2})
\]
\[
-c y_1 + (a_2 s^2 + c) y_2 = \alpha_2 \omega,
\]
(3.107)
which correspond to Eq. (3.93).

Since we are only interested in $Y_z(t)$, we can use Cramer’s rule to solve Eq. (3.107) for $y_z(s)$ and obtain

$$M_z(s) = \begin{vmatrix}
    a_1 s^2 + c & a_1 \omega - \frac{c}{\lambda} (1 - e^{-T_\lambda}) \\
    -c & a_2 \omega \\
    a_1 s^2 + c & -c \\
    -c & a_2 s^2 + c
\end{vmatrix} \quad (3.108)$$

The determinant in the denominator is

$$\Delta(s) = (a_1 s^2 + c)(a_2 s^2 + c) - c^2$$

thus a polynomial of order $2 \times 2 = 4$.

Calculating the determinants in Eq. (3.108) results in

$$M_z(s) = \frac{[a_1 a_2 s^2 + c(a_1 a_2)] \omega - \frac{c \eta}{\lambda} (1 - e^{-T_\lambda})}{\Delta^2 [a_1 a_2 s^2 + c(a_1 a_2)]} \quad (3.109)$$

Because of $Y_z(+0) = 0$, we have

$$L\left\{ Y_z^1 \right\} = s M_z = \frac{\omega}{s} - \frac{c \eta (1 - e^{-T_\lambda})}{a_1 a_2 s^2 [a_1 a_2 s^2 + c(a_1 a_2)]}$$

With the abbreviation

$$c \frac{a_1 + a_2}{a_1 a_2} = \lambda^2,$$

we obtain

$$L\left\{ Y_z^1 \right\} = \frac{\omega}{s} - \frac{\eta}{a_1 a_2} \left( \frac{1}{s^2} - \frac{1}{s^2 + \lambda^2} \right)$$

$$+ \frac{\eta}{a_1 a_2} \left( \frac{1}{s^2} - \frac{1}{s^2 + \lambda^2} \right) e^{-T_\lambda} \quad (3.110)$$
For the translation into the original space we use Table 1 and Rule II, thus

\[
\gamma_2^1(t) = c_0 + \frac{\gamma}{a_1 + a_2} \left[ \frac{1}{\lambda} \sin \lambda t - t \right. \\
+ \left( t - T - \frac{1}{\lambda} \sin \lambda (t + T) \right) u(t + T) \right]. 
\tag{3.111}
\]

3.7.4 Example of the abnormal case

The initial value problem of an abnormal system \( \{ \det[c_{n}^{\alpha \beta}] = 0 \} \) of \( N \) differential equations of order \( n \) is essentially different from the respective problem in the normal case. It can be shown that:

a) In the inhomogeneous case, the solution differs from the normal solution (3.98) in that the functions \( F_{\alpha}(t) \) do not appear only in a convolution product but also in isolated positions.

b) Consequently, since the solution has to satisfy the system of equations, certain assumptions with respect to the differentiability of \( F_{\alpha}(t) \) have to be made.

c) The initial conditions for each of the \( N \) unknown functions \( \gamma_{\beta}(t) \) cannot be given in an arbitrary way. Between the initial conditions of the unknowns \( \gamma_{\beta}(t) \) and the perturbation functions \( F_{\alpha}(t) \) exist certain relations. This implies that either less than \( nN \) initial values can be given arbitrarily, or, if \( nN \) values are given, the solution does not satisfy those but other values.

As an example we discuss the first-order system of two equations

\[
\begin{align*}
\gamma_1^1 + 4 \gamma_1 + 2 \gamma_2^1 &= F_1(t) \\
2 \gamma_1^1 + 4 \gamma_2^1 + 9 \gamma_2 &= 0
\end{align*}
\tag{3.112}
\]

The determinant

\[
\det \begin{bmatrix} c_{n}^{\alpha \beta} \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0
\]
shows that the system is abnormal. We try to solve it for

\[ Y_1(t_0) = A_1, \quad Y_2(t_0) = A_2 \]  

(3.113)

The image equations become

\[
(\alpha + 4) Y_1 + 2 \Delta Y_2 = f_1(\alpha) + A_1 + 2A_2 \\
2 \Delta Y_1 + (4\Delta + 9) Y_2 = 2A_1 + 4A_2.
\]  

(3.114)

The determinant \( \Delta(s) \) equals

\[
\Delta(\alpha) = (\alpha + 4)(4\alpha + 9) - 4\Delta^2 = 25\alpha + 36
\]

and is therefore of order \( 1 < 1 \). \( N = 2 \).

The solution of Eq. (3.114) results in

\[
Y_1(\alpha) = f_1(\alpha) \frac{4\alpha + 9}{25\alpha + 36} + 9 \frac{A_1 + 2A_2}{25\alpha + 36} \\
Y_2(\alpha) = f_1(\alpha) \frac{-2\alpha}{25\alpha + 36} + 8 \frac{A_1 + 2A_2}{25\alpha + 36}. 
\]  

(3.115)

Contrary to the normal case, the factors of \( f_1(s) \) do not here belong to the image space of the \( \mathcal{L} \)-transform*, if one considers functions only in the original space. Therefore we cannot apply the convolution theorem directly. We can, however, write

\[
\frac{4\alpha + 9}{25\alpha + 36} = \frac{4}{25} + \frac{81}{25} \frac{1}{25\alpha + 36}
\]

and obtain

\[
\frac{-2\alpha}{25\alpha + 36} = -\frac{9}{25} + \frac{72}{25} \frac{1}{25\alpha + 36}
\]

* It can be shown that necessarily \( f(s) \to 0 \) for \( s \to \infty \) in the case of real \( s \).
\[
Y_1(t) = \frac{1}{25} \left[ 4F_1(t) + \frac{81}{25} F_2(t) \ast e^{-\frac{36}{25}t} + 9(A_1 + 2A_2)e^{-\frac{36}{25}t} \right] \\
(3.116)
\]

\[
Y_2(t) = \frac{1}{25} \left[ -2F_1(t) + \frac{92}{25} F_2(t) \ast e^{-\frac{36}{25}t} + 8(A_1 + 2A_2)e^{-\frac{36}{25}t} \right].
\]

We now take the limit \( t \to +0 \) and find after elementary calculations that both equations (3.116) give the same compatibility condition for the initial values, namely

\[
8A_1 - 9A_2 = 2F_1(+0). \\
(3.117)
\]

In the strict mathematical sense, the initial value problem for the system (3.112) can only be solved if the initial values satisfy Eq. (3.117). The solution is then given by Eq. (3.116). For certain practical purposes, however, the equations (3.116) can be considered as "solutions" to the system in the case of non-compatibility, too. It is nevertheless important to remember that the solutions will then not have the given initial values. We do not discuss this problem in further detail.

Figure 10 shows the functions \( Y_1(t) \) and \( Y_2(t) \) for compatible and non-compatible initial values. The perturbation function is taken as \( F_1(t) = e^{-t} \).
\[ Y_1(0) = 1 \quad \bar{Y}_1(0) = 1 \quad Y_2(0) = 2/3 \quad \bar{Y}_2(0) = 0.88 \]

prescribed \quad \bar{Y}_1(0) = 1 \quad \bar{Y}_2(0) = 1

obtained \quad \bar{Y}_1(0) = 1.24 \quad \bar{Y}_2(0) = 0.88

Fig. 10
4. DIFFERENCE EQUATIONS

4.1 The initial value problem of the linear difference equation

Difference equations become more and more important in technical applications. In principle, they are more elementary than differential equations. The powerful infinitesimal calculus, however, can be applied to the latter, and for this reason the results become more transparent for differential equations.

If \( \omega \) is a real or complex number, then

\[
F(t + \omega) - F(t) = \Delta F(t)
\]

is called the first order difference of \( F(t) \) with respect to the step \( \omega \).

We can construct the second difference by

\[
\Delta (\Delta F(t)) = \Delta^2 F(t) = \Delta F(t + \omega) - \Delta F(t)
\]

\[
= F(t + 2\omega) - 2F(t + \omega) + F(t)
\]

and so on for higher orders.

A linear difference equation of order \( n \) with constant coefficients is defined by

\[
a_n \Delta^n Y(t) + a_{n-1} \Delta^{n-1} Y(t) + \ldots + a_1 \Delta Y(t) + a_0 Y(t) = \mathcal{G}(t)
\]

With explicit expressions for the \( \Delta^r Y(t) \), we can write

\[
c_n Y(t + n\omega) + c_{n-1} Y(t + (n-1)\omega) + \ldots + c_1 Y(t + \omega) + c_0 Y(t) = \mathcal{G}(t)
\]

By suitable substitutions \([t = \omega r, Y(\omega r) = Z(r)]\) and division by \( c_n \) we obtain the normalized equation.
\[ Y(t + n) + c_{n-1} Y(t + n - 1) + \ldots + c_1 Y(t + 1) + c_0 Y(t) = C(t). \]  

(4.3)

Y(t) and G(t) are considered as functions which are defined for all \( t \geq 0 \). There is, however, contrary to differential equations, another possibility. We can restrict \( t \) to entire values \( t = \nu \) \((\nu = 0, 1, 2, \ldots)\). Then the equation remains reasonable also in the case where \( Y(t) \) is defined for \( t = \nu \) only, in other words where \( Y(\nu) \) is a sequence. It is convenient to write \( Y_\nu, G_\nu \) instead of \( Y(\nu), G(\nu) \), or Eq. (4.3) as

\[ Y_{n+\nu} + c_{n-1} Y_{n+\nu-1} + \ldots + c_1 Y_{\nu+1} + c_0 Y_\nu = G_\nu. \]  

(4.4)

Such an equation is called a recurrence equation. One can calculate from Eq. (4.4) with \( \nu = 0 \) and given \( Y_0, Y_1, \ldots, Y_{n-1} \) the value \( Y_n \), then with \( \nu = 1 \) and given \( Y_1, Y_2, \ldots, Y_n \) the value \( Y_{n+1} \), and so on.

Equations (4.3) and (4.4) do not differ essentially from each other. For fixed \( t \) the values of \( Y(t) \) in Eq. (4.3) determine a sequence too. With \( t = \lfloor t \rfloor + \{ t \} = \nu + \{ t \} \) and \( Y(\lfloor t \rfloor + \mu) = y_{\mu} \) \(*\) we can write

\[ Y(t) = Y(\lfloor t \rfloor + \nu) \quad Y(t+1) = Y(\lfloor t \rfloor + \nu + 1) \ldots \]

These numbers are taken from a sequence of equidistant function values, which does not start with \( Y(0) \) but with \( Y(\{ t \}) \). (See Fig. 11.)

---

*) \( \lfloor t \rfloor \) is the entire, \( \{ t \} \) the fractional part of \( t \).
We, therefore, write Eq. (4.3) as

\[ \int_{v}^{t} g_{v} \, dt + c_{n-1} \int_{v+n-1}^{t} g_{v} \, dt + \ldots + c_{1} \int_{v+1}^{t} g_{v} \, dt + c_{0} g_{v} = g_{v} \]  

(4.5)

a recurrence equation which depends on a parameter \( \{ t \} \) \( 0 \leq \{ t \} < 1 \).

We now go back to Eq. (4.4). For \( n \) given values \( Y_{0}, Y_{1}, \ldots, Y_{n-1} \) we obtain a problem which is analogous to the initial value problem for differential equations. Our aim is to obtain \( Y_{n}, Y_{n+1}, \ldots \) from these values \( Y_{0}, Y_{1}, \ldots, Y_{n-1} \), not by recurrence, but by a closed expression.

In order to apply the Laplace transform, we have to construct functions \( Y(t) \) and \( G(t) \) from the sequences \( Y_{v} \) and \( G_{v} \) (we have no rule for the Laplace transform of sequences). We set

\[ Y(t) = Y_{v}, \quad G(t) = G_{v} \quad \text{for} \quad v \leq t < v+1 \]  

(4.6)

and obtain step functions which obviously satisfy an equation of type (4.3):

\[ Y(t+n) + c_{n-1} Y(t+n-1) + \ldots + c_{1} Y(t+1) + c_{0} Y(t) = G(t) \]  

(4.7)

for all \( t \geq 0 \). The initial values \( Y_{0}, \ldots, Y_{n-1} \) determine the values of \( Y(t) \) in the initial interval \( 0 \leq t < n \) in the following way

\[ Y(t) = Y_{0} \quad \text{in} \quad 0 \leq t < 1, \quad Y(t) = Y_{1} \quad \text{in} \quad 1 \leq t < 2, \]  

(4.8)

\[ \ldots, \quad Y(t) = Y_{n-1} \quad \text{in} \quad n-1 \leq t < n. \]

We need exactly these values when we apply the \( \mathcal{L} \)-transform to Eq. (4.7). With the second shifting theorem (Rule III), we obtain as image equation
\[ e^{ns} \left[ y(s) - \int_0^\infty e^{-st} Y(t) \, dt \right] + c_{n-1} e^{(n-1)s} \left[ y(s) - \int_0^{n-1} e^{-st} Y(t) \, dt \right] \\
+ c_1 e^s \left[ y(s) - \int_0^1 e^{-st} Y(t) \, dt \right] + c_0 y(s) = q(s) \] (4.9)

Since \( Y(t) \) is constant in the intervals \( v \leq t < v + 1 \), we can split the integrals into those over these intervals and have

\[ y(s) \left( e^{ns} + c_{n-1} e^{(n-1)s} + \ldots + c_1 e^s + c_0 \right) \\
= \left( e^{ns} + c_{n-1} e^{(n-1)s} + \ldots + c_1 e^s \right) \int_v^{v+1} e^{-st} Y(t) \, dt \\
+ \left( e^{ns} + c_{n-1} e^{(n-1)s} + \ldots + c_2 e^2s \right) \int_v^2 e^{-st} Y(t) \, dt \\
+ e^{ns} \int_{n-1}^\infty e^{-st} Y(t) \, dt + q(s) \] (4.10)

Because of

\[ \int_v^{v+1} e^{-st} Y(t) \, dt \overset{\text{v+1}}{=} \int_v^1 e^{-st} \, dt = \int_v^1 e^{-st} \, dt = -e^{-st} \bigg|_v^{v+1} = \frac{1 - e^{-s}}{s} \] (4.11)

we obtain, after simple calculations, the solution of the image Eq. (4.9), namely
\[ \eta(\lambda) = \frac{q(\lambda)}{e^{\lambda} + \cdots + c_1 e^{\lambda} + c_0} + \gamma \frac{e^{\lambda} - 1}{\lambda} \frac{e^{(n-1)\lambda} + \cdots + c_1 e^{\lambda} + c_0}{e^{\lambda} + \cdots + c_1 e^{\lambda} + c_0} + \gamma \frac{e^{\lambda} - 1}{\lambda} \frac{\lambda}{e^{\lambda} + \cdots + c_1 e^{\lambda} + c_0} \]  

(4.12)

We have to transform this equation to the original space.

