UNIFIED-GAUGE FORMALISM AT TWO LOOPS FOR A CLASS OF PHYSICAL GAUGES

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A B S T R A C T

A unifying prescription for axial-type gauges (light-cone, axial, temporal) is tested in Yang-Mills theory to two-loop order by evaluating a self-energy diagram and analyzing its corresponding pole structure. The calculation is free of ambiguities, but features a large number of overlapping divergences.

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In massless Yang-Mills theory, with Lagrangian density

\[ L_{\alpha} = -\frac{1}{4} (F_{\mu\nu}^{\alpha})^2 - \frac{1}{2\alpha} (n \lambda^\alpha)^2, \quad \alpha = 0, \]  

(1.1)

the axial-type gauges (pure axial, light-cone, temporal) are defined by

\[ \eta^a_{\mu} = 0, \quad \mu = 0,1,2,3, \quad n^a = n^h \cdot \mathbb{R}, \]  

(1.2)

where \( n_{\mu} = (n_0, \mathbf{n}) \) is a fixed vector and \( A_{\mu}^a \) a massless Yang-Mills field, \( a = 1,2,\ldots,N^2-1 \) for SU(N); \( \alpha \) is the gauge parameter. We use dimensional regularization and a \((+, -,-,-)\) metric. The propagator reads (for \( \alpha = 0 \))

\[ \frac{\delta^a_{\nu\mu}}{(q^2 + i\epsilon)} = \frac{(q \cdot n + q \cdot n)^a}{q \cdot n} + \frac{n^a q \cdot q}{(q \cdot n)^2}, \quad \epsilon > 0, \]  

(1.3)

and the four-gluon vertex of zero-loop order

\[ \Gamma_{abcde}^{\mu \lambda \rho} = -ig^2 \left[ \epsilon_{abcde} \delta_{\mu \lambda \rho} \frac{1}{q \cdot n} \frac{1}{q \cdot n} \right] + ig^2 \epsilon_{abcde} (\delta_{\mu \lambda \rho} \delta_{\sigma \tau} - \delta_{\mu \lambda \sigma} \delta_{\rho \tau}) \]  

(1.4)

The poles of \((q \cdot n)^{\alpha-\beta}, \beta = 1,2,\) were handled originally by the principal-value (PV) prescription \([3,4,6,9]\),

\[ \lim_{\mu \to 0} \frac{1}{q \cdot n} \int_0^\infty \frac{q \cdot n}{q \cdot n + i\epsilon}, \quad \mu > 0, \]  

but more recently by the unifying-gauge prescription for axial-type gauges, namely \([10-13]\)

\[ \lim_{\mu \to 0} \int_0^\infty \frac{q \cdot n}{q \cdot n + q \cdot n + i\epsilon}, \quad \epsilon > 0. \]  

(1.5)

Formula (1.5) is a generalization of the prescription for the light-cone gauge developed independently by Mandelstam and the author. The noncovariant vector \( \eta_\mu \) and its dual, \( \eta_\mu^* \), have the following structure:

\[ \eta_\mu = \begin{pmatrix} \eta_0 + i\eta_1 \cdot \mathbf{n}, & \eta_2 = \eta_3, \\ \eta_0^* + i\eta_1 \cdot \mathbf{n}, & \eta_2^* < \mathbb{R}, \\ \eta_0^* - i\eta_1 \cdot \mathbf{n}, & \eta_2^* > \mathbb{R} \end{pmatrix}. \]  

(1.6a)
As already indicated, the unifying prescription (1.5) is valid for the pure axial, light-cone, temporal and planar gauge and yields satisfactory results to one loop [16-27].

The purpose of the present calculation is to test the unifying prescription (1.5) in detail for the two-loop self-energy diagram, shown in Fig. 1. A partial test had already been carried out in 1980 by Höfzel, Landshoff and Taylor [28] on the Wilson loop.

Of special importance and concern are the following points:
(i) Can all two-loop integrals be calculated explicitly? In particular, can we obtain the coefficients of the single and double poles? What about the finite parts?
(ii) Are any of the integrations ambiguous?
(iii) Are there any unexpected technical problems?
(iv) Do new integrals arise at the two-loop level which are absent at the one-loop level?

Before describing the highlights of this calculation, I should like to caution that the computation for the sunset diagram Fig. 1 is not quite complete. Furthermore, I decided to calculate everything by hand, because I wanted to be "in control of the situation", ready to develop new formulae if and when the need arose. It turned out to be a wise decision.

