ELASTIC SCATTERING AND ORTHOGONAL POLYNOMIALS

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ABSTRACT

We give a series of illustrations of the role of orthogonal polynomials, especially Legendre polynomials in the study of scattering amplitudes and show the need to improve the known bounds on these polynomials to get physically interesting results. The chosen examples are: bounds on the slope of the diffraction peak, extension of the analyticity domain of scattering amplitudes by using positivity, derivation of the $(\log s)^2$ bound on the total cross-section, oscillation properties of Legendre polynomials and bounds on the phase of the scattering amplitude leading to Pomeranchuk-like theorems, equicontinuity properties of Legendre polynomials. Finally we give indications on the case of particles with spins different from zero and insist on the need for a qualitative and quantitative improvement on the bounds on associated Legendre functions.

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1. INTRODUCTION

There exists a basic object in atomic, nuclear and particle physics which is the scattering amplitude of a beam of particles by either a fixed object or another particle. The knowledge of the scattering amplitude allows us to obtain the differential cross-section and also the total cross-section, which can be measured experimentally.

The scattering amplitude depends on two variables:

\[
\begin{align*}
\theta & \quad \text{scattering angle in the centre of mass} \\
k & \quad \text{centre of mass moment}
\end{align*}
\]

There are other variables used according to the needs, for instance \( s \), square of the centre-of-mass energy, \( t \), the square of the momentum transfer:

\[
\begin{align*}
\{ s & = (\sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2})^2, \\
& \quad \text{for collisions of particles } A \text{ and } B, \\
& t = 2k^2(\cos \theta - 1)
\end{align*}
\]

(notice that \( t \) is negative for \( -1 < \cos \theta < +1 \), \( s \) and \( t \) can also be taken as independent variables. However, a third variable is useful, \( u \), defined as

\[
s + t + u = 2M_A^2 + 2M_B^2,
\]

for collisions \( A + B \rightarrow A + B \).

\( u \) is needed because the scattering amplitude can be analytically continued from the physical region for \( A + B \rightarrow A + B \) scattering to the physical region for \( A + \bar{B} \rightarrow A + \bar{B} \), where \( \bar{B} \) is the antiparticle of \( B \). Then \( u \) is the square of the centre-of-mass energy for \( A + B \rightarrow A + \bar{B} \). This is the famous “crossing property” established in full generality by Bros, Epstein and Glaser [1].

In the case where particles have spins there are several amplitudes, but for simplicity I shall most of the time speak only about spin-zero particles.

The scattering amplitude \( F(s, \cos \theta) \) is related to the differential cross-section by

\[
\frac{d\sigma}{dt} = \frac{1}{k\sqrt{s}} |F(s, \cos \theta)|^2,
\]

and the total cross-section is given by the “optical theorem”

\[
\sigma_{\text{total}} = \frac{4\pi}{k\sqrt{s}} \text{Im} F(s, \cos \theta = 1)
\]

i.e., the total cross-section is proportional to the imaginary part of the forward scattering amplitude.

Orthogonal polynomials appear because the scattering amplitude can be expanded in Legendre polynomials (for higher spins, the various amplitudes, depending on the basis chosen, can be expanded in \( P_{\ell}^{\text{na}} \), or in \( d \) functions, the functions appearing in rotation matrices):

\[
F(s, \cos \theta) = \frac{\sqrt{s}}{k} \Sigma (2\ell + 1) f_{\ell}(s) P_{\ell}(\cos \theta)
\]

This expansion converges not only for \( -1 < \cos \theta < +1 \), but also in an ellipse in the complex \( \cos \theta \) plane, with foci at \( \cos \theta = \pm 1 \). This has been shown by Lehmann for the case of elementary particles, from the axioms of field theory [2] and it is also true in potential scattering in the case of potentials decreasing faster than some exponential.
Now you may ask: why expand in Legendre polynomials and not something else? A function analytic inside an ellipse with foci $\pm 1$ can also be expanded in the Chebyshev polynomials for instance.

The reason for the special choice of Legendre polynomials is tied to invariance of physics under the rotation group. The expansion in Legendre polynomials "diagonalizes" the unitarity condition:

- In general, one has

$$\begin{cases}
\text{Im} f_\ell \geq |f_\ell|^2, \\
\text{which implies} \\
1 \geq \text{Im} f_\ell \geq 0 \\
\text{and } |f_\ell| \leq 1
\end{cases} \quad (7)$$

- if $\sqrt{s}$, the centre-of-mass energy, is such that only the reaction

$$A + B \to A + B$$

is energetically possible, i.e.,

$$A + B \to A + B + C$$

is impossible because

$$M_A + M_B + M_C > \sqrt{s},$$

then unitarity takes a much more restrictive form

$$\text{Im} f_\ell = |f_\ell|^2 \quad (8)$$

This is commonly re-expressed by writing $f_\ell = e^{i\delta_\ell} \sin \delta_\ell$, where $\delta_\ell$ is real, and called a "phase shift".

