Ref.TH.2236-CERN

A COMPLETE ACTION FOR THE SPINNING STRING

S. Deser *) and B. Zumino

CERN - Geneva

A B S T R A C T

We present an action for the Neveu-Schwarz-Ramond model from which follow both the field equations and the gauge and supergauge constraints. This is done by coupling the free-field action to two-dimensional supergravity in a geometrically clear way. The constraints arise as the supergravity field equations, the supergravity fields playing the role of Lagrange multipliers. The action is invariant under local supersymmetry transformations and, as a consequence, the field equations and the constraints are consistent. The commutator structure of the local supersymmetry algebra is exhibited. It is also shown that there exists a special gauge in which the action, the field equations and the constraints take the free-field form of the usual formulation of the Neveu-Schwarz-Ramond model.

---

*) On leave from Brandeis University, Waltham, MA; supported in part by the U.S. National Science Foundation.

Ref.TH.2236-CERN
22 September 1976

The Neveu-Schwarz-Ramond (NSR)[1] model of a spinning string is invariant under a global supersymmetry. It is the purpose of this paper to exploit this fact, together with the ideas of supergravity[2], to write a simple Lagrangian for the string. This Lagrangian incorporates the constraints in a geometrically clear way, and its consistency is guaranteed by the local supersymmetry acquired through the coupling to supergravity. In this sense, it may also be regarded as a two-dimensional model for the coupling of a matter supermultiplet to supergravity, in analogy with a similar recent formulation of the spinning particle[3].

The spinning string is described by a Minkowski vector $A_i(x^o, x^1)$ and by a two-dimensional Majorana spinor $\chi_i(x^o, x^1)$ which is also a Minkowski vector. The variables $x^\mu = (x^o, x^1)$ are the co-ordinates on the two-dimensional surface spanned by the string as it moves in space-time, the embedding of the surface in Minkowski's space being described by the co-ordinates A_i. In the following, to simplify the notation, the Minkowski indices (such as i) will be omitted; whenever two Minkowski vectors are multiplied, their product must be understood as being the Lorentz invariant inner product. The action for the spinning string[1] is usually given as[*]

$$I_o = \int \left[-\frac{1}{2}(\partial_\mu A)^2 - \frac{i}{2} \bar{\chi} \gamma \cdot \partial \chi \right] d^2x \qquad (1)$$

In this form it appears as the action of a two-dimensional field theory of massless fields, the Lorentz group in physical space playing the role of an internal symmetry group. In addition to the equations following from (1), which are

$$\Box A = 0 \quad , \quad \gamma \cdot \partial \chi = 0 \quad , \qquad (2)$$

the fields are assumed to obey the constraints that the energy momentum tensor and the spinor current vanish

---

*) Our two-dimensional gamma matrices are $\gamma^o = -i\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma_5 = \sigma_3$, and we have $\gamma_a\gamma_b = \eta_{ab} + \gamma_5\epsilon_{ab}$. All spinors are totally anticommuting two-dimensional fields. the adjoint of a spinor χ is $\bar{\chi} = \chi^T\gamma^o$. Finally we use $(\partial/\partial x_\mu) = \partial_\mu$.

$$T_{\mu\nu} \equiv \partial_\mu A \partial_\nu A - \frac{1}{2}\eta_{\mu\nu}(\partial_\rho A)^2 + \frac{i}{4}\bar{\chi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\chi = 0 \quad (3)$$

$$J_\mu \equiv \partial_\nu A\, \gamma^\nu \gamma_\mu \chi = 0 , \quad (4)$$

together with the appropriate boundary conditions for the open string. The constraints (3) and (4) are compatible with the field equations (2); one can easily verify that, if assumed valid at some particular value of the two-dimensional time variable x^0, the constraints will remain valid for all times. The theory is invariant under Lorentz transformations and also under global supersymmetry transformations in two dimensions[1]. The latter are given by

$$\delta A = i\bar{\alpha}\chi \qquad , \; \delta\chi = \partial_\mu A \gamma^\mu \alpha \quad (5)$$

where α is an infinitesimal anticommuting constant parameter. It is easy to see that the action (1) is invariant.

The constraints (3) and (4) do not follow from the action (1). In order to obtain an action which incorporates them, one must modify (1) so as to make it invariant under both general transformations of the co-ordinates x^μ on the surface and local supersymmetry transformations[4]. In earlier work by other authors[5], only general co-ordinate invariance was required: as a consequence, the gauge constraint (3) was incorporated in the action, but the supergauge constraint (4) had to be imposed by hand.