4.1.1 The inhomogeneous difference equation with vanishing initial conditions

The image equation (4.12) reads in this case

\[ \eta(\lambda) = \frac{q(\lambda)}{e^{\lambda} + \cdots + c_1 e^{\lambda} + c_0} = \frac{\tilde{q}(\lambda)}{\tilde{p}(e^{\lambda})} \]  

(4.13)

It can be proved that \( 1/p(e^{\lambda}) \) is no image function. Therefore, the convolution theorem is not applicable. We remember the partial fraction expansion of a function \( 1/p(z) \) (Section 3.2) and write for simple zeros \( \alpha_\mu \) of \( p(z) \)

\[ \frac{1}{p(z)} = \sum_{\mu=1}^{n} \frac{1}{p'(\alpha_\mu)} \frac{1}{z - \alpha_\mu} \]

thus
\[
\frac{q^{(s)}}{p(e^s)} = \sum_{\mu=1}^{\infty} \frac{1}{p'(\alpha_{\mu})} \frac{q^{(s)}}{e^{s\alpha_{\mu}}}. \tag{4.14}
\]

We can expand the second fraction into a geometric series

\[
\frac{q^{(s)}}{e^{s\alpha_{\mu}}} = q^{(s)} e^{-s} \frac{1}{1 - \alpha_{\mu} e^{-s}} = \sum_{l=1}^{\infty} \alpha_{\mu}^{l-1} q^{(s)} e^{-ls}
\]

for \( e^{\Re s} > |\alpha_{\mu}| \). Under the assumption that the inverse \( \mathcal{L} \)-transform can be exchanged with the infinite sum, we obtain with the first shifting theorem (Rule II)

\[
\sum_{l=1}^{\infty} \alpha_{\mu}^{l-1} \mathcal{L}^{-1}\{q^{(s)} e^{-ls}\} = \sum_{l=1}^{\infty} \alpha_{\mu}^{l-1} G(t-l),
\]

with the important remark that \( G(t-l) = 0 \) for \( l > t \). The sum on the right-hand side is thus finite and we write it as

\[
\left\{ \begin{array}{ll}
[t] \\
\sum_{l=1}^{[t]} \alpha_{\mu}^{l-1} G(t-l) & \text{for } t \geq 1 \\
0 & \text{for } 0 \leq t < 1.
\end{array} \right.
\]

Introducing this into Eq. (4.14) we have for \( y(s) \) the original function

\[
Y(t) = \left\{ \begin{array}{ll}
\sum_{\mu=1}^{\infty} \frac{1}{p'(\alpha_{\mu})} \sum_{l=1}^{[t]} \alpha_{\mu}^{l-1} G(t-l) & \text{for } t \geq 1 \\
0 & \text{for } 0 \leq t < 1
\end{array} \right. \tag{4.15}
\]
which is the solution of the inhomogeneous difference equation (4.4) with vanishing initial conditions.

We can simplify Eq. (4.15) considerably. According to Eq. (4.6), and when changing the order of summation, we have for $n \leq t < n + 1$ ($n \geq 1$).

\[ Y_n = \sum_{\ell=1}^{n} C^n_{n-\ell} \sum_{\mu=1}^{\ell-1} \frac{\alpha_{\mu}^{\ell-1}}{\rho'(\alpha_{\mu})} \quad , \quad Y_0 = 0 \]

In Section 3.3.1 we introduced a function

\[ Q(t) = \sum_{\mu=1}^{n} \frac{1}{\rho'(\alpha_{\mu})} e^{\alpha_{\mu}t} \]

in connection with a differential equation (3.16) which had the same coefficients $c_0, c_1, \ldots, c_{n-1}$ as the difference equation (4.4). We obtain by differentiation

\[ \sum_{\mu=1}^{n} \frac{\alpha_{\mu}^{\ell-1}}{\rho'(\alpha_{\mu})} = Q^{(\ell-1)}(0) \]

thus

\[ Y_n = \sum_{\ell=1}^{n} C^n_{n-\ell} \frac{Q^{(\ell-1)}(0)}{C^n_{n-\ell}} \]

for $n \geq 1$, $Y_0 = 0$.

Because of the relations (3.28) we can write finally
\[
Y_v = \begin{cases} 
\sum_{l=n}^{\sqrt{2}} Q^{(l-1)}(0) C_{v-l} & (v \geq n) \\
0 & (v = 0, 1, 2, \ldots, n-1)
\end{cases}
\] (4.17)

It can be shown that this sequence satisfies the difference equation (4.4) with initial values \( Y_0 = Y_1 = \ldots = Y_{n-1} = 0 \) independently of the assumptions which were required during the calculation (e.g. existence of \( \ell \)-transforms of \( Y \) and \( G \), exchange of \( \ell^{-1} \) with an infinite sum, etc.)

4.1.2 The homogeneous difference equation with arbitrary initial conditions

Here we have to set \( g(s) = 0 \) on the right-hand side of Eq. (4.12). The factors of the initial conditions in Eq. (4.12) are composed with functions

\[
\frac{e^s - 1}{s} \frac{e^{ks}}{\rho(e^s)} \quad (k = 0, 1, \ldots, n-1)
\]

Because of Eq. (3.12) we obtain the partial fraction expansion

\[
\frac{e^s - 1}{s} \frac{e^{ks}}{\rho(e^s)} = \sum_{\mu=1}^{\omega} \alpha_{\mu} \frac{k^\mu}{\rho(\alpha_{\mu})} \frac{e^s - 1}{s} \frac{1}{e^s - \alpha_{\mu}} \quad (4.18)
\]

We need, therefore, the original function of

\[
\frac{e^s - 1}{s} \frac{1}{e^s - \alpha}
\]
which is given by

\[
\alpha^{[t]} = \int_{-1} e^{s-1} \left\{ \frac{e^{s-1}}{s} - \frac{1}{e^{s} - \alpha} \right\}
\]

(4.19)

for \( \text{Re } s > \log|\alpha|, \alpha \text{ complex } \neq 0 \). This can be proved in a straightforward way by calculating

\[
\int_{0}^{\infty} e^{-st} \alpha^{[t]} dt = \sum_{v=0}^{\infty} \alpha^{v} \int_{0}^{\infty} e^{-vt} dt
\]

and summing up a geometrical series. Thus we have from Eq. (4.18)

\[
\int_{-1} e^{s-1} \frac{e^{ks}}{p(e^{s})} = \sum_{\mu=1}^{n} \frac{\alpha^{\mu}}{p^{(\mu)}(\alpha^{\mu})} \alpha^{[t]} = \sum_{\mu=1}^{n} \frac{\alpha^{[t]} + k}{p^{(\mu)}(\alpha^{\mu})} = Q^{([t]+k)}(0)
\]

which gives

\[
Y(t) = y_o \left[ Q^{([t]+n-1)}(0) + c_{n-1} Q^{([t]+n-2)}(0) + \ldots + c_1 Q^{([t])}(0) \right] + y_1 \left[ Q^{([t]+n-2)}(0) + c_{n-1} Q^{([t]+n-3)}(0) + \ldots + c_1 Q^{([t])}(0) \right] + \ldots + y_{n-1} Q^{([t])}(0)
\]
For the sequence $Y_v$ we obtain

$$Y_v = \sum_{l=0}^{n-1} \sum_{l+k=1}^{n-l} c_{l+k} Q^{(v+k-1)} \rho'(\lambda_{i+1}) \quad (c_{n=1})$$

(4.20)

or

$$Y_v = \sum_{l=0}^{n-1} \sum_{l+k=1}^{n-l} c_{l+k} \sum_{\mu=1}^{\lambda_{i+1}} Q^{(v+k-1)} \rho'(\lambda_{i+1}) \quad (4.21)$$

If some of the $\lambda_i$ are multiple roots, the results (4.17) and (4.20) do not change in principle. Instead of Eq. (3.26), Eq. (3.27) has to be taken for $Q(t)$. We therefore summarize the result as follows.

The difference equation with a discontinuous variable (the recurrence equation)

$$Y_{v+n} + c_{n-1} Y_{v+n-1} + \ldots + c_1 Y_{v+1} + c_0 Y_v = Q' \quad (4.22)$$

with given initial values $Y_0, Y_1, \ldots, Y_{n-1}$, has the solution

$$Y_v = \sum_{l=n}^{n-1} Q^{(v-l)}(0) Q_{v-l} + \sum_{l=0}^{n-1} \sum_{l+k=1}^{n-l} c_{l+k} Q^{(v+k-1)} \quad (4.23)$$
for \( n \geq n \) and \( c_n = 1 \). \( q(t) \) is the original function for \( 1/p(s) \), where

\[
P_0(n) = n^m + \sum_{n-1}^m c_i n^i + \cdots + c_1 n + c_0.
\]

Before going on to an example, we give the solution of the initial value problem for the difference equation (4.3) with a continuous variable \( t \).

For given values of a function \( Y(t) \) in \( 0 \leq t \leq n \) (in the whole interval) the solution of Eq. (4.3) results in

\[
Y(t) = \sum_{\ell=0}^{[t]} \sum_{k=1}^{(\ell-1)} Q^{(\ell-1)}(0) \cdot \ell + \ell \cdot c_\ell + \sum_{\ell=0}^{m-\ell} Y(\ell) \cdot \sum_{k=1}^{(\ell+k-1)} Q^{(\ell+k-1)}(0).
\]  

\[ (4.24) \]

4.1.3 **Examples**

We solve

\[
Y_{v+5} - 2Y_{v+3} + 2Y_{v+2} - 3Y_{v+1} + 2Y_v = 0,
\]  

\[ (4.25) \]

with the initial condition

\[
Y_0 = 0, \ Y_1 = 0, \ Y_2 = 9, \ Y_3 = -2, \ Y_4 = 23.
\]  

\[ (4.26) \]

The characteristic polynomial

\[
P_0(s) = s^5 - 2s^3 + 1s^2 - 3s + 2.
\]
can be written as
\[ \Phi(s) = (s - 1)^2 (s + 2) (s - i) (s + i). \]

The derivative is given by
\[ \Phi'(s) = 5 s^4 - 6 s^2 + 4 s - 3. \]

According to Section 3.2, we obtain for the partial fraction expansion
\[
\frac{1}{\Phi(s)} = \frac{-2/9}{s - 1} + \frac{1/6}{(s - 1)^2} + \frac{1/45}{s + 2} + \frac{(1/5) s}{s^2 + 1} + \frac{1/10}{s^2 + 1},
\]
hence
\[
Q(t) = -\frac{2}{9} e^t + \frac{1}{6} t e^t + \frac{1}{45} e^{-2t} + \frac{1}{5} \cos t + \frac{1}{10} \sin t.
\]

For the \( \lambda \)th derivative of \( Q(t) \) we find
\[
Q^{(\lambda)}(t) = -\frac{2}{9} e^t + \frac{1}{6} (\lambda + t) e^t + \frac{1}{45} (-2^\lambda) e^{-2t}
\]
\[
+ \frac{1}{5} \cos \left(t + \frac{\lambda \pi}{2}\right) + \frac{1}{10} \sin \left(t + \frac{\lambda \pi}{2}\right).
\]

Therefore for \( t = 0 \)
\[
Q^{(\lambda)}(0) = -\frac{2}{9} + \frac{\lambda}{6} + \frac{1}{45} (-2) \lambda + \frac{1}{5} \cos \frac{\lambda \pi}{2} + \frac{1}{10} \sin \frac{\lambda \pi}{2}.
\]
The homogeneous part of Eq. (4.23) gives then:

\[ Y^\gamma = (c_3 Y_2 + Y_4) Q^{(\gamma)}(0) + Y_3 Q^{(\nu + 1)}(0) + Y_2 Q^{(\nu + 2)}(0) \]

\[ = 5 Q^{(\nu)}(0) - 2 Q^{(\nu + 1)}(0) + 9 Q^{(\nu + 2)}(0). \]

After elementary calculation, we obtain for the solution of the homogeneous equation (4.25):

\[ Y^\nu = 2 \nu + (-2)^\nu - c_0 \frac{\sqrt{\nu}}{2} \quad (\nu \geq 5), \]

e.g. \( Y_5 = -22, Y_6 = 75, Y_7 = -114 \) etc.

As a second example we derive the formulae for the very common second-order recurrence relation

\[ Y_{\nu + 2} + c_1 Y_{\nu + 1} + c_0 Y_{\nu} = \mathcal{A}_\nu, \quad (4.28) \]

with given \( Y_0 \) and \( Y_1 \) and simple roots \( \alpha_1 \) and \( \alpha_2 \) of the characteristic equation

\[ \varphi(\lambda) = \lambda^2 + c_1 \lambda + c_0 = (\lambda - \alpha_1)(\lambda - \alpha_2) \]

\[ \varphi(\alpha_1) = \alpha_1 - \alpha_2 \quad , \quad \varphi(\alpha_2) = \alpha_2 - \alpha_1 \]

Because of

\[ Q(\nu) = \frac{1}{\alpha_1 - \alpha_2} \left( e^{\alpha_1 \nu} - e^{\alpha_2 \nu} \right), \]

we have

\[ Q^{(\lambda)}(0) = \frac{\lambda^\lambda - \lambda^\lambda}{\alpha_1 - \alpha_2}. \]
Putting this value into Eq. (4.23) we find the solution of Eq. (4.28)

\[
Y_v = \sum_{l=2}^{\nu} \frac{\alpha_{l-1}}{\alpha_{l} - \alpha_{2}} \frac{\alpha_{l-1}^{\nu-1}}{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}} A_{\nu - l} + Y_0 \left[ C_1 \frac{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}}{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}} + \frac{\alpha_{l}^{\nu+1} - \alpha_{2}^{\nu+1}}{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}} \right] + Y_1 \frac{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}}{\alpha_{l}^{\nu} - \alpha_{2}^{\nu}}.
\]

With \( c_1 = -\alpha_{1} - \alpha_{2} \), \( c_0 = \alpha_{1} \), \( \alpha_{2} \), this equation can be written as

\[
Y_v = \frac{1}{\alpha_{1} - \alpha_{2}} \sum_{l=2}^{\nu} (\alpha_{l}^{\nu-1} - \alpha_{2}^{\nu-1}) A_{\nu - l}
\]

(4.29)

\[-Y_0 c_0 (\alpha_{1}^{\nu-1} - \alpha_{2}^{\nu-1}) + Y_1 (\alpha_{1}^{\nu} - \alpha_{2}^{\nu})^2 \]

for \( \nu \geq 2 \). In the case of real \( c_0 \), \( c_1 \) and conjugate complex zeros \( \alpha_{1} \), \( \alpha_{2} \) we can introduce

\[
\alpha_{1} = \beta e^{i\gamma}, \quad \alpha_{2} = \beta e^{-i\gamma}
\]

and obtain, with

\[
Q^{(\lambda)}(0) = \beta^{\lambda-1} \frac{\sin \lambda \gamma}{\sin \gamma},
\]

\[
Y_v = \frac{1}{\sin \gamma} \sum_{l=2}^{\nu} \beta^{l-2} \sin [(l-1)\gamma] A_{\nu - l} \]

(4.30)

\[-Y_0 c_0 \beta^{\nu-2} \sin [(\nu-1)\gamma] + Y_1 \beta^{\nu-1} \sin \nu \gamma \]
4.2 An example of a difference equation in the image space

It is sometimes possible to obtain a special solution of a given difference equation for a continuous variable in the following way.

Let

\[ y_f(n) - y_f(n+1) = \frac{1}{2} \]  \hspace{1cm} (4.31)

be given. We consider this equation as given in the image space. The original equation can be written as (Rule IV)

\[ (1 - e^{-t}) y(t) = t \]

thus

\[ y(t) = \frac{t}{1 - e^{-t}} \]

With the definition of the \( \mathcal{L} \)-transform we obtain

\[ y_f(\lambda) = \int_0^\infty e^{-\lambda t} \frac{t \, dt}{1 - e^{-t}} \]  \hspace{1cm} (4.32)

a special solution of Eq. (4.31).
5. PARTIAL DIFFERENTIAL EQUATIONS

In Section 3 we applied the $\mathcal{L}$-transform to ordinary differential equations with constant coefficients. The procedure became very simple, because of the fact that the $\mathcal{L}$-transform removes the derivatives and gives an algebraic equation. In the case where the original equation contains derivatives with respect to two (or more) variables, say $x$ and $t$, the equation is called a partial differential equation. Here, a $\mathcal{L}$-transform with respect to $t$ will remove the derivatives for this variable, but will leave unchanged those with respect to $x$. We obtain, therefore, in the case of two variables, an ordinary differential equation. This is a very much simpler problem, which we can handle with the methods described.

In order to apply the $\mathcal{L}$-transform, we have to assume that $t$ varies in the interval $0 \leq t < \infty$. The interval of $x$ may be finite or infinite to one or both sides of the $xt$-plane. We have therefore as basic area for a partial differential equation either a half-strip, a quarter-plane or a half-plane in the $xt$-plane (Fig. 12).

\[ U(x,t) \]

In order to find a certain solution $U(x,t)$ of a partial differential equation, we have to specify the values of $U(x,t)$ or of its derivatives or of combinations of both on the boundary (or on parts of it). The values specified for fixed $x$ are functions of $t$ alone. They are called "boundary values". The values for fixed $t$ are functions of $x$ alone. They are called "initial values", especially if they are given for $t = 0$. 
The differential equation together with these conditions is called a "boundary value problem". It comes out very often that the mathematical expression of the final solution of such a problem becomes meaningless on the boundaries. It is, however, then possible to understand the boundary values as limits of $U(x,t)$ for $t$ or $x$ converging to the boundary.