Naturally when I say that in general you get an inequality for $f_\ell$, this is true if you look only at the reaction $A + B \to A + B$. If you would consider all possible multiparticle amplitudes at a given energy, you would get equations, but these equations are very difficult to handle in my opinion. Some people, however, think that multiparticle unitarity can be exploited [3,4].

Perhaps the simplest aspect of the unitarity condition is positivity, the fact that the imaginary part of the scattering amplitude has an expansion in Legendre polynomials with positive coefficients. Physicists introduce another notion, the absorptive part. For $-1 < \cos \theta < +1$ the absorptive part coincides with the imaginary part:

$$\text{Im} F(s, \cos \theta) = A_s(s, \cos \theta) \quad (9)$$

However, $A_s$ can be defined at least inside the ellipse of convergence of $F$ by

$$A_s(s, \cos \theta) = \frac{1}{2\pi} \int \left| F(s, \cos \theta) - \bar{F}(s, \cos \theta) \right| \quad (10)$$

where the bar means complex conjugation. Hence

$$A_s(s, \cos \theta) = \frac{\sqrt{s}}{k} \Sigma(2\ell + 1) \text{Im} f_\ell(s) P_\ell(\cos \theta) \quad (11)$$

In fact $A_s(s, \cos \theta)$ has often an expansion in Legendre polynomials which converges inside a larger ellipse, for instance in the case where property (6) is satisfied. Then one can show that if the semi-major axis of the ellipse of convergence of $F$ is $\cosh \theta_0$, the semi-major axis of the ellipse of convergence of $A$ is $\cosh(2\theta_0)$.

In fact, positivity can be expressed in a way which is independent of the expansion [5]. If we introduce $A$ which is a function of the unit vectors $1$ and $2$

$$A(s, \cos \theta) = A(s, 1, 2)$$

with $\cos \theta = \cos(\sqrt{2}s)$

we have

$$\int d\Omega_1 d\Omega_2 w(1) w(2) A(s, 1, 2) > 0 \quad (13)$$

where $w$ is an arbitrary complex function defined on the unit sphere.

If $A$ can be expanded in Legendre polynomials, (13) guarantees that the coefficients of the expansion are positive. It suffices to take $w(1) = Y_{\ell m}(1)$.

Amusing properties can be obtained directly from (13). For instance, Glaser [6] takes $w$ to have its support concentrated at the extremities of a regular tetrahedron. Then positivity requires

$$\begin{vmatrix}
A(1) & A(-1/3) & A(-1/3) & A(-1/3) \\
A(-1/3) & A(1) & A(-1/3) & A(-1/3) \\
A(-1/3) & A(-1/3) & A(1) & A(-1/3) \\
A(-1/3) & A(-1/3) & A(-1/3) & A(1)
\end{vmatrix} > 0$$

which implies

$$A(-1/3) \geq -\frac{A(1)}{3}$$

and hence

$$P_1(\cos \theta = -1/3) \geq -1/3 \quad (14)$$

an inequality which is saturated by $P_1(\cos \theta)$ and $P_2(\cos \theta)$.

Now I would like to present to you a certain number of problems where properties of Legendre polynomials have been useful, and where physicists have had sometimes to develop their own tools or at least to improve the existing results on Legendre polynomials.

2. UPPER BOUNDS ON THE ABSORPTIVE PART
IN THE PHYSICAL REGION

A very good review on the subject is given by S.M. Roy [7]. Here I just want to give a flavour of the subject. The problem is to find bounds on the imaginary part of the scattering amplitude, which is believed to dominate
the scattering amplitude, in terms of global data like the total cross-section \[8\] and, possibly the elastic cross-section \[9\]. For this purpose we use \(0 \leq \text{Im}f_t \leq 1\) combined with bounds on Legendre polynomials. Typical bounds found in the mathematical literature are \[10\]

\[
\begin{align*}
| P_\ell (\cos \theta) | &< 1 \\
| P_\ell (\cos \theta) | &< \sqrt{\frac{2}{\pi (\ell + 1/2) \sin \theta}}
\end{align*}
\]  
(15)

However, this is not good enough for our purpose. In the limit of very high energies we look at the limit of very small angles, for instance fixed negative \(t\), i.e., \(\theta \sim \frac{1}{\ell}\), and the second of the bounds (15) diverges unless \(\ell\) tends also to infinity. This is why it was necessary to invent a new bound \[8\] which is:

\[
| P_\ell (\cos \theta) | < B_\ell (\cos \theta) = \left[ 1 + \ell (\ell + 1) \sin^2 \theta \right]^{-1/4},
\]  
(16)

for \(-1 < \cos \theta < +1\).

