A remark on the $N \to \infty$ limit of $W_N$-algebras

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ABSTRACT

The (non-linearly) extended conformal algebras associated with $su(N)$ ($W_N$-algebras) contain operators $\tilde{W}^k(z)$ of dimensions $k = 2, 3, \ldots N$. We show that in the $N \to \infty$ limit, the quantum (commutator) algebra of the $\tilde{W}^k(z)$ reduces to their classical (Poisson bracket) algebra. In particular, this proves the closure - in the usual non-linear sense - of the quantum $W_\infty$ algebra.

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Since the discovery of Zamolodchikov and Fateev\cite{1} of the spin-3 algebra, a lot of work has been devoted to the study of general $W_N$ algebras with generators of conformal spin up to $N^{[2-5]}$. Although the commutator of two $W$ generators does not close in the Lie algebra sense, it is expressible as a (in general non-linear) polynomial of the $W$ generators. Representation theory can be based on generalized Verma modules\cite{6-7,10} and series of unitary $c < N - 1$ theories have been obtained by coset constructions based on $(A_{N-1})_k \oplus (A_{N-1})_{k+1}$. The coset construction can be extended to any simply-laced Kac-Moody algebra $\mathfrak{g}$. $W$ algebras associated with an arbitrary simple Lie algebra $\mathfrak{g}$ have been studied by means of the $\mathfrak{g}$-Toda field theories\cite{6-7}.

There are several reasons to believe that the $N \to \infty$ limit of $W$ algebras might be relevant to membrane theories\cite{11}. To be definite, we consider $W$ algebras associated with $su(N)$, containing operators $W^k(z)$, $k = 2, 3, \ldots N$.

First, membrane theories contain a doubly infinite set of generators $V_{i,n}$ of the area-preserving diffeomorphism algebra\cite{12,13}. Clearly, in the $N \to \infty$ limit, the $W$-algebra also contains a doubly infinite set of generators $W^k_n$ which are the Fourier modes (or Laurent series coefficients) of the $W^k(z)$. In particular, it was shown\cite{12,13} that a certain linear combination of the $V_{i,n}$ generates the Virasoro algebra, and hence might be identified with $W^2_n$.

Second, it was shown\cite{14-18} that the area-preserving diffeomorphism algebra on the sphere is connected to the $su(\infty)$ Yang-Mills gauge algebra. On the other hand, it is known\cite{19,20} that self-dual $su(N)$ Yang-Mills theory is closely linked to the $su(N)$ Toda theory. This relation was made even more explicit, when it was shown that the self-dual spherical membrane is an integrable system isomorphic to the $su(\infty)$ Toda theory\cite{21}.

The $su(N)$ Toda field theories give realizations of the $W_N$ algebras\cite{14-7}, and one expects again a correspondence between the $V_{i,n}$ of the area-preserving diffeomorphism algebra and the $W^k_n$ generators.

Third, it was argued in a recent paper\cite{22} that the $W^k_n$ directly provide a realization of the area-preserving diffeomorphism algebra of the plane, allowing however for modifications of the latter algebra such as extensions by the central term and by the terms which are non-linear in the $W$ generators. In our opinion, it would help if one could clarify further the relation of these modified algebras to the initial area-preserving diffeomorphism algebra. In order to do this one should have some explicit formula, relating a given realization of the $W^k_n$ (in terms of harmonic oscillators e.g.) in the $N \to \infty$ limit to the $V_{i,n}$.