5.1 The equation of heat transfer or diffusion

We discuss the problem of a linear heat conductor, which is extended from $x = 0$ to $x = l$. For a time $t$, its temperature is described at the point $x$ by a function $U(x,t)$. With a suitable normalization of $x$ and $t$ the function $U(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \quad (0 < x < l, t > 0). \quad (5.1)$$

There exists a wide choice of boundary conditions for this problem. We choose here very simple ones, namely the initial temperature $U_0(x)$ for $t = 0$ and the temperatures at the end points $x = 0$ and $x = l$, namely $A_0(t)$ and $A_1(t)$, respectively. $U_0(x)$, $A_0(t)$ and $A_1(t)$ are assumed to be continuous functions. It is physically plausible that these conditions are sufficient for the determination of $U(x,t)$. Mathematically speaking, we prescribe

$$\lim_{t \to +0} U(x,t) = U_0(x) \quad (0 < x < l) \quad (5.2)$$

$$\lim_{x \to +0} U(x,t) = A_0(t), \quad \lim_{x \to l-0} U(x,t) = A_1(t) \quad (t > 0) \quad (5.3)$$

When we apply the $\mathcal{L}$-transform with respect to $t$ to the function $U(x,t)$

$$\mathcal{L}\{U(x,t)\} = \int_0^\infty e^{-st} U(x,t) \, dt = u(x,s) \quad (0 < x < l)$$

we obtain a function $u(x,s)$. $x$ plays the role of a parameter. In order to construct the image equation of (5.1) we have to assume that

*) Here again, these assumptions are preliminary ones.
i) \( \mathcal{L} \left\{ \frac{\partial u}{\partial t} \right\} \) exists

ii) \( \mathcal{L} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = \frac{\partial}{\partial x} \mathcal{L} \left\{ u \right\} = \frac{\partial}{\partial x} u(x, \lambda) \),

that means the exchange of the operators \( \mathcal{L} \) and \( \frac{\partial}{\partial x} \) is allowed.

iii) the \( \mathcal{L} \)-transform can be exchanged with the limit operations \( x \to +0 \) and \( x \to l - 0 \).

Because of Rule V,

\[
\mathcal{L} \left\{ \frac{\partial u}{\partial t} \right\} = \lambda u(x, \lambda) - u(x, +0)
= \lambda u(x, \lambda) - u_0(x) \tag{5.4}
\]

we obtain as image equation:

\[
\frac{\partial^2 u(x, \lambda)}{\partial x^2} = \lambda u(x, \lambda) - u_0(x), \tag{5.5}
\]

or, because \( x \) is the only variable involved in differentiation:

\[
\frac{d^2 u(x, \lambda)}{dx^2} = \lambda u(x, \lambda) - u_0(x). \tag{5.6}
\]

Of course, the boundary conditions (5.3) have to be transformed as well. With the assumption (iii) we have

\[
\mathcal{L} \left\{ A_0(t) \right\} = \mathcal{L} \left\{ \lim_{x \to +0} U(x, t) \right\} = \lim_{x \to +0} \mathcal{L} \left\{ U(x, t) \right\}
= \lim_{x \to +0} u(x, \lambda)
\]

\[
\mathcal{L} \left\{ A_1(t) \right\} = \mathcal{L} \left\{ \lim_{x \to l-0} U(x, t) \right\} = \lim_{x \to l-0} \mathcal{L} \left\{ U(x, t) \right\}
= \lim_{x \to l-0} u(x, \lambda)
\]
and, because of $L \{ A_0(t) \} = a_0(s)$, $L \{ A_1(t) \} = a_1(s)$,

$$
\lim_{\lambda \to +\infty} u(x, \lambda) = a_0(s) \tag{5.7}
$$

$$
\lim_{\lambda \to -\infty} u(x, \lambda) = a_1(s)
$$

We summarize as follows:

The boundary value problem (5.1-5.3) of a partial differential equation of two variables in the original space is transformed to the boundary value problem of an ordinary differential equation (5.6-5.7) in the image space. If the latter is solved, the solution of the original one is found by the $L^{-1}$-transform.

We have therefore the following scheme

```
Scheme

Original space \{ partial diff. eq. + initial cond. + boundary cond. \} \rightarrow \text{solution}

$\mathcal{L}$-transform

Image space \{ ordinary diff. eq. + boundary cond. \} \rightarrow \text{solution}$\mathcal{L}^{-1}$-transform
```

The problem in the image space is much simpler than the original one. In fact, we have already solved it in Section 3.4. We take the results from there. Comparing Eq. (3.34)

$$
Y' = a^2 Y + F(t)
$$

with Eq. (5.6)

$$
u''(x,s) = s u(x,s) - U_0(x),
$$

we see that we have to replace in Section 3.4 $Y$ by $u$, $a$ by $\sqrt{s}$, $t$ by $x$, $F(t)$ by $-U_0(x)$, $Y(0)$ by $a_0(s)$, and $Y(\ell)$ by $a_1(s)$.
5.1.1 The linear heat conductor of finite length with vanishing initial temperature

We set $U_0(x) = 0$ and obtain for arbitrary boundary values $a_0(s)$ and $a_1(s)$ from Eq. (3.36)

$$u(x,s) = a_0(s) u_0(x,s) + a_1(s) u_1(x,s)$$  \hspace{1cm} (5.8)

with

$$U_0(x,s) = \frac{\sinh (l - x) \sqrt{s}}{\sinh l \sqrt{s}}$$

and

$$U_1(x,s) = \frac{\sinh x \sqrt{s}}{\sinh l \sqrt{s}}$$  \hspace{1cm} (5.9)

We take for $\sqrt{s}$ the main branch. It can be shown that these functions are analytic for $\Re s > 0$.

The problem is now to transform the functions $u_0(x,s)$ and $u_1(x,s)$ into the original space. In the table of Erdélyi et al. we find a formula

$$\frac{\sinh x \sqrt{s}}{\sinh l \sqrt{s}} = \sum \left\{ \frac{1}{l} \frac{\partial}{\partial x} \frac{1}{2l} \Theta_4 \left( \frac{x}{2l} \left| \frac{j\pi t}{l^2} \right. \right) \right\} \left( -l < x < l \right)$$  \hspace{1cm} (5.10)

The $\Theta_4$ function is one of the so-called theta functions. These functions are related to the elliptic functions and therefore to the elliptic integrals. They are often written with the letter $\vartheta$ instead of $\Theta$.

We will use this letter $\vartheta$, too. In the book of Erdélyi et al. on higher transcendental functions, we find the definition

$$\vartheta_4 \left( \nu | \tau \right) \equiv \vartheta_4 \left( \nu, q \right) \equiv \Theta_4 \left( \nu, e^{j\pi \tau} \right)$$

with $q = e^{j\pi \tau}$, and the remark that often by definition

*) We have replaced $p$ by $s$.
\[ \theta_4(v, q) = \theta_0(v, q) \] *.

This leads us to a formula given by Magnus et al. 3)

\[ \varphi_0^M(z, \tau) = (-i \tau)^{-1/2} \sum_{\nu = -\infty}^{\infty} e^{-i \pi (\nu + 1/2)^2 / \tau} \]

\[ \equiv \varphi_0(z, t) = \frac{1}{\sqrt{i \pi t}} \sum_{\nu = -\infty}^{\infty} e^{-(\nu + 1/2)^2 / t} \] (5.11)

\[ \equiv \Theta_4^E(z | \tau) \]

where \( \tau = j \omega t \).

For \( u_0(x, a) \), we have to transform

\[ \frac{\sinh (l - x) \sqrt{s}}{\sinh l \sqrt{s}} \]

*) Because of the fact that the definitions of their function symbols vary almost from one book to the other, to work with theta functions is rather dangerous. Extreme care has to be taken in this respect. For instance, Erdélyi et al., Magnus et al., Doetsch (Handbuch, Vol. III, p. 190), and Gradshteyn-Ryzhik use different notations. When labelling the definitions of Erdélyi, Magnus, Doetsch, Gradshteyn-Ryzhik by \( E, M, D, G \) respectively, we find that

\[ \Theta_4^E(v | \tau) \equiv \Theta_4^M(v, e^{i \pi \tau}) \equiv \varphi_0^M(v, \tau) \equiv \varphi_0^M(v, j \pi t) \]

\[ \equiv \varphi_4^D(v, t) \equiv \varphi_4^G(\pi v) \equiv \Theta_4^E(v) \]

In addition, \( \lambda = 4 \) is often replaced by \( \lambda = 0 \), or omitted altogether, so that

\[ \varphi_4 \equiv \varphi_0 \equiv \Theta_4 \equiv \Theta_0 \equiv \Theta \]

We follow here, if not stated otherwise, the notation of Doetsch and write \( \varphi_\lambda(v, t) \) for \( \lambda = 0, 1, 2, 3. \)
Using formulae (5.10) and (5.11), we write

\[
2^{-1} \left\{ \frac{\sinh (l-x) \sqrt{\frac{t}{2}}} {\sinh \frac{x}{\sqrt{2}} \sqrt{\frac{t}{2}}} \right\} = \frac{1}{\ell} \frac{\partial}{\partial (l-x)} \Theta_4 \left( \frac{l-x}{2 \ell}, \frac{t}{\ell^2} \right)
\]

\[
= \frac{1}{\ell} \frac{\partial}{\partial (l-x)} \Theta_0 \left( \frac{l-x}{2 \ell}, \frac{t}{\ell^2} \right)
\]

\[
= \frac{1}{\sqrt{\pi} t^{3/2}} \sum_{\nu = -\infty}^{\infty} e^{-\ell^2 \left( \frac{l-x}{2 \ell} + \nu - \frac{1}{2} \right)^2 / 4t} \quad (5.12)
\]

\[
= \frac{1}{2\sqrt{\pi} t^{3/2}} \sum_{\nu = -\infty}^{\infty} \frac{(-2\nu l + x)^2}{4t}
\]

\[
= \frac{1}{2\sqrt{\pi} t^{3/2}} \sum_{\nu = -\infty}^{\infty} \frac{(2\nu l + x)^2}{4t}
\]

In this calculation we have used the fact that \(-\nu\) may be replaced in the series by \(+\nu\).

In Section 2.14 we introduced the function

\[
\psi(k, t) = \frac{k}{2\sqrt{\pi} t^{3/2}} e^{-k^2/4t}
\]

With this definition, Eq. (5.12) can be written as

\[
U_0(x, t) = \sum_{\nu = -\infty}^{\infty} \psi(2\nu l + x, t) \quad (5.13)
\]
In a similar way, we can calculate the inverse of \( u_1(x,s) \)
\[
\mathcal{L}^{-1}\left\{ u_1(x,s) \right\} = \mathcal{U}_1(x,t) = \sum_{\nu = -\infty}^{\infty} \nu \left( 2\sqrt{\nu+1} - x, t \right). \tag{5.14}
\]

With the convolution theorem (Rule IX) we obtain as solution for the equation (5.1), with vanishing initial temperature, the function
\[
\mathcal{U}(x,t) = A_0(t) \ast \mathcal{U}_0(x,t) + A_\nu(t) \ast \mathcal{U}_\nu(x,t) \quad (0 < x < \ell) \tag{5.15}
\]

or
\[
\mathcal{U}(x,t) = \int_0^t A_0(t-\tau) \sum_{\nu = -\infty}^{\infty} \nu \left( 2\sqrt{\nu+1} + x, \tau \right) d\tau + \int_0^t A_\nu(t-\tau) \sum_{\nu = -\infty}^{\infty} \nu \left( 2\sqrt{\nu+1} - x, \tau \right) d\tau.
\]

It is worth while noting that another expression for \( \mathcal{U}(x,t) \) may be found by using a different series for the \( \vartheta_0 \) function. We find in Magnus et al. the relation
\[
\vartheta_0^M(v, \tau) = 1 + 2 \sum_{\nu = 1}^{\infty} (-1)^{\nu-1} v^2 \cos(2\pi \nu \tau).
\tag{5.16}
\]

With \( r = j\omega t \) and \( q = e^{j\omega t} = e^{-\pi^2 t} \) we can proceed as in Eq. (5.12) and find
\[
\mathcal{U}(x,t) = \frac{2\pi}{\ell^2} \left\{ A_0(t) \ast \sum_{\nu = 1}^{\infty} \nu \sin \frac{\pi \nu}{\ell} x \right\} e^{-\nu^2(\pi/\ell)^2 t}
\]
\[
- A_\nu(t) \ast \sum_{\nu = 1}^{\infty} (-1)^{\nu-1} \nu \sin \frac{\pi \nu}{\ell} x \right\} e^{-\nu^2(\pi/\ell)^2 t} \quad \tag{5.17}
\]

\( (0 < x < \ell) \).

Figure 13 shows the function \( U(x,t) \) for the special boundary conditions \( A_0(t) = A = \text{const.}, A_\nu(t) = B = \text{const.}, \) and \( \ell = 1 \). Formula (5.17) gives in this case
\[
\mathcal{U}(x,t) = \sum_{\nu = 1}^{\infty} \frac{\nu}{\pi} \left( \frac{A - (-1)^\nu B}{\nu} \sin \frac{\pi \nu}{\ell} x \right) e^{-\nu^2(\pi/\ell)^2 t} \left( \nu^2 - \ell^2 \right).
\]
\[ U(x,t) = \frac{2}{\pi} \sum_{v=1}^{\infty} \frac{A(-1)^v B}{v} \sin(v\pi x)(1-e^{-v^2\pi^2 t}) \]

**Fig. 13**

**Linear Heat Conductor**

*Vanishing Initial Temperature*

*Constant Boundary Temperatures* $A_0(t)=A, A_1(t)=B$
5.1.2 The linear heat conductor of finite length with vanishing boundary temperatures

In the case \( A_0(t) = A_1(t) = 0 \) and \( U_0(x) \) arbitrary, the equation (5.6) is an inhomogeneous one with vanishing boundary conditions. We already solved this problem in Section 3.4 [Eq. (3.39)] and obtained

\[
U(x, t) = \int_0^l \gamma(x, \xi; t) U_0(\xi) d\xi
\]

with

\[
\gamma(x, \xi; t) = \begin{cases} \frac{\sinh \xi \sqrt{s} \sinh (l-x) \sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & (0 \leq \xi \leq x) \\ \frac{\sinh x \sqrt{s} \sinh (l-\xi) \sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} & (x \leq \xi \leq l) \end{cases}
\]

(5.18)

(5.19)

After simple manipulation, this function can be written as

\[
\gamma(x, \xi; t) = \begin{cases} \frac{\cosh (x-\xi-l) \sqrt{s} - \cosh (x+\xi-l) \sqrt{s}}{2 \sqrt{s} \sinh l \sqrt{s}} & (0 \leq \xi \leq x) \\ \frac{\cosh (\xi-x-l) \sqrt{s} - \cosh (\xi+x-l) \sqrt{s}}{2 \sqrt{s} \sinh l \sqrt{s}} & (x \leq \xi \leq l) \end{cases}
\]

(5.20)

In the table of Erdélyi et al., we find the correspondence

\[
\int_{-1}^1 \left\{ \frac{\cosh x \sqrt{s}}{\sqrt{s} \sinh l \sqrt{s}} \right\}^2 = \frac{1}{l} \Theta_4 \left( \frac{x}{2l} \bigg| \frac{j \pi t}{l^2} \right)
\]

or, with \( x = y - \ell \) and \( \tau = j \pi t \):

\[
(- \ell \leq x \leq \ell)
\]
\[
\frac{1}{\ell} \Theta_4 \left( \frac{y-l}{2\ell} \right| \frac{t}{\ell^2} \right) = \frac{1}{\ell} \vartheta_0^M \left( \frac{y-l}{2\ell}, \frac{t}{\ell^2} \right) = \frac{1}{\ell} \vartheta_0^M \left( \frac{y-l}{2\ell} - \frac{1}{2}, \frac{t}{\ell^2} \right).
\]

In the tables of Magnus et al., one finds that

\[
\vartheta_0^M \left( z - \frac{1}{2}, \tau \right) \equiv \vartheta_3^M \left( z, \tau \right).
\]