The merits of the bound are the following:

1) \(B_\ell \leq 1\)

2) \[
\frac{d}{d\theta} B_\ell (\cos \theta) = d B_\ell (\cos \theta), \text{ at } \cos \theta = 1
\]

which means that the bound is very good in the forward direction.

3) \(B_\ell\) has the right qualitative behaviour at fixed angle:

\[
B_\ell \sim \frac{1}{\sqrt{\ell \sqrt{\sin \theta}}},
\]

i.e., we are only missing a factor \(\sqrt{2/\pi}\).

4) Finally, a technical advantage for minimisation problems is that \(B_\ell\) is monotonous in \(\ell\) and \(\theta\).

From these ingredients it is possible to obtain the bound:

\[
\frac{| A_\ell (s, \cos \theta) |}{| A_\ell (s, 1) |} < \frac{4}{3} \left( 1 + \frac{\ell}{L (\ell + 1) \sin^2 \theta} \right)^{3/4} - 1
\]

with

\[
L = \text{integer part of} \left( \frac{k^2 \sigma_{\text{total}}}{4\pi} \right)^{1/2} - 1
\]

or zero if the previous quantity is negative.

A consequence of this is an inequality on the logarithmic derivative of \(A\) in the forward direction, obtained by taking the limit \(\theta \to 0\):

\[
\left[ \frac{d}{dt} \log a_s (s, t) \right]_{t=0} \geq \frac{1}{8} \left( \frac{\sigma_\ell}{4\pi} - \frac{1}{k^2} \right)
\]  
(18)

(notice the change of variables \(\cos \theta \to t\) and the change of notations!).

To the extent that the real part contribution is small, we can compare this inequality with experiment. One finds that one gets at least the right order of magnitude. If one includes the elastic cross-section as a constraint, one gets the inequality \[8\]:

\[
\left[ \frac{d}{dt} \log a_s (s, t) \right]_{t=0} \geq \frac{1}{9} \left( \frac{\sigma_{\text{total}}^2}{4\pi \sigma_{\text{elastic}}} - \frac{1}{k^2} \right)
\]  
(19)

The remarkable fact about this inequality is that, if one neglects the real part contribution, one can compare the left-hand side and the right-hand side experimentally and see that they never differ by more than 10%.

3. Extension of the Analyticity Domain of the Scattering Amplitude by Positivity

It has been proposed long ago by Goldberger, and proved by Bogoliubov, Symanzik, and others \[11\] that the scattering amplitude satisfies "dispersion relations" which imply that it has certain analyticity properties. Defining

\[
f(s, t) = F(s, \cos \theta)
\]  
(20)

one has, for \(-T < t \leq 0\),

\[
f(s, t) = \frac{s^N}{\pi} \int_0^\infty \frac{a_s (s', t) ds'}{s'^N (s'^2 - s')} + \frac{u^N}{\pi} \int_0^\infty \frac{a_s (u', t) du'}{u'^N (u'^2 - u)}
\]

+ polynomials in \(s\) with coefficients depending on \(t\)

\[
N \text{ is the number of subtractions. You notice the introduction of the auxiliary variable } u.
\]

Equation (21) defines an analytic function in a twice cut plane, for fixed \(t\). The physical scattering amplitude \(A \bar{B} \to \bar{A}B\) is the boundary value of this function for \(s = 2 + i \epsilon, \epsilon > 0, \epsilon \to 0\) and the "crossed amplitude" \(A \bar{B} \to \bar{A}B\) is the boundary value for \(u = 2 + i \epsilon, \epsilon > 0, \epsilon \to 0\).

These dispersion relations are interesting as they are in themselves, and have been tested, especially for \(t = 0\), where the absorptive parts under the integrals are proportional to the total cross-sections for \(A \bar{B}\) and \(AB\) collisions, but it was felt that, for theoretical needs, they were insufficient. It is there that positivity has played a crucial role \[12\].