In order to find the needed modification of (1), let us study its response under the supersymmetry transformations (5), but taking now a parameter $\alpha = \alpha(x)$ with an arbitrary x dependence. One finds easily

$$\delta I_0 = -i \int \partial_\mu \bar{\alpha}\, J^\mu d^2x , \quad (6)$$

where J^μ is given by (4). This formula identifies J^μ as the Noether current of global supersymmetry and shows, in particular, that the action is invariant if α is constant. For x dependent α, in order to cancel (6), one is led to introduce a gauge field for local supersymmetry, which must be a two-dimensional Majorana vector-spinor ψ_μ transforming as

$$\delta \psi_\mu = - \partial_\mu \alpha \ , \tag{7}$$

and to add to the action a coupling term

$$I_1 = -i \int \bar{\psi}_\mu J^\mu d^2x = -i \int \bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi \partial_\nu A \, d^2x . \tag{8}$$

This, however, is not sufficient, because the current J^μ contains the derivative of the field A, so that the variation of I_1 has an additional term with a derivative of α equal to

$$\int \bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi \ \partial_\nu \bar{\alpha} \chi \, d^2x = -\frac{1}{2} \int \bar{\chi} \chi \ \bar{\psi}_\mu \gamma^\nu \gamma^\mu \partial_\nu \alpha \, d^2x \ . \tag{9}$$

The latter form is obtained with a simple Fierz rearrangement*). To compensate (9) one must introduce a contact term

$$I_2 = -\frac{1}{4} \int \bar{\chi} \chi \ \bar{\psi}_\mu \gamma^\nu \gamma^\mu \psi_\nu \, d^2x \ , \tag{10}$$

with the additional factor $\frac{1}{2}$ because of the occurrence of ψ_μ twice. The procedure is perfectly analogous to that which generates the sea-gull term in scalar electrodynamics, where the linear coupling term between the electromagnetic potential and the current is not sufficient for gauge invariance, due to the fact that the electromagnetic current contains a derivative of the charged scalar field. Finally, we observe that the action $I_0 + I_1 + I_2$ is not yet invariant under general co-ordinate transformations. This, however, can be achieved by standard methods, with the introduction of a two-dimensional "vierbein" field $e_\mu^{\ a}$. We obtain in this way our final form for the action

---

*) The rearrangement formula for any four Majorana two-component anticommuting spinors is

$$(\bar{\varphi}\chi)(\bar{\psi}\lambda) = -\frac{1}{2}(\bar{\psi}\gamma_A\chi)(\bar{\varphi}\gamma^A\lambda) \ ,$$

where the sum is over $\gamma_A = (1, \gamma_5, \gamma_a)$, $\gamma^A = (1, \gamma_5, \gamma^a)$.

$$I = \int L\, d^2x$$

$$\begin{aligned} L = &-\frac{1}{2}\partial_\mu A\, \partial_\nu A\, g^{\mu\nu} e - \frac{i}{2} e\, \bar\chi \gamma^\mu \partial_\mu \chi \\ &- ie\, \bar\psi_\mu \gamma^\nu \gamma^\mu \chi\, \partial_\nu A - \frac{e}{4} \bar\chi\chi\, \bar\psi_\mu \gamma^\nu \gamma^\mu \psi_\nu \end{aligned} \tag{11}$$

Here $e_a{}^\mu$ is the inverse vierbein, $e = \det e_\mu{}^a$, the metric tensor is $g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}$, γ^a are the gamma matrices of flat two-space-time and $\gamma^\mu = \gamma^a e_a{}^\mu$. Observe that in two dimensions the covariant derivative of a spinor has the form

$$D_\mu \chi = \partial_\mu \chi + \frac{1}{2} \omega_\mu \gamma_5 \chi \tag{12}$$