We can write therefore

\[
\sum_{n=1}^{\infty} \frac{\cosh(y-n\ell)\sqrt{s}}{\sqrt{s} \sinh n\ell \sqrt{s}} = \frac{1}{\ell} \vartheta_3 \left( \frac{y}{2\ell}, \frac{t}{\ell^2} \right) = \frac{1}{\ell} \vartheta_3 \left( \frac{y-l}{2\ell}, \frac{t}{\ell^2} \right), \quad (0 < y < 2\ell)
\]

With a formula given by Magnus et al., we obtain finally:

\[
\frac{1}{\ell} \vartheta_3 \left( \frac{y}{2\ell}, \frac{t}{\ell^2} \right) = \frac{1}{\ell} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+2\ell n)^2}{4t}}
\]

When we transform \(\gamma(x,\xi;s)\) to the original space, we obtain for the first term of the first line of Eq. (5.20) the original function \(\theta_3[ (x-\xi)/2t, t/\ell^2]\), and for the same term of the second line \(\theta_3[ (\xi-x)/2t, t/\ell^2]\). From the definition (5.23) we see that \(\theta_3(-v,t) = \theta_3(v,t)\). Therefore we can write, for the original function of \(\gamma(x,\xi;s)\):
\[ \Gamma(x, \xi; t) = \frac{1}{2l} \left[ \mathcal{D}_3 \left( \frac{x-\xi}{2l}, \frac{t}{l^2} \right) - \mathcal{D}_3 \left( \frac{x+\xi}{2l}, \frac{t}{l^2} \right) \right] \]

\[ = \frac{1}{2l \sqrt{\pi t}} \sum_{v=\infty}^{-\infty} \left[ e^{-\frac{(x-\xi+2vl)^2}{4t}} - e^{-\frac{(x+\xi+2vl)^2}{4t}} \right] \]

\[ = \frac{2}{l} \sum_{v=0}^{\infty} e^{-v^2 \left( \frac{\pi}{l} \right)^2 t} \sin \frac{\pi}{l} x \cdot \sin \frac{\pi}{l} \xi \]

or, using the definition

\[ \chi(k, t) = \frac{1}{\sqrt{\pi t}} e^{-\frac{k^2}{4t}} = \sum_{v=-\infty}^{\infty} \left\{ \frac{1}{\sqrt{l}} e^{-\frac{k^2}{4t}} \right\} \]

\[ \Gamma(x, \xi; t) = \frac{1}{2l} \sum_{v=\infty}^{\infty} \left[ \chi(x-\xi + 2vl, t) - \chi(x+\xi + 2vl, t) \right] \]

which gives (with the interchange of the \(\mathcal{L}\)-transform with the integral)

*) The last line can be proved in a similar way as for Eq. (5.17), using the relation

\[ \mathcal{D}_3 (\nu, t) = 1 + \sum_{v=1}^{\infty} e^{-\frac{\nu^2 \pi^2 t}{2}} \cos 2\nu \pi \nu = \frac{1}{\sqrt{\pi t}} \sum_{v=\infty}^{-\infty} e^{-\left( \nu + \nu \right)^2 / 4t} \]

This relation, by the way, can be found with the help of the Laplace transform (cf. Doetsch, Handbuch, Vol. II, p. 237).
(5.26)

The original function of Eq. (5.18).

Figure 14 shows \( U(x,t) \) for the initial temperature \( U_0(x) = \frac{1}{2} \cos(x/2)x \) and \( l = 1 \). In this case we obtain from Eq. (5.26)

\[
U(x,t) = \frac{A^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \cos \frac{n^2\pi^2 t}{l^2}.
\]

We can summarize as follows: the boundary value problem

\[
\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \quad (0 < x < l, t > 0)
\]

\[
\lim_{t \to +0} U(x,t) = U_0(x) \quad (0 < x < l)
\]

\[
\lim_{x \to \pm 0} U(x,t) = U_0(\pm \infty) \quad (t > 0)
\]

has, independently of the assumptions made in Section 5.1, the solution obtained by superposition of Eqs. (5.15) and (5.26), provided \( A_0(t) \), \( A_1(t) \), \( U_0(x) \) are continuous for \( t > 0 \) or \( 0 < x < l \), respectively.

5.1.3 The linear heat conductor of semi-infinite length

We discuss briefly the case of a linear heat conductor of semi-infinite length \( (l = \infty) \). Some of the following results are used later.

It can be shown that the boundary value problem in the infinite interval

\[
\begin{align*}
\gamma'' - \alpha^2 \gamma &= F(t) \\
\gamma(0), \gamma(\infty) &\text{ given}
\end{align*}
\]

(5.27)
LINEAR HEAT CONDUCTOR
INITIAL TEMPERATURE $U_0(x) = \frac{3}{2} \cos \frac{\pi}{2} x$
VANISHING BOUNDARY TEMPERATURES

$U(x,t) = \frac{12}{\nu^2} \frac{V}{\nu} \sin(\nu \pi x) e^{-\nu^2 t}$
($a^2$ complex, not real negative, $a \neq 0$, Re $a > 0$) has a solution only if $Y(\omega) = -F(\omega)/a^2$, provided $F(t)$ is continuous for $t \geq 0$ and $\lim_{t \to \infty} F(t) = F(\infty)$ exists.

This solution is then given by

$$Y(t) = Y(0) e^{-\alpha t} + \int_0^\infty Y_\infty(t, \tau; \alpha) F(\tau) d\tau$$

(5.28)

where

$$Y_\infty(t, \tau; \alpha) = \begin{cases} -\frac{1}{\alpha} e^{-\alpha \tau} \sinh \alpha \tau & (0 \leq \tau \leq t) \\ -\frac{1}{\alpha} e^{-\alpha t} \sinh \alpha \tau & (t \leq \tau < \infty) \end{cases}$$

(5.29)

Applying this theorem to our problem, we find that for continuous $U_0(x) \text{ and } U_0(\infty) \text{ existing, there is for the equation (5.6) only one boundary value possible, namely}

$$\mu_0(\infty, s) = \frac{U_0(\infty)}{s}$$

which corresponds to the original boundary value $U_0(\infty)$.

The solution of Eq. (5.6) which satisfies $u(\infty, s) = U_0(\infty)/s$ can be written as

$$\mu(x, s) = a_o(s) e^{-\sqrt{\lambda} \xi} + \int_0^\infty Y_\infty(x, \xi, \lambda) U_0(\xi) d\xi$$

(5.30)

*) This result can be obtained by constructing the limit $\ell \to \infty$ in Eqs. (3.36) and (3.39). However, mathematical objections can be made against such a procedure.
where

\[
\gamma_{\infty}(x, \xi; s) = \begin{cases}
\frac{1}{\sqrt{\xi}} e^{-x \sqrt{\xi}} \sinh \frac{\xi}{\sqrt{\xi}} & (0 \leq \xi \leq x) \\
\frac{1}{\sqrt{\xi}} e^{-\xi \sqrt{\xi}} \sinh x \sqrt{\xi} & (x \leq \xi < \infty)
\end{cases} \tag{5.31}
\]

Using the relation (5.24) and the definition of the hyperbolic functions, we obtain

\[
\frac{1}{\sqrt{\xi}} e^{-x \sqrt{\xi}} \sinh \frac{\xi}{\sqrt{\xi}} = \left\{ \frac{1}{2} \left[ \chi(x-\xi, t) - \chi(x+\xi, t) \right] \right\} \\
(0 \leq \xi < x) \tag{5.32}
\]

\[
\frac{1}{\sqrt{\xi}} e^{-\xi \sqrt{\xi}} \sinh x \sqrt{\xi} = \left\{ \frac{1}{2} \left[ \chi(x, t) - \chi(x+x, t) \right] \right\} \\
(x \leq \xi < \infty).
\]

Because of \( \chi(x-\xi, t) = \chi(\xi-x, t) \) we find for the original function of \( \gamma_{\infty}(x, \xi; s) \):

\[
\Gamma_{\infty}(x, \xi; t) = \frac{1}{2} \left[ \chi(x-\xi, t) - \chi(x+\xi, t) \right] \tag{5.33}
\]

and finally

\[
U(x, t) = A_0(t) \ast \chi(x, t) + \frac{1}{2} \int_0^\infty \left[ \chi(x-\xi, t) - \chi(x+\xi, t) \right] U_0(\xi) \, d\xi \\
= \frac{1}{2\sqrt{\pi}} \left\{ x \int_0^t A_0(t-\tau) \tau^{-3/2} e^{-x^2/4\tau} \, d\tau \\
+ t^{-1/2} \int_0^\infty \left[ e^{-(\xi-x)^2/4t} - e^{-(\xi+x)^2/4t} \right] U_0(\xi) \, d\xi \right\}. \tag{5.34}
\]
Figures 15 and 16 show the function $U(x,t) = A_0(t)\varphi(x,t)$ [$U_0(\xi) = 0$] for $A_0(t) = c \cdot \cos(\pi/2)t$ ($c = 1.5$ in Fig. 15, $c = 1$ in Fig. 16).

5.2 The telegraph equation

We discuss now a partial differential equation which appears in the theory of electric lines, and in other applications. Let an electric two-wire line, which is extended from $x = 0$ to $x = l$, be characterized by the following constant quantities which are given per unit length resistance $R$, inductance $L$, capacity $C$, leakance $G$.

If the time is $t$, then for both the current and the voltage between go and return line the following equation holds

$$\frac{\partial^2 U}{\partial x^2} = LC \frac{\partial^2 U}{\partial t^2} + (RC + LG) \frac{\partial U}{\partial t} + RGA \ U$$

(5.35)

or, with

$$LC = \alpha, \quad RC + LG = \beta, \quad RGA = \gamma,$$

$$\frac{\partial^2 U}{\partial x^2} - \alpha \frac{\partial^2 U}{\partial t^2} - \beta \frac{\partial U}{\partial t} - \gamma U = 0$$

(5.36)

$a, b, c$ are non-negative for physical reasons, only $a > 0$ is necessary for the following.

We prescribe the initial conditions as

$$U(x, +0) = 0, \quad U_t(x, +0) = \frac{\partial U}{\partial t}(x, +0) = 0,$$

(5.37)

and the boundary conditions as

$$U(+0, t) = A_0(t), \quad U(l-0, t) = A_\lambda(t).$$

The initial conditions mean in physics that the line is idle for $t = 0$; neither current nor voltage are present.
$U(x,t) = \frac{3x}{4\sqrt{\pi}} \int_0^t \cos \left[ \frac{\pi}{2} (t-\tau) \right] \tau^{-3/2} e^{-x^2/4\tau} d\tau$

SEMI-INFINITE LINEAR HEAT CONDUCTOR
VANISHING INITIAL TEMPERATURE
BOUNDARY TEMPERATURE FOR $x=0$

$A_0(t) = \frac{3}{2} \cos \frac{\pi}{2} t$
U(x,t) = \frac{X_0}{2\sqrt{m}} \left[ \frac{\pi}{2} (1 - \frac{\sqrt{t}}{x}) \right] e^{-\frac{x^2}{4mt}} + \frac{X_0}{\sqrt{2\pi m}} \int_0^t \frac{e^{-\frac{(t - \tau)^2}{4m\tau}}}{\sqrt{\tau}} d\tau.

FIG. 16

SEM-I\-FINITE LINEAR HEAT CONDUCTOR

VANISHING INITIAL TEMPERATURE FOR x = 0

A_0(t) = \cos \frac{\pi t}{2}

x = 0

x = 5

x = 10

x = 15
We can then write the image equation
\[
\frac{\partial^2 \mu}{\partial x^2} - (a \Delta^2 + b \Delta + c) \mu = 0
\]
(5.38)
with
\[
\mu (0, \lambda) = a_0 (\lambda), \quad \mu (t_0, \lambda) = a_t (\lambda)
\]

For the sake of simplicity we discuss here only the case $t = \infty$.

By comparing Eq. (5.38) with Eq. (5.27), we see that in this case it follows from $F(\infty) = 0 [F(t) = 0]$ that $u(\infty, s) = a_1 (s) = 0$, and therefore $U(\infty, t) = 0$.

From Eq. (5.30), we find the solution $u(x, s)$ of Eq. (5.38)
\[
\mu (x, \lambda) = a_\infty (\lambda) e^{-x \sqrt{a \Delta^2 + b \Delta + c}}
\]
(5.39)

We see here that an $e^{-1}$-transform by the convolution theorem is impossible. Already for $b = c = 0$, $e^{-x \sqrt{a \Delta}}$ does not belong to the image space [of functions, cf. Eq. (3.72)].

a) The loss-free line.

This special case is equivalent to the wave equation
\[
\frac{\partial^2 U}{\partial t^2} = a \frac{\partial^2 U}{\partial x^2}
\]
(5.40)

Because of $b = c = 0$, it follows that
\[
RG + LG = 0, \quad RG = 0
\]
or, since $a = LG \neq 0$,
\[
R = 0, \quad G = 0.
\]

Such a line is called "loss-free" and we can write
\[ U(x, \lambda) = \lambda \left( \frac{-x \sqrt{a}}{\lambda} \right) e^{-x \sqrt{a} \lambda} \]

and apply the first shifting theorem (Rule II), which gives

\[ U(x, t) = \begin{cases} 0 & (t < x \sqrt{a}) \\ A_0 (t - x \sqrt{a}) & (t \geq x \sqrt{a}) \end{cases} \]  

(5.41)

b) The distortion-free line

The \( e^{-1} \)-transform of Eq. (5.39) is also possible in this way if \( a \lambda^2 + b \lambda + c \) is a squared linear function. This is exactly the case if \( d = ac - \frac{(b/2)^2}{a} = 0 \).

In this case we have

\[ a \lambda^2 + b \lambda + c = \left( \sqrt{a} \lambda + \frac{b}{2\sqrt{a}} \right)^2, \]

thus

\[ U(x, \lambda) = \lambda \left( \frac{-x \sqrt{a}}{\lambda} \right) \left( \frac{b}{2\sqrt{a}} \right), \]

and

\[ U(x, t) = \begin{cases} 0 & (t < x \sqrt{a}) \\ -\frac{b}{2\sqrt{a}}x & (t = x \sqrt{a}) \\ e^{-x \sqrt{a} \lambda} \left( t - x \sqrt{a} \right) & (t \geq x \sqrt{a}) \end{cases} \]  

(5.42)

Using the line constants, we have

\[ d = LC + G - \frac{A}{4} (RC + LG)^2 = \frac{A}{4} (RC - LG)^2. \]

From \( d = 0 \) follows

\[ RC = LG \quad \text{or} \quad \frac{R}{L} = \frac{C}{G}. \]
A line with such a property is called "distortion-free". A signal \( A_0(t_0) \), which starts at a certain time \( t_0 \) at the point \( x = 0 \), arrives at the point \( x > 0 \) at a time \( t \) which is defined by

\[
 t = x \sqrt{a} = t_0 \quad \Rightarrow \quad t = t_0 + x \sqrt{a}.
\]

It needs for the distance \( x \) the time \( x \sqrt{a} \); in other words it has the velocity \( v = 1/\sqrt{a} \). It does not arrive with its full amplitude but it is damped by a factor \( \exp \left[ -(b/2\sqrt{a})x \right] \). It is, however, undistorted; this means no other signals are superposed upon it.

c) The general case

We treat now the general case \( d \neq 0 \). Here we have to use a rather complicated formula for the Bessel function \( J_0(z) \), namely

\[
 \int_0^\infty e^{-\sigma \tau} J_0 \left( k \sqrt{\tau^2 - x^2} \right) d\tau = \frac{e^{-x \sqrt{\sigma^2 + k^2}}}{\sqrt{\sigma^2 + k^2}}, \quad (5.43)
\]

\((x \geq 0, \sigma > 0, k \text{ arbitrary}).