For simplicity we forget about subtractions (i.e., take \(N = 0\)) and about the left-hand cut. Then

\[
f(s, t) = \frac{1}{\pi} \int_0^\infty \frac{a_s (s', t) ds'}{s'^2 - s}
\]

(22)

for \(-T < t \leq 0\)

Now we want to calculate the successive derivatives of \(f\), for some value \(s\) real \(\leq s_0\), at \(t = 0\):

\[
\frac{d}{dt} f(s, t) \big|_{t=0} = \lim_{T \to 0} \frac{f(s, 0) - f(s, -T)}{T}
\]

(23)
We can substitute the dispersion relation in (22) and get

$$\frac{d}{dt} f(s, t)|_{t=0} = \lim_{\delta \to 0} \int_{s_0}^{s_0+\delta} \frac{a(s', 0) - a(s', -T)}{s' - s} ds'$$

(24)

Except for unimportant complications near $s = s_0$, the arguments of $a_s$ in the integral, $(s', -T)$ correspond to a physical point, i.e., $-1 < \cos \theta < +1$, and hence from $|P_t(\cos \theta)| < P_t(1)$ we deduce that $a_s(s', 0) > a_s(s', -T)$ and that the integrand is positive. Then, from a lemma by Fatou we can take the limit under the integral and get

$$\frac{d}{dt} f(s, t)|_{t=0} = \frac{1}{\pi s_0} \int_{s_0}^{\infty} \frac{d}{ds} a(s', t)|_{t=0} ds'$$

(25)

One can repeat the argument for higher derivatives using

$$\left| \frac{d^n P_t(\cos \theta)}{(d \cos \theta)^n} \right| \leq \left| \frac{d^n P_t(\cos \theta)}{(d \cos \theta)^n} \right|_{\cos \theta = 1}$$

(26)

$$-1 \leq \cos \theta \leq +1$$

Then one gets

$$\left( \frac{d}{dt} \right)^n f(s, t)|_{t=0} = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{d}{ds} a(s', t = 0) ds'$$

(27)

However there is another piece of information from field theory [13], which is that for $s_1$ real, slightly below the beginning of the cut, $s_0$, $f(s_1, t)$ is analytic in $|t| \leq R$, so that

$$\left| \left( \frac{d}{dt} \right)^n f(s_1, t)|_{t=0} \right| < \frac{n! M}{R^\delta}$$

(28)

Then (27) can be continued in the complex plane for any complex $s$: the convergence of the positive integrand for $s = s_1$ guarantees the convergence for any complex $s$. Furthermore,

$$\left| \left( \frac{d}{dt} \right)^n f(s, t)|_{t=0} \right| < \sup_{s_0 < s' < s} \left| \frac{s' - s_1}{s' - s} \left( \frac{d}{dt} \right)^n f(s, t)|_{t=0} \right|$$

(29)

and this suffices to prove that, for any complex $s$, $f(s, t)$ is analytic in the variable $t$ for $|t| < R$, $R$ independent of $s$. In other words, $f(s, t)$ is analytic in the product $s$ cut plane $|t| < R$.

The complications arising from the presence of the second cut and from the subtractions are easily taken care of.

4. THE BOUND $a_{total} < \text{const} \times (\log s)^2$ AND THE NEED FOR LOWER BOUNDS ON LEGENDRE POLYNOMIALS FOR $\cos \theta > 1$

In 1961, Froissart [14] proved that the total cross-section cannot grow faster than $(\log s)^2$ from the assumption of the validity of the Mandelstam representation, an assumption which was neither proved nor disproved. But the analyticity domain obtained in the previous section is sufficient to obtain this result [12]. It implies that dispersion relations hold not only for $T < t \leq 0$ but also for $|t| < R$, and using again positivity one can prove that the absorptive part of the scattering amplitude is analytic in $\cos \theta$ in an ellipse with semi-major $1 + \frac{R}{\sqrt{2}}$. This is because the ellipse of convergence of a Legendre polynomial series with positive coefficients must have a singularity at the extreme right.

The validity of dispersion relations for $t = +R$ with $N$ subtraction implies

$$a_s(s, R) < s^{N/2}$$

(30)

or, more exactly,

$$\int_{s-R}^{s+R} a_s(s, R) ds < s^{N/2}$$

(31)

since $a$, strictly speaking, is a positive distribution. Then we have

$$a_s(s, R) = A_s(s, 1 + \frac{R}{2k^2}) = \sqrt{s} \frac{N}{k} \sum_{n=1}^{\infty} (2n + 1) \text{Im} f_t(s) P_t(1 + \frac{R}{2k^2}) < s^{N/2}$$

The left-hand side of (31) is a sum of positive terms.