The aim of the present note is to make a modest step in this direction. We shall show that in the $N \to \infty$ limit the structure constants of the $W_N$ algebra are greatly
simplified. In general, the $W^k(z)$ can be expressed as normal-ordered products of free fields $\partial \varphi, \partial^2 \varphi, \ldots \partial^{(k)} \varphi$. When evaluating an operator product expansion of $W^k(z)$ and $W^{k'}(z')$ a given structure constant receives contributions from simple and multiple contractions of the $\partial^{(i)} \varphi$ fields. We will show that in the $N \to \infty$ limit, and for fixed imaginary background charge, i.e. positive, real Planck constant $\hbar$, the contributions from the simple contractions ($O(\hbar)$) are dominating and the multiple contractions ($O(\hbar^p), p \geq 2$) are irrelevant.† This amounts to neglecting all loop diagrams and keeping only the tree diagrams. In other words, in the $N \to \infty$ limit, a Poisson bracket computation of the $\hat{W}$-algebra yields the same result as a complete quantum computation. This greatly simplifies the explicit determination of all structure constants. In particular, we know\cite{5,6} that the Poisson bracket algebra of all $\hat{W}_n^k$ closes in their enveloping algebra,\footnote{To be precise, we will show this property for the algebra of the $\hat{W}$-algebra. The true primary fields $W^k$ are related to the $\hat{W}^k$ by adding composite fields: $W^k = \hat{W}^k + \sum_{r_1} \cdots \sum_{r_N} \prod \theta^{r_1} \hat{W}^{r_N}$, see eq. (8) below.} whereas concerning the quantum commutator algebra (for generic finite $N$), closure is still an unproven conjecture. Hence, our result proves the closure - in the usual non-linear sense - of the quantum $W_\infty$ algebra.

The arguments we will give in the following not only apply to $W_N$ algebras associated with $su(N) = A_{N-1}$, but also to those based on any other simple Lie algebra (see e.g. refs\cite{5,6}). For simplicity however, we will only consider $su(N)$ explicitly.

Before considering the case of a general $W^k$, let us look at some simple examples. $W^2$ is nothing but the energy-momentum tensor, and in the Feigin-Fuchs-type representation in terms of free fields it reads

$$T(z) = -\frac{1}{2} : \partial \varphi(z) \cdot \partial \varphi(z) : + 2i \alpha_0 \partial^2 \varphi(z)$$ (1)

where

$$i \partial \varphi_j(z) = \sum_{n=-\infty}^{\infty} a_n^j z^{-n-1}, \quad j = 1, \ldots N - 1 \quad (2.a)$$

$$[a_n^j, a_m^i] = \delta^i_j \delta_{n,-m} \Rightarrow < \partial \varphi_i(z) \partial \varphi_j(w) >= -\delta_{ij} \frac{1}{(z-w)^2} \quad (2.b)$$

are $N - 1$ free bosonic fields. It is convenient to introduce $N$ vectors $h_a (a = 1, \ldots N)$ obeying $\sum_a h_a = 0$, $h_a \cdot h_b = \delta_{ab} - \frac{1}{N}$, $\sum_a x \cdot h_a y \cdot h_a = x \cdot y$. The $N - 1$ simple
roots of \( su(N) \) then are \( e_a = h_a - h_{a+1} \), and the fundamental weights are given by 
\[ \lambda_i = \sum_{a=1}^i h_a. \]
The Weyl vector \( \varrho \) equals
\[ \varrho = \sum_{i=1}^{N-1} \lambda_i = \sum_{a=1}^N (N - a)h_a, \quad \varrho^2 = \frac{N(N-1)(N+1)}{12} \tag{3} \]

The central charge of the Virasoro algebra generated by the Laurent coefficients of the energy-momentum tensor (1) is
\[ c = (N - 1) - 48\alpha_0^2 \varrho^2 = (N - 1) \left(1 - 4\alpha_0^2 N(N + 1)\right) \tag{4} \]

The second term, proportional to \( \alpha_0^2 \), has its origin in a simple contraction of the \( \partial^2 \varphi \) term with itself in the OPE of \( T(z)T(z') \). The first term in (4) comes from the double contraction (loop diagram) of \( (\partial \varphi)^2 \) with itself. Hence \(-1/\alpha_0^2\) is a loop counting parameter and plays the role of Planck's constant \( \hbar \). We may refer to the second term in the central charge (4) as the classical contribution, already present when computing Poisson brackets, and to the first term as a quantum correction. One sees explicitly that the classical term is of order \( N^3 \) while the quantum correction is of order \( N \) only: for \( N \to \infty \), only the classical contribution to the central charge, and hence to the structure constants of the Virasoro algebra, survives.\(^*\)

The same result can explicitly be seen to be true for the commutators of \( \hat{W}_n^2 \) with \( L_m \) and with \( \hat{W}_m^3 \). These commutators have been computed for arbitrary \( N \) in ref.\(^{[7]} \).\(^†\)