We differentiate this relation with respect to \( x \), using the formula

\[
 \frac{d}{dx} \left( \frac{h_2(x)}{h_1(x)} \right) = \frac{h_2'(x) h_1(x) - h_2(x) h_1'(x)}{h_1^2(x)} = \frac{\sigma f(\tau, x)}{\partial x} d\tau + \frac{h_2'(x) f(h_2(x), x)}{h_1(x)}
\]

\[
 - \frac{h_2'(x) f(h_2(x), x)}{h_1(x)} \right), \quad (5.44)
\]

which gives with \( J_1(z) = -J_0(z) \)

\[
 k \int_x^\infty e^{-\sigma \tau} \frac{J_1 \left( k \sqrt{\tau^2 - x^2} \right)}{\sqrt{\tau^2 - x^2}} d\tau - e^{-\sigma x} \frac{-x \sqrt{\sigma^2 + k^2}}{\sqrt{\tau^2 - x^2}} = -e \quad (5.45)
\]
With

\[ \sigma = \sqrt{a} \Delta + \frac{b}{2} \sqrt{a} , \quad \kappa^2 = d/a , \quad \sqrt{a} \tau = t \]

we obtain

\[ e^{-x \sqrt{a} \Delta^2 + b \Delta + c} = e^{-\left(\frac{b}{2} \sqrt{a}\right)x} \cdot e^{-x \sqrt{a} \Delta} \quad (5.46) \]

\[-x \sqrt{\frac{d}{a}} \int_0^\infty e^{-x \sqrt{a} \Delta} e^{-\left(\frac{b}{2} \sqrt{a}\right)t} \cdot \frac{J_1 \left( \frac{\sqrt{d}}{a} \sqrt{t^2 - a \Delta^2} \right)}{\sqrt{t^2 - a \Delta^2}} \, dt . \]

We now define a function \( V(x,t) \) by

\[ V(x,t) = \begin{cases} 0 & (0 \leq t \leq x \sqrt{a}) \\ \frac{\sqrt{d}}{a} e^{-\left(\frac{b}{2} \sqrt{a}\right)t} \cdot \frac{J_1 \left( \frac{\sqrt{d}}{a} \sqrt{t^2 - a \Delta^2} \right)}{\sqrt{t^2 - a \Delta^2}} & (t > x \sqrt{a}) \end{cases} \quad (5.47) \]

Using this definition, we can write Eq. (5.46) as

\[ e^{-x \sqrt{a} \Delta^2 + b \Delta + c} \cdot e^{-\left(\frac{b}{2} \sqrt{a}\right)x} \cdot e^{-x \sqrt{a} \Delta} = e^{\int_0^x (V(x,t))^2} . \quad (5.48) \]
Our aim was to translate Eq. (5.39) into the original space. Writing

\[ u(x, \lambda) = a_\sigma(\lambda) e^{-x\sqrt{a}\lambda^2 + b\lambda + c} \]

\[ = e^{-(k/2\sqrt{a})x} a_\sigma(\lambda) e^{-x\sqrt{a}\lambda} \]

we have, using Eq. (5.42) and the convolution theorem (Rule IX)

\[ u(x, t) = \begin{cases} 
0 & (0 \leq t < x\sqrt{a}) \\
\exp(-(k/2\sqrt{a})x) A_\sigma(t - x\sqrt{a}) & \\
-\int A_\sigma(t - \tau) V(x, \tau) \, d\tau & (t > x\sqrt{a})
\end{cases} \]  

(5.50)

We see here that, contrary to the case \( d = 0 \), not only one signal \( A_\sigma(t_0) \) alone with \( t_0 = t - x\sqrt{a} \) arrives at a time \( t \) at \( x \), but in addition a distortion composed of all previous excitations \( A_\sigma(t - \tau) \) with \( 0 \leq t - \tau < t - x\sqrt{a} \).

Figure 17 shows the function \( U(x, t) \) for all three cases and two values of \( t \).
SEMI-INFINITE TELEGRAPH LINE
VANISHING INITIAL CONDITIONS
BOUNDARY CONDITION FOR x=0
$A_0(t) = \cos \frac{\pi}{2} t$

1 - LOSS-FREE
2 - DISTORTION-FREE
3 - GENERAL CASE

$t = 1.70 \sqrt{a}$

$a = 0.5, b = \sqrt{2}, d = 2$

$t = 0.85 \sqrt{a}$

Fig. 17
6. THE INVERSION OF THE LAPLACE TRANSFORM

6.1 Tables

The inversion of the Laplace transform, that means finding the original function which corresponds to a given image function, is often the most difficult step in solving a problem by the Laplace transform. Fortunately, as we have seen in the previous sections, there exist several tables which give a wide range of corresponding functions. As a first step, therefore, one always looks into these tables in order to find a correspondence which may help to solve the problem, perhaps after applying some of the rules described in Section 2. Unfortunately, the notation varies between some of the tables, and care has to be taken in this respect. Some of the authors, for instance, introduce the Laplace transform as

\[ \mathcal{L} \begin{cases} F(t) \end{cases} = \int_0^\infty e^{-st} F(t) \, dt. \]

This has the advantage that

\[ \mathcal{L} \begin{cases} \text{const.} \end{cases} = \text{const.}. \]

This notation is often found in technical books; it is sometimes called the Carson-Laplace transform.

Various notations are used also for the functions of the original and of the image space, e.g. \( F(t), f(s), f(t), \tilde{f}(s), f(p), \tilde{f}(p) \), etc.

In many cases, however, no original function for a given image function can be found in the tables. Then one has to try to find the original function (if one exists) by other methods.

6.2 The inverse integral formula

In Section 1.1, we introduced the formula (1.19)

\[ \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{ts} f(s) \, ds = \begin{cases} F(t) & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}. \]

It is important to know under which conditions this formula can be used in order to find the function \( F(t) \). One can show that if \( \mathcal{L}\{F(t)\} = f(s) \) is absolutely convergent for \( s = x_0 \) (real) and therefore for \( \text{Re } s \geq x_0 \),
that means if
\[ \int_{0}^{\infty} e^{-x_0 t} |F(t)| \, dt < \infty \]
then the "complex inverse integral" formula
\[
\frac{F(t+0) + F(t-0)}{2} = \text{V.P.} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{ts} f(s) \, ds \]
\[
= \lim_{Y \to \infty} \frac{1}{2\pi i} \int_{x-jY}^{x+jY} e^{ts} f(s) \, ds
\]
holds for \( x \geq x_0 \) and \( t > 0 \), provided \( F(t) \) is near the point \( t \) of bounded fluctuation \( ** \). In particular, if \( F(t) \) has this property in an interval

---

*) V.P. means the Cauchy principle value of the integral, characterized by the second line of the formula, where the same variable \( Y \) appears in both the lower and the upper limit of the integral. On the contrary, the usual integral
\[
\lim_{Y_1 \to \infty} \lim_{Y_2 \to \infty} \frac{1}{2\pi i} \int_{x-jY_2}^{x+jY_2} e^{ts} f(s) \, ds,
\]
with independent \( Y_1 \) and \( Y_2 \), may not exist.

**) A function \( G(x) \) is said to be of bounded fluctuation, in an interval \((a,b)\), if for each finite set of points \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) the condition
\[
\sum_{v=1}^{\infty} |G(x_v) - G(x_{v-1})| \leq M
\]
is true for fixed \( M \).
0 \leq t \leq a, we have
\[ \frac{F(t \circ \alpha)}{2} = \text{V. P.} \left\{ \frac{1}{2 \pi i} \int_{x-j\infty}^{x+j\infty} e^{t \lambda} \lambda \, d\lambda \right\} \] (6.2)

for \( x \geq x_0 \). For all \( t < 0 \),
\[ \text{V. P.} \left\{ \frac{1}{2 \pi i} \int_{x-j\infty}^{x+j\infty} e^{t \lambda} \lambda \, d\lambda \right\} = 0 \] (6.3)

holds for \( x \geq x_0 \).

In the case where \( \mathcal{F}(F(t)) \) has no half-plane of absolute convergence, the following theorem may be useful. If \( \mathcal{F}(F(t)) = f(s) \) is convergent for a real \( s = x_0 \geq 0 \), then one has for \( t \geq 0 \)
\[ \int_{0}^{t} F(\tau) \, \tau \, d\tau = \text{V. P.} \left\{ \frac{1}{2 \pi i} \int_{x-j\infty}^{x+j\infty} e^{t \lambda} \lambda \, d\lambda \right\}, \] (6.4)

with arbitrary \( x > x_0 \geq 0 \).

In practice, formula (6.1) is used in most cases in order to calculate \( F(t) \). The conditions of Eq. (6.1) require the knowledge of certain properties of an unknown function, namely \( F(t) \), for the choice of the abscissa \( x \) in the limits of integration. In fact, \( x \) must be chosen in such a way that all singularities of \( f(s) \) lie on the left of the straight line \( Re \, s = x \).

6.2.1 Remarks on complex functions

The main advantage of Eq. (6.1) is the fact that \( f(s) \) and therefore \( e^{st} f(s) \) is an analytic function. Thus the powerful theory of complex integration of analytic functions can be applied to the inverse integral (6.1). We recall some theorems from this theory.
a) If $G$ is an arbitrary (including multiple-connected) region in the $z$-plane, then for all functions $f(z)$, which are analytic in $G$, the relation

$$\oint_{\Gamma} f(z) \, dz = 0$$  \hspace{1cm} (6.5)$$
holds, provided the closed contour $\Gamma$ and its interior region contain only points of $G$ (Fig. 18a).

b) Let $\Gamma_1$ and $\Gamma_2$ be two closed contours, which lie completely inside $G$, have no point in common, and define a double-connected bounded region inside $G$. If $f(z)$ is analytic in $G$, then (Fig. 18b)

$$\oint_{\Gamma_1} f(z) \, dz = \oint_{\Gamma_2} f(z) \, dz$$  \hspace{1cm} (6.6)$$
(Fig. 18b)

c) If $G$ is an $(n + 1)$-times-connected region [the area inside $\Gamma_0$ and outside $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ (Fig. 18c)], then for all functions $f(z)$ which are analytic in $G$,

$$\oint_{\Gamma_0} f(z) \, dz = \sum_{\nu=1}^{n} \oint_{\Gamma_{\nu}} f(z) \, dz$$  \hspace{1cm} (6.7)$$
holds, provided \( f(z) \) is continuous on the boundary of \( G \).

![Diagram](image)

**Fig. 18c**

d) The previous theorem can be applied to the case where \( f(z) \) is analytic in a single-connected and bounded region \( G \), with the exception of the points \( z_1, z_2, \ldots, z_n \). These points are then called singular points or singularities of \( f(z) \). When taking away these points from \( G \), we obtain an \((n + 1)\)-times-connected region \( \tilde{G} \). The value \( A_v \) of the contour integral around one of these singularities does not depend, because of Eq. (6.6), on the shape of the contour. It depends only on \( f(z) \) and its singularity \( z_v \). The quantity

\[
R_v = \frac{A_v}{2\pi i}
\]

is called "residue" of \( f(z) \) in \( z_v \). Using formulae (6.7), we have then Cauchy's residual theorem

\[
\frac{1}{2\pi i} \oint_{\Gamma_v} f(z) \, dz = \sum_{v=1}^{n} R_v
\]

In particular, because \( A_v \) is independent of the form of \( \Gamma_v \), we can choose circles with radii \( \rho_v \) and have
\[
\lim_{\gamma \to 0} \oint_{\gamma} f(z) \, dz = A_v
\]  
(6.10)

In the case of

\[
f(z) = \frac{f_1(z)}{f_2(z)},
\]

one can show that the residue at a point \( z_0 \) is given by

\[
R = \operatorname{Res} \left. \frac{f_1(z)}{f_2(z)} \right|_{z = z_0} = \left. \frac{f_1(z)}{f_2(z)} \right|_{z = z_0},
\]  
(6.11)

provided \( f_1(z) \) and \( f_2(z) \) are analytic in the neighbourhood of \( z_0 \), \( f_1(z_0) \neq 0 \) and \( f_2(z) \) has a zero of order one in \( z_0 \). \( f(z) \) has then a pole of order one in \( z_0 \). If, in general, \( z_0 \) is a pole of order \( \lambda \) of the function \( f(z) \), the residue at this point is given by

\[
\operatorname{Res} \left. f(z) \right|_{z = z_0} = \frac{1}{(\lambda - 1)!} \frac{d^{\lambda - 1}}{dz^{\lambda - 1}} \left[ f(z)(z - z_0)^{-\lambda} \right]_{z = z_0}
\]  
(6.12)

e) We have to consider another kind of singularity too. Some of the complex functions \( w = f(z) \) are not unique in the \( w \) plane, e.g. \( w = \log z \), \( w = z^\alpha \) with arbitrary complex \( \alpha \). This means to one value of \( z \) correspond several values of \( w \). It is nevertheless possible to make these functions unique either with the help of a so-called Riemann surface or by restricting the variable \( \phi_z \) in \( z = r_z e^{i\phi_z} \). For the function \( w = \sqrt{z} \) \([= z^{\alpha} \text{ with } \alpha = \frac{1}{2}]\), for instance, we can restrict \( \phi_z \) to \(-\pi < \phi_z \leq \pi\) and obtain a unique function. Because of \( w = r_w e^{i\theta_w} \), we obtain \( \phi_w = \frac{1}{2} \phi_z \). The value \( w = \sqrt{z} \) therefore lies in the half-plane \( \text{Re } w > 0 \), including the positive imaginary axis
Re \( w = 0 \), Im \( w \geq 0 \). 

We mark the restriction of \( \varphi_z \) in the \( z \) plane by a "branch line"; this is a cut from \( z = 0 \) to \( z = \infty \) along the negative imaginary axis. This cut cannot be passed. \( z = 0 \) and \( z = \infty \) are called branch points of \( w = \sqrt{z} \). In general, a multivalued function is made unique by cutting branch lines between pairs of the branch points **). A cut may follow an arbitrary way between these two points and is generally chosen for reasons of convenience.

The function \( w = \sqrt{z} \) is originally double-valued. The points \( z = 0 \) and \( z = \infty \) are therefore called branch points of second order. Other functions, like \( w = \log z \) or \( w = z^\alpha \) with non-real or non-rational \( \alpha \) may be infinitely many-valued; the point \( z = 0 \) is then called a branch point of order \( \infty \) or a logarithmic branch point. The function \( w = z^{p/q} \) with \( p > 0 \), \( p \) and \( q \) having no common divisor, is \( q \)-valued.

6.2.2 An example for the evaluation of the inverse formula

In Section 5.2, we found for the temperature of a semi-infinite heat conductor, with vanishing initial temperature, the formula

\[
U(x,t) = A_0(t) \times \gamma(x,t).
\]

*) It is known that for real \( z = x \) the solution of \( y^2 = x \) is double-valued, namely \( y = +\sqrt{x}, \ y = -\sqrt{x} \). This can be explained in the complex plane as follows: the real number \( x \) can be written as 

\[
z = x e^{j\varphi_x} = x e^{j(\pi / 2 + 2k\pi)}
\]

or, because of the periodicity of the exponential function, as

\[
z = x e^{j(\pi / 2 + 2k\pi)} = x e^{j\varphi_z}
\]

with

\[
\varphi_z = 2k\pi
\]

We have then

\[
\omega = \sqrt{z} = \sqrt{x} e^{j\varphi_z} = \sqrt{x} e^{j(\pi / 2 + 2k\pi)}
\]

The factor \( e^{k\pi j} \) is either +1 or -1, according to even or odd values of \( |k| \). If we restrict, however, \( \varphi_z \) to \( -\pi < \varphi_z \leq \pi \), this means \( k = 0 \), we obtain one value of \( \sqrt{x} \) only, namely

\[
\omega = \sqrt{x} e^{j\varphi_0} = +\sqrt{x}
\]

**) The cuts must join two branch points of the same order (see the following).
When taking \( U(t) = \cos \omega t \) we obtain

\[
U(x, t) = \cos \omega t \ast \psi(x, t).
\]  

(6.14)

This is a very elegant expression. It is, however, difficult to recognize the behaviour of \( U(x,t) \) from this formula. We now try to derive another expression for \( U(x,t) \) with the help of the complex inverse integral (6.1)

We recall that

\[
\mathcal{L}\{ \cos \omega t \} = \frac{s}{s^2 + \omega^2}
\]

and [from Eq. (2.25)]

\[
\mathcal{L}\{ \psi(x,t) \} = e^{-x\sqrt{s}}
\]

which yields

\[
U(x,s) = \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}}.
\]  

(6.15)

With Eq. (6.1), we obtain

\[
U(x,t) = \lim_{\gamma \to \infty} \frac{1}{2\pi i} \int_{\alpha - i\gamma}^{\alpha + i\gamma} e^{ts} \frac{s}{s^2 + \omega^2} e^{-x\sqrt{s}} ds.
\]  

(6.16)

Because of the denominator \( s^2 + \omega^2 \), \( u(x, s) \) has single poles in \( s = \pm j \omega \), and, because of the exponent \( \sqrt{s} \), it has a branch point in \( s = 0 \) (and in \( s = \infty \)). We take the negative real axis as the branch cut. Any positive number can be chosen for the abscissa \( \alpha \) in the limits of integration.