Hence, we have, for each individual term,

$$\text{Im} f_t(s) < \frac{s^{N/2}}{(2n + 1) P_t(1 + \frac{R}{2k^2})}$$

(32)

Therefore we need a lower bound on $P_t(z)$ for $1 \leq z < \infty$.

The asymptotic estimate for $n \to \infty$, $s > 1$,

$$P_t(z) \sim \sqrt{\frac{2}{\pi n^2}} \frac{1}{(z^2 - 1)^{1/4}} \frac{(z + \sqrt{z^2 - 1})^{n+1}}{\sqrt{z^2 - 1}}$$

(33)

is not a lower bound, because it is singular at $z = 1$.

A crude lower bound can be obtained from the representation
\[ P_\ell(z) = \frac{1}{\pi} \int_0^\pi (z + \cos \theta \sqrt{z^2 - 1})^\ell d\phi, \quad (34) \]

by cutting the integral at \( \phi_0 \). Hence

\[ P_\ell(z) \geq \frac{\phi_0}{\pi} (z + \cos \phi_0 \sqrt{z^2 - 1})^\ell \quad (35) \]

It is in fact possible to obtain the best possible upper bound of this kind:

\[ P_\ell(z) \geq \frac{(2N)!}{(N!)^2} \left( \frac{N + 1}{2(2N + 1)} \right) \left[ z + \frac{N}{N + 1} \sqrt{z^2 - 1} \right]^\ell \quad (36) \]

with \( N = 1, 2, 3, \ldots \).

Since this is an unpublished result, we can sketch the proof. Defining

\[ R(\ell, z) = \frac{P_\ell(z)}{(z + \cos \phi_0 \sqrt{z^2 - 1})^\ell} \quad (37) \]

for arbitrary real positive \( \ell \), we see from (35) that \( R_\ell \) has a strictly positive lower bound in the quadrant

\[ 1 < z < \infty \quad \& \quad 0 \leq \ell < \infty. \]

Therefore it has an absolute minimum or an absolute infimum. An elementary calculation gives

\[ \frac{\partial}{\partial z} R(\ell, z) = \frac{\ell}{(z^2 - 1)(z + \cos \phi_0 \sqrt{z^2 - 1})} \left[ R(\ell, z) - R(\ell - 1, z) \right] \quad (38) \]

On the other hand it is obvious from (34) and (37) that \( R(\ell, z) \) is logarithmically convex, and differentiable in \( \ell \). Hence, if \( \frac{\partial}{\partial \ell} R(\ell, z) = 0 \), \( R(\ell, z) = R(\ell - 1, z) \) and hence it is impossible to have \( \frac{\partial}{\partial \ell} R(\ell, z) = 0 \). The value of where this happens lies strictly between \( \ell - 1 \) and \( \ell \). Therefore there is no absolute minimum inside the quadrant \( 0 \leq \ell < \infty, 1 \leq z < \infty \). It is easy to see that the absolute infimum lies at infinity in \( z \). Using asymptotic behaviour of Legendre polynomials, one gets (36), which is saturated for \( \ell = N \) and \( \ell = N + 1 \), for \( z \to \infty \).

Returning to Eq. (32), we shall content ourselves with (35) to get qualitative features with \( \phi_0 = \pi/3 \). Furthermore we shall assume \( 2k^2 > R \). Then we have, with

\[ z = 1 + \frac{R}{2k^2}, \]

\[ z + \frac{1}{2} \sqrt{z^2 - 1} > \exp -1 \sqrt{2} \sqrt{z - 1} \]

and hence

\[ \text{Im} f_\ell < \frac{s^N}{2k + 1} \exp -1 \sqrt{2} \sqrt{R} \quad (39) \]

Since \( \sqrt{s} \approx 2k \), (39) shows that \( \text{Im} f_\ell \) becomes negligible for

\[ \ell > 2N \sqrt{R} k \log k = L \quad (40) \]

to bound the total cross-section we can use the unitarity limit on partial waves, \( 0 < \text{Im} f_\ell \leq 1 \) and get

\[ \sigma_{\text{tot}} \lesssim 4\pi \left( \frac{k}{L + 1} \right)^2 \approx \text{const} \left( \log s \right)^2 \quad (41) \]

which is the celebrated "Froissart" bound.

5. - OSCILLATION PROPERTIES OF LEGENDRE POLYNOMIALS AND THEIR APPLICATION TO THE PROOF OF "POMERANCHUK" THEOREMS FOR DIFFERENTIAL CROSS-SECTIONS

Cornille and I have proved [15] that the ratio of differential cross-sections for reactions \( AB \to AB \) and \( AB \to AB \) tends to unity (if it has a limit) at infinite energy, for such momentum transfers that the differential cross-section is not negligible with respect to the forward differential cross-section.