(the connection ω_μ is discussed below). When inserted into the action for the field χ, the connection term does not contribute, so we have written the Lagrangian (11) with an ordinary derivative on χ. Also, in four dimensions one would add to the matter-supergravity interaction the action for the supergravity fields $e_\mu{}^a$ and ψ_μ themselves. In two-dimensions, however, the Einstein Lagrangian (both in first and in second order form) is a divergence and integrates to zero and the Rarita-Schwinger Lagrangian vanishes identically. Therefore (11) is complete as it stands and the fields $e_\mu{}^a$ and ψ_μ take the role of Lagrange multipliers. The action (11) is invariant under general co-ordinate transfomations, local Lorentz transformations and Weyl transformations, under which the fields rescale as follows ($\Lambda = \Lambda(x)$ arbitrary function of x).

$$\begin{aligned} &A \to A \quad , \quad \chi \to \Lambda^{-1/2} \chi \\ &e_\mu{}^a \to \Lambda e_\mu{}^a \quad , \quad \psi_\mu \to \Lambda^{1/2} \psi_\mu \quad . \end{aligned} \tag{13}$$

Furthermore, because of the two-dimensional identity $\gamma_\mu \gamma^\nu \gamma^\mu = 0$, the action is not affected by a change

$$\psi_\mu \longrightarrow \psi_\mu + \gamma_\mu \varphi \tag{14}$$

where $\varphi(x)$ is an arbitrary spinor.

The action (11) is also invariant under the local supersymmetry transformations

$$\delta A = i\bar{\alpha}\chi \quad , \quad \delta\chi = (\partial_\mu A + i\bar{\chi}\psi_\mu)\gamma^\mu\alpha \tag{15}$$

$$\delta e_\mu{}^a = -2i\bar{\alpha}\gamma^a\psi_\mu \quad , \quad \delta\psi_\mu = -D_\mu\alpha \quad , \tag{16}$$

where $D_\mu \alpha$ is defined as in (12), with

$$\omega_\mu = \omega_\mu(e) - 2i\,\bar{\psi}_\mu\gamma_5\,\gamma\cdot\psi \tag{17}$$

$$\omega_\mu(e) = -\frac{1}{e}\,e_{\mu a}\,\varepsilon^{\lambda\nu}\,\partial_\lambda e_\nu{}^a \quad .$$

The explicit verification is purely a matter of relatively simple, even if somewhat lengthy, algebra*) and shall not be given here. We prefer to interpret the additional terms which appear in the variations (15) and (16) and which are absent in (5) and (7). Except for a different normalization, (15) are exactly the analogues of the variations occuring in pure supergravity in four dimensions[2]. This is especially obvious if one observes that (16) can be obtained by solving for ω_μ the equation

$$C_{\mu\nu}{}^a = 2i\,\bar{\psi}_\mu\gamma^a\psi_\nu \quad , \tag{18}$$

where the torsion $C_{\mu\nu}{}^a$ is given by

$$C_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu\,\varepsilon^a{}_b\,e_\nu{}^b + \omega_\nu\,\varepsilon^a{}_b\,e_\mu{}^b \; . \tag{19}$$

---

*) We found it convenient, after varying, to use the decomposition $\psi_\mu = \tilde{\psi}_\mu + \gamma_\mu\varphi$, $\gamma\cdot\tilde{\psi} = 0$. One has the useful identities $\gamma_\mu\tilde{\psi}_\nu = \gamma_\nu\tilde{\psi}_\mu$, $\gamma_5\tilde{\psi}_a = -\epsilon_{ab}\tilde{\psi}^b$, $2\bar{\tilde{\psi}}_a\psi_b = \eta_{ab}\,\bar{\tilde{\psi}}\cdot\tilde{\psi}$ and $\bar{\tilde{\psi}}_\mu\gamma_\nu\tilde{\psi}_\lambda = 0$.

As in supergravity, ω_μ has the property that its own variation under (16) does not contain derivatives of the parameter α, these derivatives cancelling between the two terms on the right-hand side of (17). The same is true of the expression $\partial_\mu A + i\bar{\chi}\psi_\mu$ which enters in the variation of $\delta\chi$. In other words, the variations are entirely constructed in terms of supercovariant derivatives, i.e derivatives covariant not only under general co-ordinate and local Lorentz transformations but also under local supersymmetry transformations.