In order to calculate Eq. (6.16), we construct the following closed contour \( \Gamma \) (Fig. 19).
This contour $\Gamma$ consists of several pieces*):

i) a straight line $AB$ parallel to $\text{Re } s = 0$ and passing through the real axis at $s = \alpha$;

ii) circular arcs $BCD$, $GHA$ (radius $R$), and $EF$ (radius $\varepsilon$);

iii) the straight lines $DE$ and $FG$ above and under the negative real axis (we cannot pass the branch cut).

* There are many possibilities for the choice of $\Gamma$. To find the most suitable one for a given integral is sometimes very difficult and very much a matter of experience.
Inside this contour the function \( u(x,s) \) is unique and analytic with the exception of the poles of first order at \( s = \pm j \omega \). According to Cauchy's theorem (6.9) we can write

\[
\frac{1}{2\pi j} \int_\Gamma e^{ts} u(x,s) \, ds = \text{Res} \{ e^{ts} u(x,s) \}^{s=j\omega}_{s=-j\omega} \tag{6.17}
\]

Because of

\[
e^{ts} u(x,s) = e^{ts} \frac{e^{-x\sqrt{j\omega}}}{s^2 + \omega^2} = \frac{f_1(s)}{f_2(s)}
\]

we have \( f_2'(s) = 2s \) and obtain, with formula (6.11):

\[
\frac{1}{2\pi j} \int_\Gamma e^{ts} u(x,s) \, ds = e^{tj\omega} \frac{e^{-x\sqrt{j\omega}}}{2j\omega} + e^{-tj\omega} \frac{(-j\omega)e^{-x\sqrt{-j\omega}}}{-2j\omega}
\]

\[
= \frac{1}{2} \left( e^{j\omega t - x\sqrt{j\omega}} + e^{-j\omega t - x\sqrt{-j\omega}} \right). \tag{6.18}
\]

Using \( \sqrt{j} = (1+j)/\sqrt{2} \), \( \sqrt{-j} = (1-j)/\sqrt{2} \) we can simplify this expression and obtain

\[
\frac{1}{2\pi j} \int_\Gamma e^{ts} u(x,s) \, ds = e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2}) \tag{6.19}
\]
Our aim is to calculate the integral in Eq. (6.16). We go back to Fig. 19 and note that the integral over the straight line AB goes to \( U(x,t) \) if the radius \( R \) goes to infinity. Because we know the integral over the closed contour [Eq. (6.19)], the problem would be solved if we knew the integral over the contour BCDEFGHA.

It is now characteristic for this kind of contour integration that one tries to show that the integrals over the arcs BD, GA and EF vanish in the case \( R \to \infty \) and \( \varepsilon \to 0 \), respectively. Fortunately, there exist several theorems in this connection. We use here the following one: Let \( f(s) = f(r \, e^{i\varphi}) \) be a function which in the left half-plane \( \text{Re} \, s < 0 \) \((\pi/2 \leq \varphi \leq 3\pi/2)\) tends uniformly in \( \varphi \) to zero for \( r \to \infty \). If \( \Omega \) is a left semi-circle with radius \( r \) and origin zero, then

\[
\int_{\Omega} e^{ts} f(s) \, ds \to 0
\]

is true for \( r \to \infty \), provided that \( t > 0 \). \( \Omega \) may be also a partial arc of this semi-circle, provided its central angle remains unchanged for \( r \to \infty \).

We note in our case that for \( s \) with \( \text{Re} \, s < 0 \), we have \( \text{Re} \sqrt{s} \geq 0 \), hence \( (x \geq 0) \)

\[
|e^{-x\sqrt{s}}| = e^{-x \text{Re} \sqrt{s}} \leq 1
\]

and for \( |s| \to \infty \)

\[
\frac{1}{\Delta^2 + \omega^2} \sim \frac{1}{\Delta}
\]

It follows that \( f(s) = u(x,s) \) satisfies the conditions of the above theorem and we can write

\[
\int_{CD} e^{ts} u(x,s) \, ds \to 0, \quad \int_{CA} e^{ts} u(x,s) \, ds \to 0.
\]

*\) Uniformly in \( \varphi \) means that for \( r \to \infty \) \(|f(s)| < \delta\), with \( \delta \) independent of \( \varphi \).
In addition, we see that the integrals over the arcs BC and HA tend to zero, too. The functions $e^{s}t$ and $e^{-x\sqrt{s}}$ are bounded on these contours (Re $s$ is bounded) and the factor $s/(s^{2} + \omega^{2})$ ensures uniform convergence to zero. Since the contour is bounded, it follows that the integral tends to zero as well. It can also be seen in a similar way that the integral over EF goes to zero for $\epsilon \to 0$.

Taking these results into account, we can write

$$
\frac{1}{2\pi \imath} \int_{\Gamma} e^{st} u(x,s) \, ds =
$$

$$
= \frac{1}{2\pi \imath} \int_{\alpha - \imath \infty}^{\alpha + \imath \infty} e^{st} u(x,s) \, ds + \frac{1}{2\pi \imath} \left( \int_{DE} + \int_{FG} \right) e^{st} u(x,s) \, ds
$$

$$
= e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2})
$$

or

$$
U(x,t) = e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2})
$$

$$
- \frac{1}{2\pi \imath} \int_{-\infty + \imath \infty}^{\infty + \imath \infty} e^{st} u(x,s) \, ds = \frac{1}{2\pi \imath} \int_{-\infty - \imath \infty}^{\infty - \imath \infty} e^{st} u(x,s) \, ds
$$

On the upper side of the cut we have $s = r e^{\imath \pi}$ on the lower one $s = r e^{-\imath \pi}$. This leads to
\[
\begin{align*}
\int_{-\infty}^{\infty} e^{ts} \mu(x,s) ds &= \int_{-\infty}^{0} e^{tr} e^{j\pi} e^{j\pi} e^{-x\sqrt{r}} e^{\frac{1}{2}j\pi} dr \\
&= -\int_{0}^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} e^{-j\pi x\sqrt{r}} dr
\end{align*}
\]

and to

\[
\begin{align*}
\int_{0}^{\infty} e^{ts} \mu(x,s) ds &= \int_{0}^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} e^{j\pi x\sqrt{r}} dr
\end{align*}
\]

We introduce these results into Eq. (6.20) and obtain finally

\[
\begin{align*}
U(x,t) &= e^{-x\sqrt{\omega/2}} \cos(\omega t - x\sqrt{\omega/2}) - \frac{1}{\pi} \int_{0}^{\infty} e^{-tr} \frac{r}{r^2 + \omega^2} \sin x\sqrt{r} dr \\
&= \cos \omega t \times \psi(x,t)
\end{align*}
\]

This expression for \(U(x,t)\) gives a much deeper insight into the behaviour of this function than does the original expression. The integral on the right-hand side tends to zero for \(t \to \infty\). It is a Laplace integral, only with variables \(t\) and \(r\) instead of \(s\) and \(t\), respectively. Theorems of the Laplace transform, especially asymptotical properties, can be applied to this integral. We discuss this later.

Here we note from Eq. (6.21) that \(U(x,t)\) is composed of a "steady state" solution with respect to \(t\) (the first term), which, at a place \(x > 0\), is an oscillation with the same frequency \(\omega\) as the exciting function, but damped with \(e^{-x\sqrt{\omega/2}}\) and shifted by \(x\sqrt{\omega/2}\). The second component of \(U(x,t)\) (the integral) describes a smoothing process. It dies away for \(t \to \infty\). Figure 20 shows these functions for \(x = \frac{1}{2}\) and \(\omega = \pi/2\).
$U_1(x,t) = e^{-x^2t/4} \cos\left(\frac{\pi}{2} t - x^2 \frac{\pi}{4}\right)$

$U_2(x,t) = -\frac{1}{\pi} \int_0^\infty e^{-tr} \frac{r}{r^2 + (\pi/2)^2} \sin x \sqrt{r} \, dr$

$U(x,t) = U_1(x,t) \ast U_2(x,t)$

$x = \frac{1}{2}$

Semi-infinite linear heat conductor
vanishing initial temperature
boundary temperature for $x=0$

$A_0(t) = \cos \frac{\pi t}{2}$

Fig. 20
6.3 **Series expansions**

It is often possible to find the original function $F(t)$ of a given image function $f(s)$ by a series expansion of $f(s)$. Writing

$$f(\alpha) = \sum_{\nu=0}^{\infty} f_{\nu}(\alpha)$$

with image functions $f_{\nu}(s)$, we can deduce formally, integrating term by term and using $f_{\nu}(s) = \mathcal{L} \{ F_{\nu}(t) \}$, the series

$$F(t) = \sum_{\nu=0}^{\infty} F_{\nu}(t)$$

It is clear that this procedure may give wrong results, because we exchange an infinite sum with an integral, in this case even with an improper one. There are, however, types of series, where this procedure is allowed. We discuss some of them here.

a) **Power series in 1/s**

In the case where $f(s)$ can be expressed by a power series in 1/s, which converges for $|s| > R$

$$f(s) = \sum_{\nu=0}^{\infty} \frac{a_{\nu}}{s^{\nu+1}}$$

(6.22)

the original function $F(t)$ is given by

$$F(t) = \sum_{\nu=0}^{\infty} \frac{a_{\nu}}{\nu!} t^{\nu}$$

(6.23)

This series converges for all real and complex $t$. It can be shown that $f(s)$ is, in this case, for $s = \infty$ an analytic function with $f(\infty) = 0$. $F(t)$ is a function of exponential type, that means it is possible to find constants $M, \gamma$ such that $|F(t)| < Me^{\gamma|t|}$. 
The previous result can be extended to powers of $s$ with non-integer exponents. If a series

$$f(s) = \sum_{\nu=0}^{\infty} \frac{a_{\nu}}{\lambda_{\nu}^s}$$

(6.24)

with $0 < \lambda_0 < \lambda_1 < \ldots \rightarrow \infty$ converges absolutely for $|s| > R$, then we can translate this series term by term and have

$$F(t) = \sum_{\nu=0}^{\infty} a_{\nu} \frac{t^{\lambda_{\nu}-1}}{\Gamma(\lambda_{\nu})}.$$ \hspace{1cm} (6.25)

We apply these procedures to some examples. Let

$$f(s) = \frac{1}{\sqrt{\lambda^2 + 1}}$$

be given. We use the binomial expansion and write

$$f(s) = \frac{1}{\sqrt{\lambda^2 + 1}} = \frac{1}{\lambda} (\lambda + \lambda^{-2})^{-\frac{1}{2}} = \sum_{\nu=0}^{\infty} \left( \frac{-\frac{1}{2}}{\nu} \right) \frac{1}{\lambda^{2\nu + 1}}$$

(6.26)

This series converges for $|s| > 1$. The binomial coefficient can be written as

$$\left( \frac{\frac{1}{2}}{\nu} \right) = \frac{(-\frac{1}{2})(-\frac{3}{2}) \ldots (-\nu + \frac{1}{2})}{\nu!} = \frac{(-1)^{\nu-1} \cdot 3 \cdot \ldots \cdot (2\nu-1)}{\nu! \cdot 2^\nu}$$

$$= \frac{(-1)^{\nu}(2\nu)!}{\nu! \cdot 2^\nu \cdot 4 \cdot \ldots \cdot 2\nu} = \frac{(-1)^{\nu} (2\nu)!}{\nu! \cdot 2^{2\nu} \cdot \nu!}$$

(6.27)
Using Eq. (6.23), we obtain

\[
F(t) = \sum_{\nu = 0}^{\infty} \left( -\frac{1}{2} \right)^{\nu} \frac{t^{2\nu}}{(2\nu)!} = \sum_{\nu = 0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \left( \frac{t}{2} \right)^{2\nu} = J_0(t),
\]  

(6.28)

the Bessel function \( J_0(t) \).

As an example for the case of non-integer exponents we treat

\[
f(s) = \frac{1}{\sqrt{s}} e^{-\frac{1}{4s}} = \sum_{\nu = 0}^{\infty} \frac{(-1)^\nu}{\nu! 4^\nu \Gamma(\nu + 1/2)}
\]  

(6.29)

It is clear that this series converges absolutely for \(|s| > R\). We obtain

\[
F(t) = \sum_{\nu = 0}^{\infty} \frac{(-1)^\nu}{\nu! 4^\nu \Gamma(\nu + 1/2)} t^{\nu - 1/2}
\]  

(6.30)

Using a formula for the gamma function (Legendre's theorem of duplication)

\[
\sqrt{\pi} \Gamma(2z) = 2^{\frac{2z - 1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2})
\]  

(6.31)

we obtain, with \( z = \nu + \frac{1}{2} \)

\[
\nu! 4^\nu \Gamma(\nu + 1/2) = \sqrt{\pi} \Gamma(2\nu + 1) = \sqrt{\pi} (2\nu)!
\]

and

\[
F(t) = \sum_{\nu = 0}^{\infty} \frac{(-1)^\nu}{\sqrt{\pi} (2\nu)!} t^{\nu - 1/2}
\]  

(6.32)
This can be written as
\[ F(t) = \frac{\cos \sqrt{t}}{\sqrt{\pi \cdot t}} \] (6.33)
and we have proved the relation
\[ \mathcal{L}\left\{ \frac{\cos \sqrt{t}}{\sqrt{\pi \cdot t}} \right\} = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{4s}} \] (6.34)

b) Series of exponential functions

In Section 3.2 we discussed the \( \mathcal{L}^{-1} \)-transform of rational functions \( f(s) = r_1(s)/r_2(s) \) [degree \( (r_1) < \) degree \( (r_2) \)]. We found by partial fraction expansion that for single zeros \( \alpha_v \)
\[ \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{ \sum_{v=1}^{n} \frac{\tau_1(\alpha_v)}{\tau_2(\alpha_v)} \frac{1}{s-\alpha_v} \right\} \] (6.35)
\[ = \sum_{v=1}^{n} \frac{\tau_1(\alpha_v)}{\tau_2(\alpha_v)} e^{\alpha_v t} = F(t) \]

This is exactly the result which we can obtain by the residue theorem (6.9)
\[ \frac{1}{2\pi i} \int_{C} e^{ts} \frac{\tau_1(\lambda)}{\tau_2(\lambda)} d\lambda = \sum_{v=1}^{n} \text{Res} \left\{ e^{ts} \frac{\tau_1(\lambda)}{\tau_2(\lambda)} \right\}_{\lambda = \alpha_v} = \alpha_v \]
The contour $\Gamma$ can be chosen as a straight line $\text{Re } s = \alpha > \max(\text{Re } \alpha_j)$ and a circular arc with radius $R > \max |\alpha_j|$ (Fig. 21).

According to the theorem given in Section 6.2.2, the integral over the arc vanishes for $R \to \infty$.