2. Structure of sunset graph

The self-energy graph in Fig. 1, sometimes called the sunset diagram, is

\[
\Pi_\mu^\nu(p) = \frac{1}{3!} \int \frac{dqdk}{(2\pi)^2} v^{abcd} (q) \, \Pi_{\rho\sigma}^{\rho\sigma} (p-k.\, q) \, \Pi_{\sigma\chi}^{\sigma\chi} (k) \, v^{dfe} \, \epsilon_{abc}^{def}. \tag{2.1}
\]

\[G_{\mu\nu}^{ab} \text{ and } v^{abcd} \text{ are given in eqs. (1.3) and (1.4), respectively. Let us write}
\]

\[
\Pi_\mu^\nu(p) = \delta_{\mu\nu} \left[ \Pi_\mu^\nu(n^2=0) + \Pi_\mu^\nu(n^2=0) \right], \tag{2.2}
\]

where the "light-cone" part reads [29]

\[
\Pi_\mu^\nu(n^2=0) = 3\lambda(N)(2\mu-3) \left[ A(p) \delta_{\mu\nu} + B(p) (k^\mu n^\nu + k^\nu n^\mu) - C(p) n^\mu n^\nu \right]. \tag{2.3}
\]

while

\[
\Pi_\mu^\nu(n^2=0) = \lambda(N)n^2 \left[ \Pi(1) \delta_{\mu\nu} + \Pi(2) k^\mu n^\nu + \Pi(3) n^\mu n^\nu \right. \nonumber
\]

\[+ \left. \Pi(4) (k^\mu n^\nu + k^\nu n^\mu) + \Pi(5) (k^\mu k^\nu + n^\mu n^\nu) \right]. \tag{2.4}
\]

with \( N = 3 \) for SU(3). The coefficients \( A, B, C \), are the same as in ref. [29]:

\[A(p) = \int \frac{dq \, dk}{(2\pi)^2} \left[ Q_{\mu}(q^2)Q_{\nu}(q^2) \right]^{-1}, \tag{2.5a}\]

\[B(p) = \int \frac{dR(k-n)^{-1}, \tag{2.5b}}{dR(k-n)^{-1}} \]

\[C(p) = \int \frac{dR(k-n)^{-1}, \tag{2.5c}}{dR(k.n)^{-1}} \]

It is convenient to re-write the five terms in eq. (2.4) as follows:

\[
\Pi(1) \delta_{\mu\nu} = 4\Pi_1^\mu^\nu \quad \Pi(2) \delta_{\mu\nu} = \Pi_2^\mu^\nu + 2n^\mu \Pi_3^\mu^\nu.
\]
\[ \Pi(2)_{\mu\nu} = -2R_{\mu\nu} + 4n^\alpha \Pi_{\alpha\mu\nu} - 4n^\alpha \Pi_{\mu\alpha\nu} + 3n^\alpha n^\beta\Pi_{\alpha\mu\beta\nu}, \]
\[ \Pi(3)_{\rho\mu\nu} = -\Pi_{\mu\rho\nu}, \]
\[ \Pi(4)_{(\kappa\rho\alpha\beta \mu\nu)} = 4n^\kappa \rho + 4n^{\rho\alpha} + 2n^{\rho\beta} \Pi_{\alpha\beta\mu\nu} - 4n^{\rho} \Pi_{\mu\nu} - 2n^{\rho} \Pi_{\mu\nu}, \]
\[ + \mu \leftrightarrow \nu, \]
\[ \Pi(5)_{(\rho^\mu\kappa\nu\kappa\mu)} = 3n^\rho \mu - 3n^\kappa \Pi_{\kappa\mu\nu\mu} + 2n^\kappa \Pi_{\kappa\mu\nu\mu} - \frac{1}{2}n^\mu n^\mu \]
\[ + \mu \leftrightarrow \nu. \]

The double integrals \( \Pi_j^{\mu\nu} \), \( j = 1, \ldots, 17 \), are listed in Appendix A. For example,

\[ \Pi_1^{\mu\nu} = \delta^{\mu\nu} \int dR \ k^2 (k \cdot n)^{-2}, \]
\[ \Pi_2^{\mu\nu} = \delta^{\mu\nu} \int dR \ q \cdot k (q \cdot n - \mu)^{-1}, \]
\[ \Pi_3^{\mu\nu} = \delta^{\mu\nu} \int dR \ (q \cdot k)^2 [(q \cdot n)^2 (k \cdot n)^2]^{-1}. \]

All integrals contain overlapping divergences, the more difficult ones being \( \Pi_j^{\mu\nu} \), \( \Pi_j^{\mu\nu}, \Pi_{1,\mu\nu}, \Pi_{1,\mu\nu} \), \( \Pi_{1,\mu\nu} \), and \( \Pi_{1,\mu\nu} \).