It is interesting to compute the commutator of two local supersymmetry transformations of parameters α_1 and α_2. From (15) one finds immediately, for the commutator on the field A,

$$[\delta_2,\delta_1]A = 2i\bar{\alpha}_1\gamma^\mu\alpha_2\partial_\mu A - 2\bar{\alpha}_1\gamma^\mu\alpha_2\bar{\Psi}_\mu\chi \quad . \tag{20}$$

This can be interpreted as the combination of a general co-ordinate transformation of parameter $\xi^\mu = 2i\bar{\alpha}_1\gamma^\mu\alpha_2$ and of local supersymmetry transformation of (field dependent) parameter $\alpha = 2i\bar{\alpha}_1\gamma^\mu\alpha_2\psi_\mu$. The computation of the commutator on the other fields is a little more complicated and (for χ) requires use of the field equations. The result has the same interpretation as for A, but in addition one finds also a local Lorentz transformation of (field dependent) parameter $i\omega_\mu\bar{\alpha}_1\gamma^\mu\alpha_2$. For instance

$$\begin{aligned}[\delta_2,\delta_1]\chi &= 2i\bar{\alpha}_1\gamma^\mu\alpha_2\partial_\mu\chi + i\omega_\mu\bar{\alpha}_1\gamma^\mu\alpha_2\gamma_5\chi \\ &+ 2i\bar{\alpha}_1\gamma^\mu\alpha_2\ \gamma^\nu\psi_\mu(\partial_\nu A + i\bar{\chi}\psi_\nu) .\end{aligned} \tag{21}$$

In this sense the algebra closes[4),2),6)].

Variation of the fields A and χ in the action (11) yields the differential equations

$$\partial_\mu(eg^{\mu\nu}\partial_\nu A) + i\partial_\mu(e\bar{\Psi}_\nu\gamma^\mu\gamma^\nu\chi) = 0 \tag{22}$$

and

$$\gamma^\mu(\partial_\mu + \tfrac{1}{2}\omega_\mu(e)\gamma_5)\chi + \gamma^\mu\gamma^\nu\psi_\mu\partial_\nu A - \tfrac{i}{2}\chi\,\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu = 0 \tag{23} ,$$

while variation of the Lagrangian multipliers $e_\mu^{\ a}$ and ψ_μ gives the constraints*)

$$\frac{1}{e}\frac{\delta I}{\delta e_\mu{}^a} \equiv \hat{T}_a{}^\mu = 0 \quad , \hat{J}^\mu \equiv \partial_\nu A \gamma^\nu \gamma^\mu \chi - \frac{i}{2} \bar{\chi}\chi \, \gamma^\nu \gamma^\mu \psi_\nu = 0 \tag{24}$$

Equations (22) to (24) are consistent with each other as a consequence of the invariances of the action. The consistency can also be verified directly, and we have checked it. For instance the divergence of the spinor current turns out to vanish as a consequence of the equations (22) to (24) themselves.

We now wish to prove that there exists a special gauge in which our action and our equations take the simpler form given by (1) to (4). The identity

$$2\bar{\psi}_\mu \gamma_5 \varphi \, \gamma_5 \alpha = \bar{\alpha}\varphi \, \psi_\mu - \bar{\alpha}\gamma_5 \varphi \, \gamma_5 \psi_\mu + \gamma_\mu \gamma^\nu \alpha \, \bar{\psi}_\nu \varphi \quad , \tag{25}$$

where $2\varphi = \gamma\cdot\psi$, shows that the terms bilinear in ψ_μ in (16) can be interpreted as a combination of a Weyl transformation, a local Lorentz transformation and a transformation (14). Since the action is invariant under these transformations we can shift these terms from $\delta\psi_\mu$ to the variations of the other fields and retain only the metric covariant derivative

$$\delta' \psi_\mu = D_\mu' \alpha = \left(\partial_\mu + \frac{1}{2}\omega_\mu(e)\gamma_5\right)\alpha \; . \tag{26}$$

Now, a vector-spinor in two dimensions can always be written in the form

---

*) We refrain from writing explicitly the obvious form of $\hat{T}_a{}^\mu$. Observe that, if one introduces the supercovariant derivative of χ, which obviously is $\mathcal{D}_\mu \chi = D_\mu \chi + (\partial_\nu A + i\bar{\chi}\psi_\nu)\, \gamma^\nu \psi_\mu$, the Dirac equation (20) takes the extremely simple form $\gamma^\mu \mathcal{D}_\mu \chi = 0$, as one verifies with a simple Fierz rearrangement.

$$\psi_\mu = D'_\mu \lambda + \varepsilon_\mu{}^\nu \gamma_5 D'_\nu \beta \, . \tag{27}$$