In the proof there are two ingredients: the fact that for fixed \( t \) the scattering amplitude satisfies a dispersion relation as we have seen, the boundary value above the right-hand cut being the amplitude for \( AB \to AB \), and the boundary value above the left-hand cut being the complex conjugate of the \( AB \to AB \) amplitude:

\[ \frac{AB \to AB}{(\text{energy})^w} = \left( \frac{AB \to AB}{(\text{energy})^w} \right)^* \]

The theorem that Cornille and I proved [15] is:

\[ \lim_{s \to \infty} \left| f(s, t) \right|_{AB \to AB} = 1. \]

if it exists, and if the phases of \( f_{AB \to AB} \) and \( f_{AB \to AB} \) are bounded.

The phase of \( f \) has to be defined by continuity. Naturally this can be done unambiguously only in a region where \( |f| \) does not vanish.

In the special case of \( t = 0 \), i.e. exactly forward direction, \( \text{Im} f_{AB \to AB} > 0 \) and \( \text{Im} f_{AB \to AB} > 0 \). This means that the phase of \( AB \to AB \) as well as the phase of \( AB \to AB \) remain between 0 and \( \pi \), and therefore [16]

\[ \lim_{\omega \to 0} \frac{\partial}{\partial \omega} f_{AB \to AB} = 1 \]

The problem is to find a bound on the phase of the scattering amplitude for \( t < 0 \). For \( t < 0 \), i.e., \( \cos \theta < 1 \), the Legendre polynomials have no definite sign, and there is absolutely no guarantee that the amplitude has a positive imaginary part.
To get a bound on the phase, we want to exert some control on the oscillations of the absorptive part of the scattering amplitude from $\theta = 0$, where it is positive, to some physical angle. With Cornille [17] we have obtained the following property

$$\left| \sum_{n=0}^{N} (-1)^{n} P_{\ell}(x_{n}) \right| < c\sqrt{N + 1}$$

with $1 > x_{0} > x_{1} > \ldots > x_{N} > -1$

$c$ being independent of $\ell$.

What is non-trivial in (44) is that the right-hand side behaves like $\sqrt{N}$ and not $N$. From (44) we get, using (11)

$$\left| \sum_{n=0}^{N} (-1)^{n} A_{s}(s, \cos \theta_{n}) \right| < c\sqrt{N + 1} A_{s}(s, \cos \theta = 1)$$

$$0 < \theta_{1} < \theta_{2} \ldots < \theta_{n} < \pi$$

(45)

Before giving the proof of (44) let us explain why (45) restricts the phase of $F(s, \cos \theta)$.

Suppose the phase of $F$ varies from $0 < \varphi(0) < \pi$ for $\theta = 0$ to $N\pi < \varphi(\theta) < (N + 1)\pi$. We take, for instance, $N > 0$. Then $\varphi$ takes successively, at least once, in that order, the values $\frac{\pi}{N}, \frac{2\pi}{N} \ldots N\pi - \frac{\pi}{N}$, for $\theta = \theta_{1} < \theta_{2} \ldots < \theta_{N-1}$. Then we have

$$-\text{Im} F(\cos \theta_{1}) = |F(\cos \theta_{1})|$$

$$+ \text{Im} F(\cos \theta_{2}) = |F(\cos \theta_{2})|$$

$$(-1)^{k} \text{Im} F(\cos \theta_{k}) = |F(\cos \theta_{k})|$$

and therefore, from (45),

$$|F(\cos \theta_{1})| + |F(\cos \theta_{2})| + \ldots + |F(\cos \theta_{N-1})|$$

$$< C\sqrt{N} \text{Im} F(\cos \theta = 1) < C\sqrt{N} |F(\cos \theta = 1)|$$

(46)

if we call

$$|F|_{\text{min}}(\cos \theta) = \inf_{0 < \theta' < \theta} |F(\cos \theta')|$$

we get from (46):

$$\sqrt{N} - 1 < \frac{|F(1)|}{|F|_{\text{min}}(\cos \theta)}$$

(47)

therefore

$$\varphi(\theta) < \pi \left[ 2 + \frac{|F(1)|}{|F|_{\text{min}}(\cos \theta)} \right]^{2}$$

(48)

This is not the best possible result, but it shows that the knowledge of the modulus of the amplitude in the interval $0 - \theta$ gives a bound on its phase.

The "diffraction peak" is precisely that region where the right-hand side of (47), and hence of (48), remains bounded as the energy goes to infinity.