Initial steps in computation

(a) The first step is to apply the decomposition formula [30]

\[ \frac{1}{q \cdot (p-q \cdot n)} = \frac{1}{p \cdot n} \left[ \frac{1}{q \cdot n} + \frac{1}{(p-q \cdot n)^2} \right], \ p \cdot n \neq 0, \]

(3.1)
to all integrals containing multiple, noncovariant factors in the denominator.

(b) It is convenient to defer any vector or tensor integrals, whenever possible, to the second integration. Take, for instance, the term

\[ I^{\mu\nu} = \int dq \ \int d\mathbf{k} \ k^{\mu} \Pi^{(q-k)^2}(q-k)^2, \quad D = D(p,q,k,n). \]

(3.2)

Rather than perform the \( k \cdot \) integration in (3.2), we first let \( q_{\mu} \leftrightarrow k_{\mu} \).

\[ I^{\mu\nu} \rightarrow \int dq \ q^{\mu} q^{\nu} \int d\mathbf{k} (q-k)^2 = I^{\mu\nu}_0, \]

(3.3)
and only then integrate over \( \mathbf{k} \) in \( I^{\mu\nu}_0 \), by applying the procedure in part (c) below.

(c) To evaluate

\[ \int d\mathbf{k} (q-k)^2 = q_{\nu} q_{\lambda} \left( \int d\mathbf{k} k^{\nu} k^{\lambda} \right), \]

(3.4)
or any integral having the general form

\[ \int d\mathbf{k} (k_{\mu} k_{\lambda} k_{\rho} k_{\alpha} k_{\beta} k_{\gamma} \ldots), \]

(3.5)
we apply the tensor method, which is the only viable technique for two-loop integrals, in my opinion. The tensor method is trivial for covariant gauges, but requires "additional work" for noncovariant-gauge integrals. Two examples follow.

(1) For the integral \( I_\mu = \int d\mathbf{k} k_\mu / D \) we make either the ansatz (11)

\[ I_\mu = A_{\rho} p_\mu + B_{\rho} n_\mu + C_{\rho} F_\mu, \]

(3.6a)
where \( F_\mu \) is related to the dual vector \( n_\mu^\rho \) or the ansatz

\[ I_\mu = A_{\rho} p_\mu + B_{\rho} n_\mu + C_{\rho} n_\mu^\rho. \]

(3.6b)
Since \( F_\mu \) is a null vector, the structure (3.6a) is computationally more convenient than (3.6b). In the temporal gauge, for instance,

\[ I_{\mu}^{\text{comp}} = (\delta_{\mu} / \sigma, \ n_\mu / \sigma, 1), \quad \sigma = n_\mu \sigma, \]

in which the double integrals contain overlapping divergences, the more difficult ones being \( \Pi_j^{\mu\nu} \), \( \Pi_j^{\mu\nu}, \Pi_{1,\mu\nu}, \Pi_{1,\mu\nu} \), \( \Pi_{1,\mu\nu} \), and \( \Pi_{1,\mu\nu} \).
Multiplication of (3.6a) successively by $p_\mu$, $n_\mu$, $F_\mu$ yields three equations which can be solved uniquely for A, B, C. We note that integration of $I_\mu$ in component form is not a viable alternative, simply because it becomes too messy, especially by the time the second integration over $dq$ is completed.

(II) Next we demonstrate the tensor method for the integral

$$K_{\mu
u} = \int \frac{dk \, k_\mu k_\nu}{k^2(p-k)^2(k-n)^2}.$$  \hspace{1cm} (3.7)

The proper ansatz turns out to be

$$K_{\mu\nu} = [A_{\mu\nu} + B(n_\mu F_\nu + n_\nu F_\mu) + \epsilon_{\mu\nu}],$$

$$+ (Dn_\mu k_\nu + K(p_\mu k_\nu + p_\nu k_\mu) + \epsilon_{\mu\nu}),$$

$$\frac{F_\mu F_\nu}{p_\mu p_\nu + H(p_\mu F_\nu + p_\nu F_\mu)},$$

where $K_{\mu\nu}$ and (finite terms) $= f_{\mu\nu}$ contain the pole part and finite portion of $K_{\mu\nu}$, respectively. The coefficients A, B, ..., H, are found in the usual way.

Successful application of the tensor method to the "first" integration over $dk$ yields about a dozen distinct types of divergent integrals. We shall show in Section 4 how to represent the poles of these integrals in terms of the two scalar integrals X and Y.