Using the invariance (26) one can obtain $\lambda = \beta$ or

$$\psi_\mu = \gamma_\mu \, \gamma \cdot D' \beta \qquad . \tag{28}$$

Then, because of (14), one can go to the gauge $\psi_\mu = 0$. For the vierbein, one chooses the co-ordinates x^μ and the local Lorentz frames in such a way that $e_{\mu a} = \Lambda(x)\eta_{\mu a}$, which is always possible in two dimensions. Finally, by means of a Weyl transformation one reaches the gauge where $e_{\mu a} = \eta_{\mu a}$. The equations and the action have now the form (1) to (4). The boundary conditions for the open strings can be obtained from (11) and studied either in their general form or in the special gauge. They have two solutions, which correspond respectively to the Neveu-Schwarz and to the Ramond model.

Recently P.A. Collins and R.W. Tucker[7)] have constructed a Lagrangian for the N.S.R. model which incorporates correctly the constraints. Their approach is very different from ours and it is interesting to see whether one can establish a connection. It turns out that their Lagrangian can be obtained from (11) by going to a particular gauge. First observe that it is always possible to choose the local Lorentz frames so that the component $\mu = 1$, $a = 0$ of the vierbein $e_\mu{}^a$ vanishes. Then, using the Weyl invariance (13) one can rescale the vierbein field so that $\det e_\mu{}^a = 1$. In this way the four quantities $e_\mu{}^a$ are reduced to two. Observe also that, using the invariance (14), one can transform ψ_μ so that it satisfies $\gamma^\mu \psi_\mu = 0$. This equation reduces the number of independent components of $\psi_a = e_a{}^\mu \psi_\mu$ from four to two. Now the Lagrangian has lost the manifest two-dimensional space-time symmetry but one has the advantage of a smaller number of Lagrange multipliers, two bosonic and two fermionic functions. A final rescaling of the field χ gives the Lagrangian of Ref 7).

The superspace formalism provides an alternative method for constructing a Lagrangian for the N.S.R. model[8)]. The relation between the superspace method and the present work will be discussed separately.

ACKNOWLEDGEMENTS

One of us (B.Z.) wishes to acknowledge the early work done in collaboration with J. Wess[4),6)] during which the idea of local supersymmetry was first developed, with the aim of giving a Lagrangian derivation of the constraints of the Neveu-Schwarz-Ramond model.

After the present work was completed[*)], we received a preprint by S. Ferrara, J. Scherk and P. van Nieuwenhuizen and one by the same authors with F. Gliozzi[9)], describing the coupling of supergravity to matter supermultiplets in four dimensions. These authors define supercovariant derivatives and emphasize their importance. The coupling to the vector-spinor supermultiplet was also obtained by us, using the first order formalism for supergravity[10)].

---

*) The present work was presented at the Scottish Universities Summer School, August 1976.

REFERENCES

(1) A. Neveu and J.H. Schwarz, Nucl. Phys. B31, 86 (1971);
P. Ramond, Phys. Rev. D3, 2415 (1971);
Y. Aharonov, A. Casher and L. Susskind, Phys. Letters 35B, 512 (1971);
J-L. Gervais and B. Sakita, Nucl. Phys. B34, 633 (1971).

(2) D. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D13, 3214 (1976);
S. Deser and B. Zumino, Phys. Letters 62B, 335 (1976).

(3) L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. Howe, CERN preprint TH.2208 (1976), to be published in Phys. Letters B.

(4) B. Zumino, Lectures at the 1973 Capri Summer School, published by Plenum Press 1974 with the title: " Renormalization and Invariance in Quantum Field Theory", E. Caianiello editor.

(5) Y. Iwasaki and K. Kikkawa, Phys. Rev. D8, 440 (1973);
L.N. Chang, K. Macrae and F. Mansouri, Phys. Letters 57B, 59 (1975).

(6) J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974).

(7) P.A. Collins and R.W. Tucker, University of Lancaster preprint (1976)

(8) B. Zumino, Proceedings of the Conference of Gauge Theories and Modern Field Theory, Boston, (1975), p.281; (M.I.T. Press, Cambridge MA. 1976).

(9) S. Ferrara, J. Scherk and P. van Nieuwenhuizen, Ecole Normale preprint, Paris (1976), PTENS 76/17;
Same authors with F. Gliozzi, PTENS 76/19;

(10) S. Deser and B. Zumino, in preparation.