When solving partial differential equations and in other connections, instead of rational functions with a finite number of poles, so-called meromorphic functions appear as image functions. These functions are, like the rational functions, analytic in the whole finite plane, with the exception of isolated poles. Meromorphic functions, however, may have an infinite number of poles and a more complicated behaviour than rational functions for $z = \infty$ (a so-called essential singularity). Our aim is to find in this case a procedure which extends Eq. (6.35) to $n \to \infty$. 
It should be noted that we cannot generalize the finite sum in Eq. (6.35) to an infinite sum without further considerations. We remember that in Section 3.2 the partial fraction expansion was developed under an important restriction, namely that the degree of the denominator was assumed to be higher than the degree of the numerator. In the general case, we would have to add to the partial fraction expansion a polynomial in $s$. If we have a meromorphic function, the consideration of degrees becomes meaningless. One can show, however, that a meromorphic function $f(z)$ is composed in general of a (finite or infinite) sum of partial fraction parts

$$
\gamma_{\nu}(z) = \frac{a_{\nu 1}}{z - z_{\nu}} + \frac{a_{\nu 2}}{(z - z_{\nu})^2} + \ldots + \frac{a_{\nu k_{\nu}}}{(z - z_{\nu})^{k_{\nu}}}
$$

the so-called principal parts of the poles $z_{\nu}$ of order $k_{\nu}$, and, in addition, of a so-called integral function $g(z)$. These integral functions are, in a certain sense, generalizations of the integral rational functions (the polynomials), in such a way that the "degree" of the "polynomial" may be $\infty$. Integral functions have no singularities in the finite plane. Therefore, they can certainly not be expressed by partial fraction expansions. In other words, a straightforward generalization of Eq. (6.35) to $n \to \infty$ is only possible if one knows that $g(z) \equiv 0$. This is often very difficult to prove. Proceeding in a careless way, however, may lead to remarkably wrong results. One can show for instance, that the function

$$\gamma(t) = -\pi e^t \sin \pi t e^t$$

has an integral function $y(s)$ as image function. If such a $y(s)$ is

*) This means they have an essential singularity for $z = \infty$. Examples are $e^z$, $\sin z$, etc.

**) $\mathcal{L}\{f(t)\}$ converges for $\Re s > 0$. By partial integration one can show that $y(s)$ satisfies the functional equation

$$y(s) = 1 - \frac{\Delta(s+1)}{\pi^2} y(s+2)$$

This proves the statement.
hidden in a given meromorphic image function, it will just be neglected
when going with \( n \to \infty \) without care. The result is easy to realize.

We try another way here and write

\[
U(x, t) = \lim_{Y \to \infty} \frac{1}{2\pi i} \int_{\alpha^+ + iY} \int_{\alpha^- - iY} e^{ts} \mu(x, s) ds
\]

and construct auxiliary contours \( \Gamma_1, \Gamma_2, ..., \Gamma_n \) beginning at \( Y_1, Y_2, ..., Y_n \) and ending at \(-Y_1, -Y_2, ..., -Y_n\) in such a way that all the poles
\( \alpha_\mu (\mu = 1, 2, ..., \nu) \) lie between the contour \( \Gamma_v \) and the straight line
\(-Y_v, Y_v\). These contours may be half-circles, or rectangular or other
shapes (Fig. 22).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig22.png}
\caption{Fig. 22}
\end{figure}

We then obtain

\[
\frac{1}{2\pi i} \int_{\alpha^+ + iY_n} \int_{\alpha^- - iY_n} e^{ts} \mu(x, s) ds + \frac{1}{2\pi i} \int_{\Gamma_n} e^{ts} \mu(x, s) ds
\]

\[= \sum_{v=1}^{\nu} \text{Res} \left\{ e^{ts} \mu(x, s) ds \right\}_{s=\alpha_v}
\]
If \( \alpha_{\nu} \) is a pole of order one, then \( u(x,s) \) can be written near this pole as

\[
\frac{b_{\nu}(x)}{s - \alpha_{\nu}(x)}
\]

and the residue of \( e^{ts} u(x,s) \) in \( \alpha_{\nu}(x) \) is \( b_{\nu}(x) e^{\alpha_{\nu}(x)t} \). We obtain, therefore, from Eq. (6.36)

\[
\frac{1}{2\pi i} \int_{\Gamma_n} e^{ts} u(x,s) ds = \sum_{\nu=1}^{\infty} b_{\nu}(x) e^{\alpha_{\nu}(x)t} - \frac{1}{2\pi i} \int_{\Gamma_n} e^{ts} u(x,s) ds.
\]

(6.37)

If it is now possible to show that for \( n \to \infty \) and \( \gamma_n \to \infty \)

\[
\frac{1}{2\pi i} \int_{\Gamma_n} e^{ts} u(x,s) ds \to 0,
\]

but only in this case, we can write

\[
U(x,t) = \sum_{\nu=1}^{\infty} b_{\nu}(x) e^{\alpha_{\nu}(x)t}.
\]

(6.38)

As an example, we consider the function

\[
U_6(x,t) = \frac{\sinh (\pi - x) \sqrt{3}}{\sinh (\pi \sqrt{3})} = \frac{e^{(\pi - x) \sqrt{3}} - e^{-(\pi - x) \sqrt{3}}}{e^{\pi \sqrt{3}} - e^{-\pi \sqrt{3}}}
\]

(6.39)
which appeared in Section 5.1.1 ($\ell = \pi$). In spite of the exponent $\sqrt{s}$, this function is unique in the $s$ plane. This can be seen by replacing $\sqrt{s}$ by $-\sqrt{s}$. The only singularities of $u_0(x,s)$ are poles, given by the zeros of the denominator.

\[
e^{-\pi \sqrt{s}} - e^{\pi \sqrt{s}} = 0 \text{ or } e^{\frac{2\pi \sqrt{s}}{v}} = 1
\]

It follows that

\[
2\pi \sqrt{s} = 2\pi v^j_i \quad (v=0, \pm 1, \pm 2, \ldots)
\]

or

\[
s = -v^2.
\]

Since for $v = 0$ the numerator vanishes too and the negative $v$ give the same poles as the positive ones, we have as poles only

\[
\alpha_v = -v^2 \quad (v = 1, 2, 3, \ldots)
\]

The residue of $u_0(x,s)$ is then calculated, using Eq. (6.11), to be

\[
\text{Res } u_0(x,s) = \left\{ \frac{e^{(\pi - x)\sqrt{s}} - e^{-(\pi - x)\sqrt{s}}}{\frac{\pi}{2\sqrt{s}} \left( e^{\pi \sqrt{s}} + e^{-\pi \sqrt{s}} \right)} \right\}
\]

\[
= \frac{e^{(\pi - x)v^j_i} - e^{-(\pi - x)v^j_i}}{\frac{\pi}{2v^j_i} \left( e^{\pi v^j_i} + e^{-\pi v^j_i} \right)}
\]

\[
= \frac{2j \sin (\pi - x)v}{\frac{\pi}{v^j_i} \cos \pi v} = \frac{2}{\pi} v \sin v x = F_v(x)
\]
Using formula (6.38), we obtain

\[ U_0(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \nu \sin \nu x \ e^{-\nu^2 t}, \quad (6.40) \]

which corresponds to the first sum in Eq. (5.17). Since we have obtained previously the same result by another method, we can conclude that in this case the integral

\[ \int_{-\infty}^{\infty} e^{-st} U_0(x, s) \, ds \to 0 \]

for \( n \to \infty. \)

c) **Expansions with arbitrary functions**

In the previous cases we considered special expansions, namely power series and exponential series. We quote now a more general theorem, which is often useful.

Supposing \( f(s) \) can be expressed in \( \text{Re} \, s \geq x_0 \) by an infinite sum of image functions \( f_\nu(s) \)

\[ f(s) = \sum_{\nu=0}^{\infty} f_\nu(s), \quad f_\nu(s) = \int_{0}^{\infty} F_\nu(t) e^{-st} \, dt, \]

that means all the integrals

\[ \int_{0}^{\infty} e^{-st} F_\nu(t) \, dt = f_\nu(s), \]

exist in the half-plane \( \text{Re} \, s \geq x_0. \) If, in addition, the integrals

\[ \int_{0}^{\infty} e^{-st} |F_\nu(t)| \, dt = \varphi_\nu(s) \quad (\nu = 0, 1, 2, \ldots) \]
exist and the series

\[ \sum_{\nu=0}^{\infty} \varphi_{\nu}(\lambda) \]

converges, then

\[ F(t) = \sum_{\nu=0}^{\infty} F_{\nu}(t) \]

is absolutely convergent for \( t \geq 0 \) and \( F(t) = \mathcal{L}^{-1}\{f(s)\} \).

An example is the so-called factorial series

\[ f(\lambda) = \sum_{\nu=0}^{\infty} \left( \frac{a_{\nu} \nu!}{\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+\nu)} \right) = \sum_{\nu=0}^{\infty} a_{\nu} f_{\nu}(\lambda). \tag{6.41} \]

If this series converges anywhere, then it is always an image function.

The corresponding image function to \( f_{\nu}(\lambda) \) is

\[ F_{\nu}(t) = (1 - e^{-t})^\nu \]

With the substitution \( e^{-t} = \tau \), we obtain

\[ \int_{0}^{\infty} e^{-\lambda t}(1 - e^{-t})^\nu dt = \int_{0}^{1} \tau^{\nu-1} (1 - \tau)^\nu d\tau = B(\lambda, \nu+1). \]

The function

\[ B(x, y) = \int_{0}^{1} \tau^{x-1} (1 - \tau)^{y-1} d\tau \tag{6.42} \]

\((\text{Re } x > 0, \text{Re } y > 0)\)

is called beta function or Euler's integral of first kind\(^*\). One can show that

\(^*\) The gamma function \( \Gamma(x) \) is also called Euler's integral of second kind.
\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = B(y, x) \]  
\[ (6.43) \]

hence

\[ \mathcal{L} \left\{ (1 - e^{-t})^\nu \right\} = B(\alpha, \nu+1) = \frac{\Gamma(\alpha) \Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} \]
\[ \quad = \frac{\nu!}{\alpha (\alpha+1)(\alpha+2) \ldots (\alpha+\nu)} = f_\nu(\alpha). \]

We have therefore

\[ F(t) = \sum_{\nu=0}^{\infty} a_\nu F_\nu(t) = \sum_{\nu=0}^{\infty} a_\nu (1 - e^{-t})^\nu. \]  
\[ (6.44) \]

6.4 Numerical methods

To invert the Laplace transform numerically does not seem to be a very promising procedure. Recently, the subject was discussed in the following way. Let

\[ f(s) = \int_0^\infty e^{-st} F(t) \, dt \]  
\[ (6.45) \]

be given for real s. The function F(t) is wanted. Eq. (6.45) is therefore an integral equation for F(t). We make the substitution \( x = e^{-t} \) and obtain
\[ f(s) = \int_0^1 x^{s-1} F(-\log x) \, dx. \quad (6.46) \]

Here the integral is taken over a finite interval only, which can be written as

\[ f(s) = \int_0^1 x^{s-1} G(x) \, dx. \quad (6.47) \]

With an appropriate integration formula, e.g. a Gauss formula, we can approximate the integral by a sum and obtain

\[ f(s) \approx \sum_{v=1}^N \omega_v x_v^{s-1} G(x_v). \quad (6.48) \]

Letting \( s \) assume \( N \) different values \( s_k \), we obtain a linear system of \( N \) equations with \( N \) unknowns, \( G(x_v) \), \((k,v = 1,2,\ldots,N)\), namely

\[ \sum_{v=1}^N \omega_v x_v^{s_k-1} G(x_v) = f(s_k). \quad (6.49) \]

In particular, if the \( s_k \) are natural numbers, special integration formulae

\[ \int_0^1 x^w G(x) \, dx = \sum_{v=1}^N \omega_v G(x_v) \quad (6.50) \]

can be used.
There is, however, a property of the Laplace transform, which is very disadvantageous with respect to the above procedure. The Laplace inverse is a so-called unbounded operator. This means that arbitrary small changes in \( f(s) \) can produce arbitrary large changes in \( F(t) \). The matrix \( \left\| w_j x_j^k \right\| \) is very ill-conditioned, its determinant is very small. This worsens rapidly with increasing \( N \).

7. **ASYMPTOTICS**

Investigations on the asymptotic behaviour of functions become more and more important in practically all fields of theory and application. They are one of the most important subjects of research in present day analysis. The Laplace transform is a very powerful tool in this field. It may be noted that the solution of equations, etc., as presented in the previous sections can be treated also by other methods of operational calculus, which are based on algebraic theories. Asymptotic relations between original and image functions, however, can be found at present only by the Laplace transform.

7.1 **Definitions**

Two functions \( f(z) \) and \( g(z) \) are asymptotically equivalent (or equal)

\[
\frac{f(z)}{g(z)} \sim 1 \quad (z \to z_0)
\]

(7.1)

if the quotient of both functions tends to 1

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1 \quad (7.2)
\]

The variable \( z \) may be restricted to a certain region when it goes to \( z_0 \), e.g. \( \text{Re} \ z > z_0 \), \( z \) inside a sector of the complex plane, etc.
Examples are:

\[ e^z + z \sim e^z \quad (z \to +\infty) \]
\[ e^z + z \sim z \quad (z \to -\infty) \]
\[ \sin z \sim z \quad (z \to 0) \]
\[ 1 - e^{-z} \sim z \quad (z \to 0). \]

Asymptotic equations can be multiplied, divided and raised to a power \( a \), but in general not be added or subtracted. Subtracting

\[ 1 + z \sim 1, \quad 1 - z^2 \quad (z \to 0) \]

for instance, gives

\[ 1 + z - 1 \sim z^2 \quad (z \to 0) \]

which is wrong. In general, one is not allowed either to replace the argument of a function by an asymptotic equivalent, e.g. we can write

\[ z + 1 \sim z \quad (z \to \infty) \]

but not

\[ e^{z + 1} \sim e^z \quad (z \to \infty) \]

It is often advantageous to use the so-called Landau order symbols \( O \) and \( o \).

One says

\[ f(z) = O(g(z)) \quad (z \to z_0) \quad (7.3) \]

if a constant \( M \) exists such that

\[ |f(z)| \leq M |g(z)| \quad (7.4) \]
for \( z \) sufficiently near to \( z_0 \), and

\[
\frac{f(z)}{g(z)} = \mathcal{O}\left(\frac{g(z)}{z_0}\right) \quad (z \to z_0) \tag{7.5}
\]

if, for arbitrary \( \varepsilon > 0 \),

\[
\left| \frac{f(z)}{g(z)} \right| \leq \varepsilon \left| g(z) \right| \tag{7.6}
\]

In the first case, \( |f(z)/g(z)| \leq M \); in the second case, \( f(z)/g(z) \to 0 \).

In the case where \( f(z) \) and \( g(z) \) depend on a parameter \( \lambda \), one says that Eqs. (7.3) or (7.6) are valid uniformly if \( M \) or \( \varepsilon \) does not depend on \( \lambda \).

It is possible to find calculation rules for the order relations, e.g.

\[
\mathcal{O}\left(\frac{g(z)}{z_0}\right) + \mathcal{O}\left(\frac{g(z)}{z_0}\right) = \mathcal{O}\left(\frac{g(z)}{z_0}\right)
\]
\[
\mathcal{o}\left(\mathcal{O}\left(\frac{g(z)}{z_0}\right)\right) = \mathcal{o}\left(\frac{g(z)}{z_0}\right)
\]

The symbols \( O(1) \), \( o(1) \) mean that an expression is, for \( z \to z_0 \), bounded or vanishing, respectively, e.g. \( \sin z = O(1) \ (z \to \infty) \), \( \sin z = o(1) \ (z \to 0) \).