4. Reduction of one-loop integrals

The general aim is to express the pole parts of all necessary one-loop integrals in terms of two scalar integrals $Y$ and $X$:

$$Y(p) = \int \frac{dq}{q^2(p-q)^2(2q,n)^2}.$$  \hspace{1cm} (4.1)

$$X(p) = \int \frac{dq}{(q-p)^2(q,n)^2}.$$  \hspace{1cm} (4.2)

We shall demonstrate the procedure for the integral $K_{\mu\nu}$ given in eqs. (3.7) - (3.8):

$$K_{\mu\nu} = \{A_{\mu\nu} + B(n_\mu F_\nu + n_\nu F_\mu) + \epsilon_{\mu\nu}\}.$$  \hspace{1cm} (4.3)

We know from explicit computation [11] that only the first three coefficients, in square brackets, are proportional to simple poles. In other words, to obtain the divergent part of $K_{\mu\nu}$ it suffices to determine A, B, C. Multiplication of (4.3) in turn by $\delta^{\mu\nu}$, $n_\mu n_\nu$, $p_\mu p_\nu$, $p_\mu p_\nu$ yields four equations:

$$[2wA + 2Bn \cdot F + 0] + \delta^{\mu\nu} e_{\mu\nu} = \int \frac{dk}{(k-p)^2(k-n)^2} = X.$$  \hspace{1cm} (4.4)

$$[n^2A + 2n^2 n \cdot F B + \epsilon (n \cdot F)^2] + n_{\mu} n_{\nu} e_{\mu\nu} = \int \frac{dk}{k^2(p-k)^2} = Y.$$  \hspace{1cm} (4.5)

$$[p \cdot n A + (n^2 p \cdot F + p \cdot n \cdot F) B + n \cdot F p \cdot FC] + n_{\mu} p_{\nu} e_{\mu\nu} = \int \frac{dk \cdot p \cdot k}{k^2(p-k)^2(k-n)^2} = Z.$$  \hspace{1cm} (4.6)

$$[p^2A + 2p \cdot n p \cdot F B + (p \cdot F)^2C] + p_{\mu} p_{\nu} e_{\mu\nu} = \int \frac{dk(p-k)^2}{k^2(p-k)^2(k-n)^2} = W.$$  \hspace{1cm} (4.7)

or

$$[2wA + 2Bn \cdot F + 0] - \delta^{\mu\nu} e_{\mu\nu} = X.'$$  \hspace{1cm} (4.8)

$$[n^2A + 2n^2 n \cdot F B + \epsilon (n \cdot F)^2] - n_{\mu} n_{\nu} e_{\mu\nu} = Y.'$$  \hspace{1cm} (4.9)

$$[p \cdot n A + (n^2 p \cdot F + p \cdot n \cdot F) B + n \cdot F p \cdot FC] - Z - n_{\mu} p_{\nu} e_{\mu\nu} = Z.'$$  \hspace{1cm} (4.10)
\[ [p^2 A + 2 \cdot n \cdot p \cdot F B + (p \cdot F)^2 C] - \frac{\phi}{\phi} - p \mu p_{\nu} = \chi. \] (4.11)

First we solve system (4.8) – (4.10) for A and B, constructing the ratio
\[ \left( \phi \right) \left( 4.8 \right) \left( 4.10 \right); \] next we solve system (4.8), (4.10), (4.11) for A and B, constructing \( \left( \phi \right) \left( 4.8 \right), \left( 4.10 \right), \left( 4.11 \right) \), and then finally equate the two ratios.

The equality
\[ \left( \phi \right) \left( 4.8 \right) \left( 4.10 \right) = \left( \phi \right) \left( 4.8 \right), \left( 4.10 \right), \left( 4.11 \right) \] (4.12)

leads to two identities:
\[ \frac{p \cdot F Y}{n \cdot F} Z - (4.13) = 0, \]
\[ 2(\mu - 1) n \cdot F (p \cdot F Z' - n \cdot F Y') + D_{\mu} X' = 0, \] (4.14)

which, in turn, yield the following relations among the four scalar integrals \( X, Y, Z, W \):
\[ X = (p \cdot F/n \cdot F) Y + Z_\text{finite}. \] (4.15)
\[ W = \frac{1}{(n \cdot F)^2} \left[ \frac{1}{2} (p \cdot F)^2 Y + \frac{D_{\mu}}{2(\mu - 1)} X + Z'/_\text{finite}, \right. \] (4.16)

where
\[ Z'_\text{finite} = \eta_{\mu} \eta_{\nu} \cdot (p \cdot F/n \cdot F) n_{\mu} \gamma_{\nu} \mu_{\nu}, \] (4.17)
\[ Y'_{\text{finite}} = \left( p_{\mu} p_{\nu} - (n \cdot F)^2 \left[ (p \cdot F)^2 n_{\mu} n_{\nu} + \frac{D_{\mu}}{2(\mu - 1)} \delta_{\mu \nu} \right] \right) \gamma_{\mu} \mu_{\nu}, \] (4.18)
\[ D_{\nu} = p_{\mu} p_{\nu} - 2 p_{\nu} n \cdot F \cdot p \cdot F + n_{\nu} (p \cdot F)^2. \] (4.19)

