PARTICLE LOSS IN CROSSING A RESONANCE IN A NON-LINEAR SYSTEM

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1. INTRODUCTION

There exists now a fair knowledge of the behaviour of particles in accelerators when subject to linear and non-linear resonances, excited by perturbations of the guiding field, provided the parameters of the system are constant in time. The case of parameters changing with time, in particular the dynamic crossing of a resonance, is more difficult to treat. It is, however, of considerable importance, as (i) crossing of resonances may play a part in real accelerators due to imperfections and energy oscillations, and (ii) intentional crossing of resonances may be considered in some designs, e.g. isochronous cyclotrons.

For linear oscillations, the crossing of a resonance is a soluble problem. If the oscillations are non-linear, peculiar phenomena can happen, due to the dependence of oscillation frequency on amplitude. In particular, it can happen that the change in frequency due to system parameters is balanced by the change of frequency with amplitude, in such a way that the particle remains in resonance (i.e. is “locked” to the exciting frequency) at least for some time. This phenomenon might lead to unusual particle loss in accelerators when a resonance is passed in the appropriate way.

Crossing of a resonance of a non-linear oscillator was studied experimentally, using an electromechanical model described earlier. The results permit some conclusions on the percentage of particles lost by lock-on, as a function of perturbation and rate of change of frequency.

2. PARAMETERS OF THE SYSTEM AND PROCEDURE OF MEASUREMENTS

The system studied is characterized by the following differential equation for the particle displacement $x$:

$$\frac{d^2x}{dt^2} + \omega_0^2 x - k x^3 = F \cos \int \omega(t) dt .$$

or, after suitable scaling of variables

$$\frac{d^2\xi}{d\theta^2} + \xi - \xi^3 = \frac{\alpha}{2} \cos \int \frac{\omega_0}{\omega} d\theta$$

The changing parameter is the frequency of the external perturbation, $\omega$, rather than the restoring force of the oscillator, which would correspond more closely to the situation in an accelerator. This is, however, not a fundamental modification.

So far, only excitation forces independent of $x$ have been studied. Excitation forces depending on $x$ produce sub-resonances which are by no means less interesting for accelerators.

In the electromechanical model used, the system is simulated by a quartz pendulum, oscillating in vacuum with negligible friction. Non-linear forces, excitation forces, and pick-up of displacements are effected electrostatically. The oscillations are displayed on an oscilloscope screen as trajectories in $(x, \frac{1}{\omega_0} \frac{dx}{dt})$ phase space; more precisely, only one representative point is displayed in each period of the exciting force, always appearing at the same phase of the excitation voltage.

It would be desirable to watch the evolution of a given initial distribution of points in phase space. However, the phase angles are difficult to measure because of their rapid charge, unless $\omega$ and $\omega_0$ are very nearly equal. Therefore, the following procedure was adopted.

In crossing the resonance, the phase angle becomes stationary at one instant, corresponding to exact
Fig. 1 Phase-space maps corresponding to departures from (top parts) and crossing of (bottom part) the resonance for a perturbation $\alpha = 5.9 \times 10^{-4}$. 
Fig. 2 Phase-space maps corresponding to departures from (top part) and crossing of (bottom part) the resonance for a perturbation. $a = 9.8 \times 10^{-4}$. 

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synchronism between oscillation and exciting force. At this instant, amplitude and phase are measured. Furthermore, the amplitudes can be measured sufficiently far from resonance, where they are practically not affected by the perturbation and therefore constant.

Covering all possible amplitudes and phases at synchronism, one also covers all possible amplitudes and phases far from resonance. Thus, a sort of quasi-phase space diagram can be plotted, connecting, by curves, points of synchronism belonging to equal amplitudes far from resonance. This can be done in either of the cases of moving into or moving out of resonance.

Examples of such diagrams are given in Figs. 1 and 2. It is indicated on the figures how the excitation frequency is moving, as well as in what sense the oscillation frequency is changing with amplitude. In addition, circles representing equal off-resonance amplitudes are drawn in.

3. PROPORTION OF PARTICLES LOCKED TO RESONANCE

It may be noted that for weak perturbations, the phase plane pattern defined by the points of synchronism is essentially similar to the off-resonance pattern, apart from some shifts and deformations (see Fig. 1 for illustration). This is true for either sense of movement of frequency. For strong perturbations, there is a characteristic difference between the two directions of frequency change (Fig. 2). In the direction of possible lock-on, limiting boundaries appear in the points-of-synchronism plane; points beyond these boundaries show indefinite growth of amplitude when \( \omega \) moves away from resonance.

The points of synchronism, connected to curves in Figs. 1 and 2, do not represent a rigorous mapping of the off-resonance plane in the sense of Liouville's theorem, because different points reach synchronism after different times. It is found, however, that area is conserved to within 10\% for small perturbations, the deviation attaining 40\% for large perturbations.

Assuming area conservation, the diagrams can be used for estimating the percentage of particles remaining within a given amplitude-confining circle. Fig. 3, top part, gives, as example, the fraction of particles conserved in moving out of a resonance in the lock-on sense, as a function of the perturbing force. Parameter is the maximum amplitude.

**Fig. 3** Top part. Fraction of particles conserved within a given amplitude confining chamber, when moving out of a resonance in the lock-on sense, as a function of the perturbing force. Parameter is the maximum amplitude. Bottom part. Fraction of particles caught by lock-on as a function of the perturbing force. Parameter is the maximum initial amplitude.
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force. Parameter is the maximum amplitude. The values obtained should be upper limits, as some particles might have transgressed the maximum amplitude only temporarily.

Fig. 3, bottom part, gives the result of a similar evaluation of the fraction of particles caught by lock-on, as defined by the limiting boundaries mentioned with regard to Fig. 2. A rather abrupt onset of lock-on with increasing excitation can be recognized clearly. At the lock-on threshold the perturbation force is about $5 \times 10^{-3}$ of the restoring forces, if the amplitude is such that the non-linear force is $4 \times 10^{-3}$ of the restoring force. More details can be found in a CERN report.  

LIST OF REFERENCES


RADIATION EFFECTS IN ELECTRON SYNCHROTRONS

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The effects of radiation loss on the motion of electrons in a synchrotron may be analyzed as damping or anti-damping of the modes of oscillation due to the average radiation loss, and the excitation of the modes of oscillation due to the emission of the radiation of quanta.

It may be shown 1) that the total damping rate, due to radiation loss, of the three modes of oscillation in an electron synchrotron is given by $2P_r/E_0$ and the damping rate of the vertical betatron oscillation is given by $\frac{1}{2} P_r/E_0$, for any accelerator in which the vertical oscillations are independent of the oscillations in the radial plane.

If the momentum compaction factor is very small compared with unity, the average magnetic field must increase proportional to the energy variation during a synchrotron oscillation. The instantaneous radiation loss varies as $E^2 B^2$. In an alternating gradient magnet structure with the principle orbit on an isomagnetic line, the variation in radiation loss may be expressed as a first order expansion of the variation in magnetic field and energy from the ideal values. In this case the average radiation loss will increase proportional to the fourth power of the energy. The damping rate of the synchrotron oscillation will then be $2P_r/E_0$. Then the radial betatron oscillation must be anti-damped at a rate of $\frac{1}{2} P_r/E_0$ in order that the total damping rate is $2P_r/E_0$.

An exact calculation which includes the change in average radius with energy variation gives a damping rate for the synchrotron oscillations of

(*) See note on reports, p. 696.