Now we shall try to give a sketch of the proof of (44), in the simpler case where all the $x_{i}'s$ are in the forward hemisphere:

$$0 < x_{1} < x_{2} < \ldots < x_{N} < 1$$

(49)

We define

$$A_{n,\ell} = \text{sup} \left[ -P_{\ell}(x_{1}) + P_{\ell}(x_{2}) + \ldots + (-1)^{n} P_{\ell}(x_{n}) \right]$$

(50)

Then it is easy to show that if $n < \frac{\ell}{2}$, the $x_{i}'s$ have to lie at the successive minima and maxima of $P_{\ell}$. Then

$$A_{n,\ell} = m_{1,\ell} + m_{2,\ell} + m_{3,\ell} + \ldots + m_{n,\ell}$$

(51)

where the $m_{k,\ell}$ are the successive maxima of $|P_{\ell}|$ in decreasing order. It is well known that these maxima form a monotonous sequence [10].

There is a less well-known property which we have proved [17] which is:

$$m_{k,\ell} > m_{k,\ell+1} > \ldots > m_{k,\infty}$$

(52)

where $m_{k,\infty}$ is the $k$th maximum of the Bessel function $|J_{0}|$. It follows from (52) that

$$A_{n,\ell} \leq A_{n,2n} \quad \text{if} \quad \frac{\ell}{2} \geq n$$

(53)

if, on the other hand, $\frac{\ell}{2} < n$ the optimum choice consists in taking the first $\left[ \frac{n}{2} \right]$ $x_{i}$'s at the extrema of $P_{\ell}$ and the remaining ones coincident, i.e., not contributing to the sum. Therefore

$$A_{n,\ell} = A_{\left[ \frac{n}{2} \right],\ell} \quad \ell < 2n$$

and therefore

$$A_{n,\ell} \leq \sup_{p \leq n} A_{p,2p}$$

(54)

What is left is to estimate $A_{n,2n}$. Again we use property (52) which implies

$$m_{k,\ell} \leq m_{k,2k} = |P_{2k}(0)| < \sqrt{\frac{2}{\pi(2k + 1/2)}}$$

(55)

and summing (51) with $\ell = 2n$, we get
\[ A_{n,2n} < \sqrt{\frac{\pi}{2}} \sqrt{2n + 1/2} \]  
(56)

and

\[ A_{n,\ell} < \sqrt{\frac{\pi}{2}} \sqrt{2n + 1/2} \]  
(57)

This inequality could be slightly improved quantitatively but it has the correct qualitative behaviour that we need to prove inequality (45) and get a bound on the phase.

6. EQUICONSERVENCY OF FUNCTIONS OF POSITIVE TYPE

In the problem of reconstructing the scattering amplitude from the knowledge of the differential cross-section at a given energy, one line of attack is to use the Leray-Schauder fixed point theorems [18]. These theorems apply to a set of equiconfinuous functions. We want to show that if a set of functions of \( \cos \theta \) are of positive type in the sense of (19), and if all functions take the same value at \( \cos \theta = 1 \), this set is equiconfinuous and in fact satisfies a Lipschitz condition if one excludes a small interval near \( \theta = 0 \).

We have

\[ A(\cos \theta) = \sum \alpha_\ell P_\ell(\cos \theta), \quad \alpha_\ell \geq 0 \]  
(58)

Now, what we need is a bound on \( P_\ell(\cos \theta_1) - P_\ell(\cos \theta_2) \) which is independent of \( \ell \).

First of all we have, trivially:

\[ |P_\ell(\cos \theta_1) - P_\ell(\cos \theta_2)| < \sqrt{\pi(\ell + 1/2)\left[ \frac{1}{(\sin \theta_1)^{1/2}} + \frac{1}{(\sin \theta_2)^{1/2}} \right]} \]  
(59)

But, on the other hand, we can integrate a bound on \( P_\ell'(\cos \theta) \):

\[ |P_\ell'(\cos \theta)| < 1.225 \sqrt{\ell + 1/2} \frac{1}{(\sin \theta)^{3/2}} \]  
(60)

This bound is already present in a qualitative way in Szegö's book [10] but the numerical constant was obtained during this Conference as we shall explain in the last section.