Systematic application of this technique to the other integrals enables us to express the divergent part of all one-loop integrals in terms of \( X(p) \) and \( Y(p) \). (See Appendix B.) For example, \( K_{\mu \nu} \) in eq. (3.7) becomes
\[ K_{\mu \nu}(p) = \frac{Y(p)}{(n \cdot F)^2} F_{\mu \nu}. \]

5. Final integration

We illustrate the second integration, i.e. the integration over \( dq \), by computing the double integrals \( H_{\mu \nu} \) and \( H_{\mu \nu} \).

5.1 The integral \( H_{\mu \nu} \)

\[ H_{\mu \nu} = Q \left\{ \int \frac{dq}{(q \cdot n)^2} Y(p \cdot q) \cdot Q - (p \cdot q)^2 \mu_{\nu} \right\}, \] (5.1)

Substituting for \( Y(p \cdot q) \) from eq. (4.1), we get
\[ H_{\mu \nu} = \frac{1}{(n \cdot F)^2} \int \frac{d\mu}{\Gamma(\mu - 2)} \frac{\Gamma(\mu - 2)}{\Gamma(\mu - 2)} \] (5.3)

The remaining one-loop integral
\[ \int dq \left[ (p \cdot q)^2 \right]^{2-\omega} (q \cdot n)^2 = X(2-\omega), \] (5.4)

can be computed explicitly to yield
\[ H_{\mu \nu} = Q \left\{ \frac{1}{(n \cdot F)^2} \frac{\Gamma(\mu - 2)}{\Gamma(\mu - 2)} \right\}. \]
\[
\begin{aligned}
&-\frac{i\pi}{\rho F(2-\omega)} \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-2}}{(1-\beta\xi)^{3/2}} \{g(\xi\xi)^{2\omega-1} + \Gamma(\xi-\omega) \left( \frac{2}{(1-\beta\xi)^{3/2}} + 21\rho F(\xi\xi)^{2\omega-1} \right) \right.

\end{aligned}
\]

\[
\begin{aligned}
&-\rho F^2 \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-1}}{(1-\beta\xi)^{3/2}} \{g(\xi\xi)^{2\omega-4} \right. \\
&\left. \right\},
\end{aligned}
\]

(5.5)

where

\[
\begin{aligned}
g(\xi\xi) &= F(\xi\xi) + \frac{(\xi\xi)^2}{\rho^2 (1-\beta\xi)} \cdot B - 1 - \alpha^2/\rho^2, \\
\rho_{\text{temp}} &= \rho_{\text{ax}} = \frac{n}{n_1}, \quad \rho_{\text{fc}} = \frac{n}{n_1}; \quad \vec{p}_1 = \vec{P}_1, \quad P_3 = P_4, \quad P_4 = P_4, \\
\sigma^2 &= n_1^2 + n_2^2, \quad \vec{n}_1 = \vec{n}_1, \quad n_3 = n_4, \quad n_4 = n_4.
\end{aligned}
\]

(5.6)

Clearly the double integral \( \Pi^{\mu\nu}_{1} \) possesses only a \textbf{simple} pole as \( \omega \to 2^+ \).

5.2 The integral \( \Pi^{\mu\nu}_{1} \)

A more challenging overlapping divergence occurs in the integral \( \Pi^{\mu\nu}_{1} \).

\[
\Pi^{\mu\nu}_{1} = \int^\infty \frac{dq}{q^2} \frac{d\kappa}{(q-k)^2} \left[ \frac{d\xi}{q^2(q\xi)^2} \right],
\]

(5.8)

Using formula (4.20) for the \( k \)-integral, we obtain

\[
\Pi^{\mu\nu}_{1} = \int^\infty \frac{dq}{q^2} \frac{d\kappa}{(q-k)^2} \left[ \frac{\Gamma(p\cdot q)}{(p\cdot F)^2 F\cdot F} \right] X(p\cdot q) + \frac{1}{2(\omega-1)} \left\{ \delta_{\alpha\lambda} - \frac{(n\cdot F + n\cdot F)}{(p\cdot F)^2 F\cdot F} + k_{\alpha\lambda}(\text{finite}) \right\},
\]