Although the notation used there is slightly different from the present one, it is not difficult to see that the ratio of the contribution to a given structure constant coming from a double (triple) contraction and of the corresponding contribution coming from a simple (double) contraction is of order \( \hbar N^{-2} \sim \frac{1}{\alpha_0^2} N^{-2} \). Again, in the \( N \to \infty \) limit only the simple contractions survive. Finally, looking at the commutator of \( L_n \) and of \( \hat{W}_n^3 \) with certain basic primary fields \( \psi_a \) (the generalizations of the \( \phi(2,1) \) fields of BPZ\(^{[22]} \)) one finds again\(^{[6,7]} \) that for \( N \to \infty \) the dominant terms are those of lowest order in \( \hbar \) (simple contractions).

\(^*\) In order that the central charge remains positive when \( N \to \infty \) it is of course necessary that \(-1/\alpha_0^2 \sim \hbar > 0\) : This is the regime naturally arising in the Toda field theory.\(^{[4-7]} \)

\(^†\) In ref.\(^{[9]} \) a slight error crept in eq. (3.53) : in the contribution of order \( \hbar^2 \) to the central term one should replace the factor 6\( n^2 + 1 \) by 5\( Nn^2 + n^2 + 1 \).
Having motivated our claim by specific examples, let us now proceed to show it in the general case. The \( W^k(z) \) are defined, starting from a differential operator, by

\[
(2\alpha_0 \partial - H_N(z))(2\alpha_0 \partial - H_{N-1}(z)) \ldots (2\alpha_0 \partial - H_2(z))(2\alpha_0 \partial - H_1(z)):
\]

\[
= \sum_{k=0}^{N} (-1)^{k+1} \tilde{W}^k(z)(2\alpha_0 \partial)^{N-k}
\]  

(5)

where

\[
H_a(z) = ih_a \cdot \partial \varphi(z), \quad \sum_{a=1}^{N} H_a = 0
\]  

(6)

One has \( \tilde{W}^0 = -1 \) and \( \tilde{W}^1 = 0 \). One readily verifies that \( \tilde{W}^2 \) coincides with the energy-momentum tensor defined in eq.(1):*

\[
-\tilde{W}^2 = \sum_{a > b} H_a H_b - 2\alpha_0 \sum_{a=1}^{N} (N - a) \partial H_a = \frac{1}{2} (\partial \varphi)^2 - 2i\alpha_0 \varphi \cdot \partial^2 \varphi = -T
\]  

(7)

In general, the \( \tilde{W}^k(z) \) so defined are not primary fields of the Virasoro algebra. However, one can define primary fields \( W^k(z) \) by adding fields composite in the \( \tilde{W}^r(z) \), \( r < k \):

\[
W^k(z) = \tilde{W}^k(z) + \sum d(s_i, r_i) \times \partial^{s_i} \tilde{W}^{r_1}(z) \ldots \partial^{r_k} \tilde{W}^{r_k}(z) \times
\]  

(8)

with well-chosen coefficients \( d(s_i, r_i) \). The sum is over \( r_i, s_i < k \) such that \( \sum s_i + \sum r_i = k \). One has, e.g.,

\[
W^3(z) = \tilde{W}^3(z) - (N - 2)\alpha_0 \partial \tilde{W}^2(z)
\]  

(9)

The double crosses \( \times \ldots \times \) in eq.(8) denote a normal ordering with respect to the Fourier modes of the \( \tilde{W}^r \), not with respect to the modes of the oscillators \( a_n^* \). In the following we will only consider the \( \tilde{W}^k \)'s, since once the structure constants for the \( \tilde{W} \) algebra are known, it is straightforward to deduce those of the \( W \) algebra.‡

* We do not write : \ldots : any longer, normal ordering with respect to the \( a_n^* \) oscillator modes is implicitly understood.