From (60) we get

\[ |P_\ell(\cos \theta_1) - P_\ell(\cos \theta_2)| < 1.225 \sqrt{\ell + 1/2} \int_{\ell_1}^{\ell_2} \frac{d \cos \theta}{(\sin \theta)^{3/2}} \]  
(61)

\[ < c|\ell_1^{1/2} - \ell_2^{1/2}| \]

Multiplying (60) and (61), we eliminate \( \ell + 1/2 \) and get, in the end

\[ |P_\ell(\cos \theta_1) - P_\ell(\cos \theta_2)| < c\frac{\sqrt{\ell_1 - \ell_2}}{(\sin \theta_1 \sin \theta_2)^{1/4}} \]  
(62)

Therefore we get

\[ |A(\cos \theta_1) - A(\cos \theta_2)| < c\frac{\sqrt{\theta_1 - \theta_2}}{(\sin \theta_1 \sin \theta_2)^{1/4}} A(\cos \theta = 1) \]  
(63)

which guarantees that if we have a set of functions with the same \( A(1) \), they are equiconfinuous for \( \theta > \epsilon > 0 \).

7. GENERALIZATIONS TO HIGHER SPINS. THE NEED FOR IMPROVEMENTS ON EXISTING BOUNDS ON THE \( P_\ell^m \)

For higher spins one has more than one amplitude. One can take several possible bases, the normal basis, where the spin is quantized perpendicularly to the scattering plane, or the helicity basis where initial spin is quantized along the initial direction of the particles and the final spin is quantized along the final direction of the particles [19]. In the latter case the amplitudes are expanded in \( d \) functions which are products of \( \sin \theta/2 \), \( \cos \theta/2 \) and Jacobi polynomials of \( \cos \theta \). Bounds can be obtained. For details, see for instance Cornille [20], Mahoux and Martin [21]. It happens that in recombining helicity amplitudes to get amplitudes corresponding to a given parity, derivatives of Legendre polynomials appear, and bounds on derivatives of Legendre polynomials are needed.

Here, as a physicist, I would like to complain that bounds existing in the mathematical literature are often far from being optimal. For instance, Gradstein and Rysik [22] give

\[ |P_\ell^m(\cos \theta)| < \sqrt{\frac{8}{\ell \pi}} \frac{\Gamma(\ell + m + 1)}{\Gamma(m + 1)} \frac{1}{(\sin \theta)^{m+1/2}} \]  
(64)

while qualitatively Szegö [10] gives

\[ |P_\ell^m(\cos \theta)| < c\frac{\ell^{m-1/2}}{(\sin \theta)^{1/2}} \]  
(65)

with \( P_\ell^m(\cos \theta) = (-1)^m(\sin \theta)^{m/2} d^m P_\ell(\cos \theta) \).

We believe that the best possible answer would be

\[ |P_\ell^m(\cos \theta)| < \sup_{x < \infty} \frac{1}{\Gamma(\ell + 1/2)} \frac{\Gamma(\ell - m + 1)}{(\sin \theta)^{1/2}} \]  
(66)

This is based on the integral representation [23].
\[ P_{\ell}^{-m}(\cos \theta) = \frac{1}{\Gamma(\ell + m + 1)} \int_0^\infty \exp(-t \cos \theta) J_m(t \sin \theta) t^\ell dt \]  

In the meantime, I have found a bound on $\frac{dP_{\ell}^{-m}}{d \cos \theta}$, after having been taught "Mathematica" by Professors Prevost and Gilewicz, here in Erice.

Using the fact that [10]

\[ |u(\theta)|^2 + \frac{1}{\phi(\theta)} |u'(\theta)|^2, \]

with

\[ u(\theta) = (\sin \theta)^{1/2} P_{\ell}(\cos \theta), \]

and

\[ \phi(\theta) = (\ell + 1/2)^2 + \frac{1}{4 \sin^2 \theta}. \]

is increasing with $\theta$ we get

\[ (\sin \theta)^{3/2} |P_{\ell}^\prime| < \sqrt{\frac{2}{\pi(\ell + 1/2)}} \left[ ((\ell + 1/2)^2 + \frac{1}{\sin^2 \theta})^{1/2} + 1 \right] \left( \frac{1}{2 \sin \theta} \right) \]  

(68)

and, on the other hand, from the fact that

\[ f(x) = (1 - x^2)(P_{\ell}^2 + \ell(\ell + 1))(P_{\ell}^2 \right] \]

is increasing in $x$ [10]:

\[ (\sin \theta)^{3/2} |P_{\ell}^\prime| < (\sin \theta)^{3/2}(\ell + 1/2) \]  

(69)

Both bounds are of the same order of magnitude for $\sin \theta = 0(\ell + 1/2)$. Taking the best possible value we find

\[ (\sin \theta)^{3/2} |P_{\ell}^\prime| < 1.225 \]  

(70)

and this can be further improved by refining (69).

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