(5.9)

where

\[
\begin{aligned}
X &= X_{\text{pole}} + X_{\text{finite}}, \\
X_{\text{pole}}(p\cdot q) &= \frac{i\pi n^2}{\rho F(2-\omega)} \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-1}}{(1-\beta\xi)^{3/2}} \{g(\xi\xi)^{2\omega-1} + \Gamma(\xi-\omega) \left( \frac{2}{(1-\beta\xi)^{3/2}} + 21\rho F(\xi\xi)^{2\omega-1} \right) \right. \\
&\left. \right\}, \\
X_{\text{finite}}(p\cdot q) &= \frac{i\pi^n}{\rho^2 (1-\beta\xi)} \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-1}}{(1-\beta\xi)^{3/2}} \{g(\xi\xi)^{2\omega-4} \right. \\
&\left. \right\},
\end{aligned}
\]

To extract the coefficients of the double or single poles from (5.9) we need to compute the following two integrals:

\[
\begin{aligned}
(1) \quad K^{\alpha\lambda}(2-\omega) &= \int_0^1 \frac{dq}{q^2} \frac{d^\lambda}{(p\cdot q)^2} X_{\text{pole}}(p\cdot q) - \frac{1}{2(\omega-1)} \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-1}}{(1-\beta\xi)^{3/2}} \right. \\
&\left. \right\} \\
(2) \quad \chi^{\alpha\lambda}(p\cdot q) &= \int_0^1 \frac{dq}{q^2} \frac{d^\lambda}{(p\cdot q)^2} X_{\text{pole}}(p\cdot q) - \frac{1}{2(\omega-1)} \left\{ \frac{1}{2} \int_0^1 \frac{d\xi 2^{\nu-1}}{(1-\beta\xi)^{3/2}} \right. \\
&\left. \right\},
\end{aligned}
\]

with \( n = \rho^2 (1-\beta\xi) - \rho^2 + n^2 \xi; \rho_{\text{temp}} = \rho_{\text{ax}} = \frac{n}{n_1}; \rho_{\text{fc}} = \frac{n}{n_1}. \quad K^{\alpha\lambda}(2-\omega) \) and \( \chi^{\alpha\lambda} \) may be evaluated with the help of the tensor method and the appropriate identities.

It is also worth noting, in the context of dimensional regularization, that [31]
\[
\int \frac{dq}{q^2(q-n)^2} = 0
\]  
(5.12)

as expected, but that
\[
\lim_{\omega \to 0} \int \frac{dq}{q^2[(p-q)^2]^{2-\omega}(q-n)^2}
\]  
(5.13)
is different from zero and finite.

6. Summary

The Yang-Mills self-energy \( \Pi^{\text{af}}_{\mu \nu} \) has been calculated in a unified-gauge formalism which includes the axial, temporal and light-cone gauge, and — with an appropriate change of the gauge-fixing term — the planar gauge. The corresponding unified-gauge prescription
\[
\frac{1}{q \cdot n} = \lim_{\epsilon \to 0} \frac{q \cdot n}{q \cdot n \cdot n^\mu n_\mu + \epsilon}, \quad \epsilon > 0,
\]
works to two loops for both simple and multiple poles, \((q \cdot n)^{-\beta}, \beta = 1, 2, 3, \ldots\).

The main results are:

(a) \( \Pi^{\text{af}}_{\mu \nu}(p) = \frac{e^{2\epsilon}}{2} \sum_{j=0}^{\epsilon} \left[ A_j q^2 + B_j p_\mu p^\mu + C_j (p_\mu p_\nu + p_\nu p_\mu) + D_j p_\mu n_\nu + \right. \\
+ E_j (p_\mu n_\nu + p_\nu n_\mu) + F_j (n_\mu n_\nu + n_\nu n_\mu) + \left. H_j n^\mu n^\nu \right], \quad \epsilon = 2 - \omega.

The coefficients \((A_0, \ldots, H_0)\) can be obtained explicitly; the coefficients \((A_1, \ldots, H_1)\) and \((A_2, \ldots, H_2)\) are derivable but not necessarily in closed form.

(b) All single and double integrals are well defined. Ambiguities do not arise, and integrals of the type
\[
\int \frac{dq}{q^2[(p-q)^2]^{2-\omega}(q-n)^2}
\]
can be evaluated explicitly.