‡ This would not be true if the double cross normal ordering \( \times \ldots \times \) in eq.(8) was replaced by the normal ordering : \ldots : with respect to the oscillators.
From the defining equations (5) we find the explicit form of \( \tilde{W}^k \):

\[
\begin{align*}
-\tilde{W}^k &= \sum_{i_1 > \ldots > i_k}^N H_{i_1} \ldots H_{i_k} - 2\alpha_0 \sum_{i_1 > \ldots > i_{k-1}}^N \sum_{l=1}^{k-1} (N - i_l - l + 1) \partial_{i_l} H_{i_1} \ldots H_{i_{k-1}} \\
&\quad + (2\alpha_0)^2 \sum_{i_1 > \ldots > i_{k-2}}^N \sum_{l,m=1}^{k-2} \frac{1}{2} (N - i_l - l + 1)(N - i_m - m) \partial_{i_l} \partial_{i_m} H_{i_1} \ldots H_{i_{k-2}} - \ldots + \\
&\quad = - \sum_{r=0}^{k-1} \tilde{W}^k, r 
\end{align*}
\]

where the indices \( i_1, \ldots, i_k \) run from 1 to \( N \), and

\[
\tilde{W}^k, r = (-)^{r+1}(2\alpha_0)^r \sum_{i_1 > \ldots > i_{k-r}}^N \sum_{l_1, \ldots, l_r=1}^{k-r} \frac{1}{r!} \prod_{s=1}^r (N - i_{l_s} - l_s - s + 2) \partial_{i_{l_s}} \ldots \partial_{i_{l_r}} H_{i_1} \ldots H_{i_{k-r}} 
\]

(10b)

Here \( \partial_{i_l} \) means \( \partial^z \) acting on \( H_{i_l} \). \( \tilde{W}^k, r \) is the sum of all terms containing \( r \) derivative operators acting on \( k - r \) fields \( H \).

Consider now the OPE of \( \tilde{W}^k(z) \) and \( \tilde{W}^{k'}(z') \):

\[
\tilde{W}^k(z)\tilde{W}^{k'}(z') = \sum_{r=0}^{k-1} \sum_{r'=0}^{k'-1} \tilde{W}^k, r(z)\tilde{W}^{k'}, r'(z')
\]

(11)

Using \(< H_a(z)H_b(z') > = (\delta_{ab} - \frac{1}{N})/(z - z')^2 \), the contributions to the OPE of \( \tilde{W}^k, r(z) \) and \( \tilde{W}^{k'}, r'(z') \) from simple contractions are easily seen to be:

\[
(-)^{r+r'}(2\alpha_0)^{r+r'} \sum_{i_1 > \ldots > i_{k-r}}^N \sum_{j_1 > \ldots > j_{k'-r'}} \sum_{l_1, \ldots, l_r=1}^{k-r} \sum_{l'_1, \ldots, l'_r=1}^{k'-r'} \frac{1}{r!} \frac{1}{r'!} \\
\times \prod_{s=1}^r (N - i_{l_s} - l_s - s + 2) \partial_{i_{l_s}} \prod_{s'=1}^{r'} (N - j_{l'_s} - l'_s - s' + 2) \partial_{j_{l'_s}} \\
\times \sum_{m=1}^{k-r} \sum_{m'=1}^{k'-r'} : \prod_{p=1}^{k-r} H_{i_p}(z) \prod_{q=1}^{k'-r'} H_{j_q}(z') : \frac{(\delta_{im,m'} - \frac{1}{N})}{(z - z')^2}
\]

(12)

Again, \( \partial_{i_{l_s}} \) acts on \( H_{i_{l_s}}(z) \) if this field has not been contracted. It acts on \( (\delta_{im,m'} - \frac{1}{N})/(z - z')^2 \) if the \( H \)-field has been contracted \( (l_s = m) \), and similarly for \( \partial_{j_{l'_s}} \).
The term (12) contributes to \( \hat{W}^{k+k'-1,r+r'+1}(z')/(z - z') \), \( \hat{W}^{k+k'-2,r+r'}(z')/(z - z')^2 \), \( \hat{W}^{k+k'-1,r+r'-1}(z')/(z - z')^3 \), ..., \( \hat{W}^{k+k'-r-r'-2}(z')/(z - z')^{r+r'+2} \). Other contributions of the same form (same number of \( H \) and same number of derivatives) to these operators arise from simple contractions in the OPE of \( \hat{W}^{k,r+q}(z) \) and \( \hat{W}^{k',r'+q}(z') \).