The relation (7.1) can be written as

\[
f(z) = g(z) + o\left(\frac{g(z)}{z_0}\right)
\]

or

\[
f(z) = g(z) \left(1 + o(1)\right)
\]

In general

\[
f(z) \sim a g(z)
\]

which is reasonable for \( a \neq 0 \) only, can be written as

\[
f(z) = a g(z) + o\left(\frac{g(z)}{z_0}\right)
\]
which is also meaningful for \( a = 0 \). Furthermore, the equation

\[
1 - e^{-\frac{z}{a}} = z + O(z^2) \quad (z \to 0)
\]

or better

\[
1 - e^{-\frac{z}{a}} = z - \frac{1}{2} z^2 + o(z^2) = z - \frac{1}{2} z^2 + O(z^3) \quad (z \to 0),
\]

gives an essentially better indication of the error than does the simple relation

\[
1 - e^{-\frac{z}{a}} = z + o(z).
\]

The right-hand side of Eq. (7.7) can be developed further and we obtain a so-called asymptotic series or asymptotic expansion. This is a series of the form

\[
f(z) = \sum_{v=0}^{\infty} a_v g_v(z) + o(g_v(z)) \quad (z \to z_0)
\]

(7.8)

for \( n = 0, 1, 2, \ldots \), provided \( g_v(z) \) has the property

\[
g_{v+1}(z) = o(g_v(z))
\]

(7.9)

for all \( v \). Instead of Eq. (7.8), one writes also

\[
f(z) = \sum_{v=0}^{\infty} a_v g_v(z) \quad (z \to z_0).
\]

(7.10)

The convergence of this series is not necessary. Asymptotic series with the same functions \( g_v(z) \) can be added, subtracted and multiplied by a
constant, in the term by term way. For some kinds of functions $g_\nu(z)$, asymptotic series may be multiplied, too *)

The most common asymptotic expansions for finite $z_0$ are series with increasing powers of $z - z_0$:

$$
\varphi(z) \sim \sum_{\nu=0}^{\infty} c_\nu (z - z_0)^\lambda_\nu
$$

$$
(-N < \lambda_0 < \lambda_1 < \ldots \rightarrow \infty),
$$

and for $z_0 = \infty$ series with decreasing powers of $z$:

$$
\varphi(z) \sim \sum_{\nu=0}^{\infty} \frac{c_\nu}{z^{\lambda_\nu}}
$$

$$
(-N < \lambda_0 < \lambda_1 < \ldots \rightarrow \infty).
$$

According to Eq. (7.8) we have, for the latter case

$$
\varphi(z) = \sum_{\nu=0}^{n} \frac{c_\nu}{z^{\lambda_\nu}} + o \left( \frac{1}{z^{\lambda_n}} \right)
$$
or

$$
z^{\lambda_n} \left[ \varphi(z) - \sum_{\nu=0}^{n} \frac{c_\nu}{z^{\lambda_\nu}} \right] \rightarrow 0
$$

for $z \rightarrow \infty$. This means that not only the error $\varphi(z) - \sum_{\nu=0}^{n}$ but also its product with $z^{\lambda_n}$ tends to zero.

*) Convergent power series are examples of asymptotic expansions. Roughly one can say that an asymptotic series gives for fixed $n$ the better result the nearer $z$ is to $z_0$, the convergent series give for fixed $z$ the better result the nearer $n$ is to $\infty$. 
7.2 Abel and Tauber theorems

We discuss some theorems on asymptotic properties of the Laplace transform.

Let \( F(t) \) and \( G(t) \) have image functions \( f(s) \) and \( g(s) \), and let \( G(t) \) be for all \( t > 0 \) positive and continuous. If

\[
F(t) \sim G(t) \quad (t \to +0)
\]

then

\[
f(s) \sim g(s) \quad (s \to +\infty)
\]

Theorems of this kind are called Abel theorems, even if the transform is not the Laplace one. Another Abel theorem is the following. If (with the same assumptions as above)

\[
F(t) \sim G(t) \quad (t \to +\infty)
\]

then

\[
f(s) \sim g(s) \quad (s \to +0)
\]

Two properties of these theorems should be noted:

a) They change the place where the asymptotic properties are referred to.

b) They are, in general, not reversible. Eq. (7.13) does not follow from Eq. (7.14) nor does Eq. (7.15) from Eq. (7.16).

We apply the first theorem to an example. Let the beta function
\[ B(s, \alpha) = \frac{\Gamma(s) \Gamma(\alpha)}{\Gamma(s+\alpha)} = \int_0^\infty e^{-st} (1 - e^{-t})^{\alpha-1} dt \]

be given. We want the asymptotic behaviour for \( s \to \infty \) with \( \text{Re} \alpha > 0 \).

Since \( \mathcal{L}\{t^{\alpha-1}\} = \Gamma(\alpha)s^{-\alpha} \) we have obviously

\[ F(t) \sim t^{\alpha-1} \quad (t \to +0), \quad f(s) \sim \Gamma(\alpha)s^{-\alpha} \quad (s \to +\infty) \]

Because of

\[ 1 - e^{-t} \sim t \quad (t \to +0) \]

we obtain

\[ G(t) = (1 - e^{-t})^{\alpha-1} \sim t^{\alpha-1} = F(t) \]

therefore, with Eqs. (7.13) and (7.14)

\[ B(s, \alpha) \sim \Gamma(\alpha)s^{-\alpha} \quad (s \to +\infty) \]

With sufficient assumptions on the original function it is possible to prove certain inversions of the Abel theorems. That means one can deduce asymptotic properties of the original function from such properties of the image function. Theorems that deduce asymptotic properties of the original function of a functional transform, from such properties of the image function with additional conditions for the original function, are called Tauber theorems. They are in general more difficult to prove than the Abel ones. An example is the following.

Let \( \mathcal{L}\{F(t)\} = f(s) \) be convergent for all \( s > 0 \). If \( f(s) \to l \) for \( s \to +0 \) and \( F(t) \geq 0 \), then

\[ \int_0^t F(\tau) d\tau \to l \quad (t \to +\infty) \]

holds.
7.2.1 Asymptotic expansion of the image function

We now consider theorems on asymptotic expansions which are of great practical importance.

Let $f(F(t)) = f(s)$ be convergent anywhere. If $F(t)$ has the asymptotic expansion

$$F(t) \sim \sum_{\nu=0}^{\infty} c_{\nu} t^{\lambda_{\nu}} \quad (t \to \infty, \text{real}), \ (-1 < \Re \lambda_0 < \Re \lambda_1 < \ldots),$$

then $f(s)$ has the asymptotic expansion

$$f(s) \sim \sum_{\nu=0}^{\infty} c_{\nu} \frac{\Gamma(\lambda_{\nu} + 1)}{\delta^{\lambda_{\nu} + 1}} \quad (s \to \infty), \ (1 \text{arc } s | \leq \Psi < \frac{\pi}{2}).$$

We apply this theorem to a previous result. In Eq. (6.21) we found, in connection with the inversion of

$$\mu(x, s) = \frac{A}{\Delta^{2} + \omega^{2}} e^{-x \sqrt{s}},$$

the integral (with a change of variable names):

$$f(s) = -\frac{1}{\pi} \int_{0}^{\infty} e^{-st} \frac{t}{t^{2} + \omega^{2}} \sin x \sqrt{t} \, dt \quad (7.20)$$

Because of
\[ \frac{t}{t^2 + \omega^2} = \frac{t}{\omega^2} \frac{1}{1 + \left(\frac{t}{\omega}\right)^2} = \sum_{v=0}^{\infty} (-1)^v \frac{t^{2v+1}}{\omega^{2v+2}} \]
\[ = \frac{t}{\omega^2} - \frac{t^3}{\omega^4} + \frac{t^5}{\omega^6} - \ldots \quad (|t| < \omega) \]

and

\[ \sin \sqrt{t} = \frac{x}{1!} t^{1/2} - \frac{x^3}{3!} t^{3/2} + \frac{x^5}{5!} t^{5/2} - \ldots \quad (\text{all } t), \]

we obtain by multiplication:

\[ \frac{t}{t^2 + \omega^2} \sin \sqrt{t} = \frac{x}{\omega^2} t^{3/2} - \frac{x^3}{3! \omega^2} t^{5/2} + \left( \frac{x^5}{5! \omega^2} - \frac{x}{\omega^4} \right) t^{7/2} + \ldots \]
\[ (|t| < \omega). \]

According to Eq. (7.17), \( f(s) \) has for \( s \to \infty \) the asymptotic expansion

\[ f(s) \approx -\frac{1}{\pi} \left\{ \frac{x}{\omega^2} \frac{\Gamma(5/2)}{\sqrt{s}^{5/2}} - \frac{x^3}{3! \omega^2} \frac{\Gamma(7/2)}{\sqrt{s}^{7/2}} + \left( \frac{x^5}{5! \omega^2} - \frac{x}{\omega^4} \right) \frac{\Gamma(9/2)}{\sqrt{s}^{9/2}} + \ldots \right\} \]

We can replace \( s \) by \( t \) and obtain, for Eq. (6.21)

\[ U(x, t) = e^{-x\sqrt{\omega^2}} \cos(\omega t - x\sqrt{\omega}^2) \]

\[ -\frac{1}{\pi} \left\{ \frac{x}{\omega^2} \frac{\Gamma(5/2)}{t^{5/2}} - \frac{x^3}{3! \omega^2} \frac{\Gamma(7/2)}{t^{7/2}} + O(t^{-9/2}) \right\} \quad (t \to \infty). \]
Another theorem is the following. Let $\mathcal{L}\{P(t)\} = f(s)$ be convergent anywhere. If $P(t)$ has the asymptotic expansion

$$F(t) \sim -\log t \sum_{\nu=0}^{\infty} c_{\nu} t^{\lambda_{\nu}} \quad (t \to 0) \quad (7.23)$$

$$(-1 < \Re \lambda_0 < \Re \lambda_1 < \ldots),$$

then $f(s)$ has for $s \to 0^+$ the expansion

$$f(s) \sim \sum_{\nu=0}^{\infty} c_{\nu} \frac{\Gamma(\lambda_{\nu} + 1)}{\lambda_{\nu} + 1} \log s - \Psi'_{\nu}(s) \quad (s \to 0^+, \text{ real}) \quad (7.24)$$

where

$$\Psi'(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (7.25)$$

is the logarithmic derivative of the gamma function.

7.2.2. Asymptotic expansion of the original function

In practical problems it is often more important to obtain an asymptotic expression for the original function $F(t)$ for $t \to \infty$, than for the image function $f(s)$ for $s \to \infty$. There exist a great number of theorems in this respect, referring to the complex inverse integral (the $\mathcal{L}^{-1}$-transform)\(^*)\).

\(^*)\) It is important to note that these theorems are classified in the books under Abel's asymptotics of the $\mathcal{L}^{-1}$-transform, and not under Tauber's asymptotics of the $\mathcal{L}$-transform.
We start with the $\mathcal{L}^{-1}$-transform

$$\mathcal{L}^{-1}(\mathcal{L}(f)) = \lim_{\eta \to \infty} \frac{1}{2\pi j} \int_{x-j\eta}^{x+j\eta} e^{st} f(t) \, ds$$  \hfill (7.26)

and assume that there is only one singularity $\alpha_0$ of $\gamma(s)$ having the largest real part of all singularities. We assume, furthermore, that the original straight line of integration may be replaced by a contour $\gamma$, consisting of a circular arc at the right of $\alpha_0$ and two straight lines inclined under the angle $\vartheta$ ($\frac{\pi}{2} < \vartheta \leq \pi$) (Fig. 23a)

$$Y(t) = \frac{1}{2\pi j} \int_{\Gamma} e^{st} \gamma(s) \, ds$$
Then one can show that if $y(s)$ can be expanded near $\alpha_0$ into an asymptotic power series

$$y(s) \sim \sum_{\nu=0}^{\infty} c_{\nu} (s - \alpha_0)^{\lambda_{\nu}}$$

$$\quad (-\infty < \lambda_0 < \lambda_1 < \ldots \rightarrow \infty),$$

for $Y(t)$ the following asymptotic expansion

$$Y(t) \sim e^{\alpha_0 t} \sum_{\nu=0}^{\infty} \frac{c_{\nu}}{\Gamma(-\lambda_{\nu})} t^{-\lambda_{\nu} - 1} \quad (7.28)$$

holds. $1/\Gamma(-\lambda_{\nu})$ has to be replaced by 0 if $\lambda_{\nu}$ is a non-negative integer, because of $\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \ldots = \infty$. This means that terms with non-negative integers as exponents in Eq. (7.27) have no influence in Eq. (7.28).

There remains the case where $y(s)$ has more than one singularity $\alpha_\mu$ with the same largest real part. Then one has to try to see whether the original integration path can be replaced by the contour $\Xi$ as given in Fig. 23b. If this is the case, one takes the above expansion for each of the $\alpha_\mu$ and gets the final result by superposition.

As an example, we consider again a result of the semi-infinite heat conductor

$$U(x, t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} e^{s} \frac{1}{s^2 + \omega^2} e^{-x\sqrt{s}} ds \quad (\alpha > 0) \quad (7.29)$$

which we have discussed already in several sections. Here, there are three singularities $\alpha_1 = +j\omega$, $\alpha_2 = -j\omega$, $\alpha_3 = 0$, all with the same real part $\Re \alpha_\mu = 0$. It can be shown that in this case the deformation of the integral path to the contour $\Xi$ is possible, and we construct the expansions (7.27). We have for $s = j\omega$.
\[ M_i(\eta) = \frac{1}{(s^2 + \omega^2)^2} e^{-\sqrt{s}/2} = \frac{1}{s - j\omega} \frac{1}{s + j\omega} e^{-\sqrt{s}/2} = \frac{1}{s - j\omega} \sigma(s). \]

The function \( \sigma(s) \) is analytic in the neighbourhood of \( s = j\omega \) and can be expanded therefore into a Taylor series

\[ \sigma(s) = \sigma(j\omega) + \frac{\sigma'(j\omega)}{1!} (s-j\omega) + \frac{\sigma''(j\omega)}{2!} (s-j\omega)^2 + \ldots \]

When dividing \( \sigma(s) \) by \( s - j\omega \), we see that all terms except the first one have integer exponents \( \geq 0 \).

Because of Eq. (7.28) we have to consider only the first term

\[ \frac{\sigma(j\omega)}{s - j\omega} = \frac{j\omega}{j\omega + j\omega} e^{-\sqrt{j\omega}/2} \frac{1}{s - j\omega} = \frac{e^{-\sqrt{j\omega}/2}}{2(s-j\omega)}. \]  

Introducing this result into Eq. (7.28), we obtain \((\lambda_0 = -1)\)

\[ U_1(x,t) \sim e^{j\omega t} \frac{e^{-\sqrt{j\omega}/2}}{2}. \]  

Similarly, for \( s = -j\omega \), Eq. (7.28) yields

\[ U_2(x,t) \sim e^{-j\omega t} \frac{e^{-\sqrt{-j\omega}/2}}{2}. \]

The singularity at \( s = 0 \) gives (multiplication of power series)
\[
\mathcal{U} \left( \frac{A}{\Delta^2 + \omega^2} e^{-x \sqrt{2}} \right) = \frac{1}{\omega^2} \Delta \frac{x}{\lambda^2} \Delta^3 + \frac{x^2}{2 \lambda \omega^2} \Delta^2 \frac{x^3}{3 \lambda \omega^2} \Delta^{5/2} + \left( \frac{x^4}{4 \lambda^2 \omega^2} - \frac{1}{\omega^4} \right) \Delta^3 + \ldots
\]

and, with Eq. (7.28)

\[
\mathcal{U}_3 (x, t) \sim \frac{x}{\omega^2} \frac{1}{\Gamma \left( \frac{3}{2} \right)} \frac{1}{t^{-5/2}} - \frac{x^3}{3 \lambda \omega^2} \frac{1}{\Gamma \left( \frac{7}{2} \right)} \frac{1}{t^{-7/2}} + \ldots \quad (7.33)
\]

Using a relation for the gamma function [cf. Eq. (6.31)]

\[
\frac{1}{\Gamma \left( \frac{1}{2} - \nu \right)} = \frac{(-1)^\nu}{\Gamma (\nu + \frac{1}{2})} = \frac{(-1)^\nu (2 \nu)!}{4 \sqrt{\pi} \nu \sqrt{4 \nu + 1}}
\]

we can write

\[
\mathcal{U} \left( \frac{\nu}{\lambda} \right) \sim \frac{1}{\pi^3/2} \left\{ \frac{x}{\omega^2} \frac{1}{t^{-5/2}} - \frac{x^3}{3 \lambda \omega^2} \frac{1}{t^{-7/2}} + \ldots \right\}
\]

\[
\sim \frac{1}{\pi^{3/2}} \left\{ \frac{x}{\omega^2} \frac{3}{4} t^{-5/2} - \frac{x^3}{3 \lambda \omega^2} \frac{15}{8} t^{-7/2} + \ldots \right\}
\]

The complete asymptotic expansion of \( \mathcal{U}(x, t) \) is the superposition

\[
\mathcal{U}(x, t) \sim \mathcal{U}_1 (x, t) + \mathcal{U}_2 (x, t) + \mathcal{U}_3 (x, t) \quad (7.35)
\]

It is clear that \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) may be combined together and we get the expression (7.22).
BIBLIOGRAPHY

Books and Tables used for preparing the lecture, occasionally directly referenced in the text


OTHER RELEVANT BOOKS (NO COMPLETE LIST)


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