(c) No additional problems are caused by overlapping divergences.

(d) The basic approach to the evaluation of two-loop noncovariant-gauge integrals is:

(i) to derive identities between certain one-loop integrals; for instance,
\[
Z = (p \cdot F/n \cdot F)Y + Z \text{ finite, i.e.}
\]
\[
\int \frac{dq}{q^2(p-q)^2(q \cdot n)^2} \quad \frac{p \cdot F}{n \cdot F} \int \frac{dq}{q^2(p-q)^2 + \text{Finite Part}}
\]

(ii) to exploit these identities to express the divergent parts of all one-loop integrals in terms of the scalar integrals
\[
Y = \int dq(q^2(p-q)^2)^{-1}, \quad \text{and} \quad X = \int dq[(p-q)^2(q \cdot n)^2]^{-1}.
\]

(iii) to apply the tensor method to the first as well as the second integrals, i.e. to both the \(k\) and \(q\) integrations.

As for future work in this area, the most urgent task is to get the coefficients of all the poles. Knowledge of the coefficients \((A_0, \ldots, H_0)\) of the double poles should give us further insight into the structure of the counterterms and hence into the renormalization program. An intriguing
question in this connection concerns the non-local terms: can they be handled consistently, for instance, in the framework of the Becchi-Rouet-Stora formalism? [32]

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Appendix A

The Double Integrals $\Pi_{\mu\nu}^j$

We list the seventeen double integrals $\Pi_{\mu\nu}^j$ referred to in eq. (2.6) in the main text. These integrals (see also ref. [33]) constitute the self-energy part $\Pi_{\mu\nu}^{(n^2=0)}$ defined in eq. (2.4).

$\Pi_{\mu\nu}^1 = \delta^{\mu\nu} \int dR \, k^2 (k-n)^{-2}$,

$\Pi_{\mu\nu}^2 = \delta^{\mu\nu} \int dR \, q \cdot k (q-n) (k-q-n)^{-1}$,

$\Pi_{\mu\nu}^3 = \delta^{\mu\nu} \int dR (q \cdot k)^2 (q-n)^2 (k-n)^2)^{-1}$.
\[ n_{\mu} = \int d\mathbf{r} \frac{q \cdot Q \cdot Q}{(q \cdot n)^2 (k \cdot n)^2 (Q \cdot n)^2} \eta_{\mu}^2. \]

Appendix B

Application of the technique developed in Section 4 leads to the following one-loop integrals as functions of the scalar integrals \( Y \) and \( X \).

\[ \int \frac{dk}{k^2 (q - k)^2} = Y(q), \]
\[ \int \frac{dk}{(q - k)^2 (k \cdot n)^2} = X(q), \]
\[ \int \frac{dk}{k^2 (q - k)^2 k \cdot n} = \frac{Y(q)}{n \cdot F} \eta_{\mu}^2 + F.P., \]
\[ \int \frac{dk}{k^2 (q - k)^2 (k \cdot n)^2} = \frac{X(q)}{(n \cdot F)^2} \eta_{\mu}^2 F_{\mu} F_{\nu} + \]
\[ + \frac{2q \cdot F}{2(\omega - 1)} \left[ \eta_{\mu}^2 \frac{\eta_{\nu} F_{\nu}}{n \cdot F} + \eta_{\nu}^2 \frac{\eta_{\mu} F_{\mu}}{n \cdot F} \right] + F.P., \]
\[ \int \frac{dk}{(q - k)^2 k \cdot n} = \frac{Y(q)}{n \cdot F} + F.P., \]
\[ \int \frac{dk}{k^2 (q - k)^2 (k \cdot n)^2} = - \frac{X(q)}{q^2} + F.P., \]
\[ \frac{4q \cdot F}{(n \cdot F)^2} Y(q) + \left[ q^2 + \frac{2D_0}{(\omega - 1)(n \cdot F)^2} \right] X(q) + F.P., \]
\[ \frac{4q \cdot F}{(n \cdot F)^2} \left( q \cdot n - \eta_{\mu}^2 \frac{q \cdot F}{n \cdot F} \right) X(q). \]

where \( D_0 = n^2 (q \cdot F)^2 - 2q \cdot n \cdot Fq \cdot F + q^2 (n \cdot F)^2 \), and F.P. means "finite part."