On the other hand, one also obtains the same kind of terms in the OPE of \( \hat{W}^{k,r-q-(n-1)}(z) \) and \( \hat{W}^{k',r'-q-(n-1)}(z') \) by taking n-fold contractions of the \( H \)-fields. For \( q = 0 \) e.g. the result reads (\( R \equiv r - n + 1 \) and \( R' \equiv r' - n + 1 \))

\[
(-)^{r+r'}(2\alpha_0)^{R+R'} \sum_{i_1 > ... > i_k \leq R}^{N} \sum_{j_1 > ... > j_{k'} \leq R'}^{N} \sum_{l_1, ..., l_R = 1}^{k-R} \sum_{l_1', ..., l_{R'} = 1}^{k'-R'} \frac{1}{R!} \frac{1}{R'!} \prod_{s=1}^{R} (N - i_s - l_s - s + 2) \partial_{i_s}^{R'} \prod_{s'=1}^{R'} (N - j_{s'} - l_{s'} - s' + 2) \partial_{j_{s'}}^{R'}
\]

\[
\times \sum_{m_1, ..., m_n = 1}^{k-R} \sum_{m_1', ..., m_n' = 1}^{k'-R'} \prod_{p \neq m_1, ..., m_n}^{k-R} H_{i_p}(z) \prod_{q \neq m_1', ..., m_n'}^{k'-R'} H_{j_q}(z') : \prod_{t=1}^{n} \left( \delta_{m_t,j_{m_t}} - \frac{1}{N} \right) (z - z')^2 \quad (13)
\]

Only the first two sums range from 1 to \( N \). There are \( k - R + k' - R' = k - r + n - 1 + k' - r' + n' - 1 \) indices summed from 1 to \( N \). However, they are constrained by \( n \) Kronecker deltas \( \delta_{m_t,j_{m_t}} \) leaving \( k + k' - r - r' + n - 2 \) summations. Since there are \( k + k' - r - r' - 2 \) fields \( H \), there are \( n \) sums over "free" indices, resulting in a factor \( N^n \). There are also \( R = r - n + 1 \) factors \( (N - i_s - l_s - s + 2) \) and \( R' = r' - n + 1 \) factors \( (N - j_{s'} - l_{s'} - s' + 2) \). Altogether this is of order \( N^{n+r-n+1+r'-n+1} = N^{r+r'+2-n} \). The same result is obtained for \( q \neq 0 \).

We conclude that the contributions to a given \( \hat{W}^{k+k'-l,r+r'-l+2}(z')/(z - z')^l \) arising from the n-fold contractions in the OPE of \( \hat{W}^{k,r+q-(n-1)}(z) \) and \( \hat{W}^{k',r'-q-(n-1)}(z') \) are proportional to \( (2\alpha_0)^{r+r'-2(n-1)} \) and are of order \( N^{r+r'+2-n} \). Hence the ratio of the contributions of n-fold contractions to those of single contractions is

\[
\frac{\text{n-fold contractions}}{\text{single contractions}} \sim (2\alpha_0)^{-2(n-1)} N^{1-n} \quad (14)
\]

In fact, all explicit computations indicate, that due to the identity \( \sum_{a=1}^{N} H_a = 0 \), there are indeed only \( \left[ \frac{k-r}{2} \right] \) independent summations over indices in \( \hat{W}^{k,r} \), and that

\* \( [z] \) denotes the integer part of \( z \).
(13) contains only $\left[ \frac{n}{2} \right]$ summations over "free" indices. This tends to show, that one could even establish a more severe result, replacing $N^{1-n}$ in eq. (14) by $N^{1-n-[n/2]}$. In any case, equation (14) is enough to establish the property claimed above:

$$(2\alpha)^{-2}$$ is a loop-counting constant ($\sim \hbar$), and in the $N \to \infty$ limit only the contributions of the simple contractions (no loop) to a given structure constant of the $\hat{W}$-algebra survive. This allows to obtain all $\hat{W}$-algebra structure constants by Poisson bracket computations. In particular, the classical result of ref.\cite{9} shows that in the $N \to \infty$ limit the quantum $\hat{W}$-algebra, and hence also the quantum $\hat{W}$-algebra, closes - in the usual non-linear sense.

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