Appendix C

Consider the double integral

\[ \eta^{\mu \nu} = \int dq \int dk \, g^{\mu \nu}(k, p, q, n), \]
and define

\[ \eta^{\mu \nu} = \int dk g^{\mu \nu}(k, p, q, n) = \eta^{\mu \nu, \text{pole}} + \eta^{\mu \nu, \text{finite}}. \]

Then

\[ \eta^{\mu \nu} = \int dq (\eta^{\mu \nu, \text{pole}} + \eta^{\mu \nu, \text{finite}}) = \eta^{\mu \nu, \text{sub-div}} + \eta^{\mu \nu, \text{sub-finite}}, \]
with
In this appendix we list all $\Pi_j^{\mu\nu}(\text{sub-div}), j = 1, \ldots, 16$, the expressions for $\Pi_j^{\mu\nu}(\text{sub-finite})$ being discussed elsewhere.

$$\Pi_1^{\mu\nu}(\text{sub-div}) = \delta^{\mu\nu}
\left[ \frac{dq}{(p-q)n^2} \ Y(q) \right].$$

$$\Pi_2^{\mu\nu}(\text{sub-div}) = \delta^{\mu\nu}
\left[ \frac{dq}{(p-q)^2(p-q)n} \ \frac{Y(q)}{n-F} \ p^\sigma \right].$$

$$\Pi_3^{\mu\nu}(\text{sub-div}) = \delta^{\mu\nu}
\left[ \frac{dq}{(p-q)^2(p-q)n} \ \frac{Y(q)}{n-F} \ p^\sigma \right] \ F_A F_A
\frac{Y(q)}{(n-F)^2} \ X(q) +
\frac{1}{2(w-1)} \ \left[ \delta^\sigma_A \ \left( n_F h_A + n_F^F \right) + \frac{n^2}{(n-F)^2} \ F^\sigma F_A \right] X(q) \right].$$

$$\Pi_4^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_5^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ X(q) \right].$$

$$\Pi_6^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_7^{\mu\nu}(\text{sub-div}) = \frac{1}{2(w-1)} \ \left[ \delta^\sigma_A \ \left( n_F h_A + n_F^F \right) + \frac{n^2}{(n-F)^2} \ F^\sigma F_A \right] X(q) \right].$$

$$\Pi_8^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_9^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{10}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{11}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{12}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{13}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{14}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

Finally, we consider the sub-divergent contributions to the renormalization of the soft radiation.

$$\Pi_{15}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

$$\Pi_{16}^{\mu\nu}(\text{sub-div}) = \left[ \frac{dq}{(p-q)^2(p-q)n} \ Y(q) \right].$$

where

$$q = (p-q)_{\mu}, \quad Q_\mu = (p-q)_{\mu}.$$
\[ F_i \text{div}(q) = \frac{4q \cdot F \cdot Y(q)}{(u-1)(n-F)^2} \left\{ n^2(q \cdot F)^2 - 2q \cdot n \cdot F \cdot q \cdot F - (4u-7)q^2(n-F)^2 \right\} + \]
\[ + 2 X(q) \left\{ -2q^2 + \frac{D_6(q)}{(u-1)(n-F)^2} \right\} = P_{13} \text{div}(q), \]
\[ P_{11} \text{div}(q) = \frac{2q \cdot F}{n \cdot F} \left\{ \frac{q \cdot (p-q) \cdot F}{q \cdot n} + \frac{(p-q) \cdot F}{n \cdot F} \right\} Y(q) + \]
\[ + \frac{X(q)}{(n-F)^2} \left\{ \frac{(p-q) \cdot n \cdot F}{n-F} - q \cdot (p-q)(n-F)^2 \right\} + q \cdot n \cdot F \cdot (p-q) \cdot F - n^2q \cdot F \cdot (p-q) \cdot F \right\}, \]
\[ p^\mu \text{div}(q) = \left\{ \frac{q \cdot (p-q)}{n \cdot F} + \frac{(p-q) \cdot F}{n \cdot F} \right\} q^\mu Y(q) + \]
\[ + \frac{(p-q)^2}{(n-F)^2} \left\{ \frac{q \cdot F \cdot n \cdot F}{n-F} - \frac{q \cdot F}{n-F} \right\} \frac{\varepsilon_\lambda}{\varepsilon_{\lambda-1}} \left( n^\mu \varepsilon_\lambda + n_\lambda F^\mu \right) + \]
\[ + n \cdot F(q_3^\mu + q^\mu F_3) + \left\{ \frac{n^2F}{n-F} - q \cdot n \right\} \varepsilon_\lambda \varepsilon_{\lambda-1} Y(q) ; \]
\[ D_6(q) = n^2(q \cdot F)^2 - 2q \cdot n \cdot F \cdot q \cdot F + q^2(n-F)^2. \]

References


Figure Caption

Fig. 1 Two-loop Yang-Mills self-energy