SOIL: Simulation of Oide Limit

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Abstract
In linear colliders in the TeV region, the synchrotron radiation occurring in the final quadrupole magnet limits the smallest (vertical) r.m.s. beam size which could be reached at the interaction point (Oide limit). On the other hand, due to the same effect, the distribution does not remain Gaussian, so that the estimate of the luminosity should also be modified. A computer code SOIL (Simulation of Oide Limit) is made which simulates these effects. It is found that this deviation helps the luminosity: it is much larger than what is expected from the r.m.s. beam size (but much smaller than the nominal value.) The method of the program, its results and theoretical considerations are given.

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1 Introduction

Recently, Oide[1] found that the synchrotron radiation limits the smallest possible (r.m.s.) beam size which can be reached at the interaction point (IP) of a linear collider for a given emittance. This fact seems to discourage efforts to achieve lower values of the beta-function at the IP ($\beta^*$). The present paper shows that such efforts can still be useful.

For the final focus design, it is only important that final distribution be optimized such that the luminosity becomes maximum. It should be noted that the final distribution function can be very different from a Gaussian due to the strong radiation effect. In this case, the nature of the radiation spectrum becomes important.

When the beam is very different from a Gaussian, we cannot apply the usual formula for the luminosity $L$;

$$L \neq L_G \propto \frac{1}{\sigma},$$

where $\sigma$ is the r.m.s. vertical beam size and $L_G$ stands for the luminosity obtained from the Gaussian-distribution approximation.

A computer code SOIL (Simulation of Olde Limit) was made to simulate this effect. The aim of the present paper is to present the program and its results with some theoretical considerations concerning the problem.
2 The Problem

As shown in Ref.[1], the perturbation $Y$ of the vertical coordinate $y$ at the IP due to the final quadrupole magnet can be written as

$$y = y_0 + Y = y_0 + K y_0' \int_0^L ds \frac{u(s)}{E_0} G(s), \quad G(s) = \int_0^s ds_1 g^2(s_1).$$ (2.1)

where $y_0$ and $y_0'$ are the nominal values (without radiation) of $y$ and $y'$ at the IP, $K$ the (constant) focusing gradient of the lens, $L$ its length, $E_0$ the nominal energy of the particle, and

$$g(s) = \frac{\sin \sqrt{K} s}{\sqrt{K}} + l^* \cos \sqrt{K} s.$$  

Here $l^*$ is the distance between the face of the magnet and the IP, and $u$ is a stochastic variable representing the energy of the emitted photon (see Sect.2.2).

The distribution of $y_0$ and $y_0'$ are Gaussian with standard deviations

$$\sigma_0 = \sqrt{\varepsilon \beta^*}, \quad \sigma_0' = \sqrt{\varepsilon / \beta^*},$$

respectively. Here $\varepsilon$ is the nominal emittance of the beam. In this connection, it is useful to define the effective $\sigma$ by the relation

$$\sigma_{\text{eff}} = \sigma_0 \frac{L_0}{L} = \sigma \frac{L_0}{L},$$

where $L_0$ is the nominal luminosity which do not consider the radiation effect. That is, from $\sigma_{\text{eff}}$, we can obtain the real luminosity by the usual formula for the Gaussian distribution.

2.1 The Characteristic Function and Luminosity

As is clear from Eq.(2.1), $Y$ is statistically independent\(^\dagger\) of $y_0$. It is thus convenient to introduce the characteristic functions

$$\tilde{\rho}(k) = \langle e^{iky} \rangle, \quad \tilde{\rho}_0(k) = \langle e^{iky_0} \rangle, \quad \text{and} \quad \tilde{\rho}_Y(k) = \langle e^{ikY} \rangle,$$

where $\langle \rangle$ is the average over distributions of $y$, $y_0$ and $Y$, respectively. That is, they are Fourier transforms of the corresponding distribution functions. From

$$\tilde{\rho}(y) = \int \int dy_0 dY \rho_0(y_0) \rho_Y(Y) \delta(y - y_0 - Y),$$

it is shown that

$$\tilde{\rho}(k) = \tilde{\rho}_0(k) \tilde{\rho}_Y(k).$$ (2.2)

Since

$$L \propto \int \rho(y)^2 dy \propto \int \tilde{\rho}(k) \tilde{\rho}(-k) dk,$$

\(^\dagger\)Actually, $Y$ depends also on $y_0$. In Eq.(2.1), this is rightly neglected. The neglect is justified whenever $\beta^* \ll l^*$. 

3
we can express $L$ as

$$L = L_0 \frac{dk \hat{\rho}(k) \hat{\rho}(-k)}{dk \hat{\rho}_0(k) \hat{\rho}_0(-k)} = L_0 \frac{2\sigma_0}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-\frac{\sigma_0^2}{\sigma^2} k^2} \hat{\rho}_Y(k) \hat{\rho}_Y(-k),$$

where we used the assumption that the distribution of $y_0$ is Gaussian:

$$\hat{\rho}_0(k) = \exp\left(-\frac{\sigma_0^2}{2} k^2\right).$$

It will be convenient to define the "luminosity enhancement factor" $D$ (which will be shown less than one) as

$$D = \frac{L}{L_0} = \frac{\sigma_0}{\sigma_{eff}}.$$

Since $Y$ is real, we have

$$D = \frac{2\sigma_0}{\sqrt{\pi}} \int_0^\infty \int_0^\infty e^{-\frac{\sigma_0^2}{\sigma^2} k^2} |\hat{\rho}_Y(k)|^2 dk. \quad (2.3)$$

This formula clearly shows a remarkable fact:

**Lemma 1** When a distribution $y_0$ is perturbed by a stochastic variable $Y$, which is statistically independent of $y_0$, the luminosity is always reduced:

$$L \leq L_0.$$

The equality holds if and only if $Y$ is a constant.

This is due to the fact that

$$|\hat{\rho}_Y(k)| \leq 1. \quad (2.4)$$

In the present case, we can assume that $Y$ has a symmetric distribution so that

$$|\hat{\rho}_Y(k)|^2 = \frac{1}{\sigma_{eff}} \cos kY.$$

The essence of the problem lies in this quantity. The program SOIL evaluates it. It will be shown, by the simulation, that

$$L > L_G.$$

### 2.2 Campbell-Rice Law of Radiation

Here we will discuss what can be known without simulation, i.e., only by theoretical considerations. We will consider the stochastic properties of $Y$.

Note that $Y$ in Eq.(2.1) is stochastic in two ways. In what follows, we will denote as follows:

- $\langle \cdot \rangle$: The first average is over all possible ways of the emission of photons, $u(s)$,

$$u(s) = \sum u_i \delta(s - s_i). \quad (2.5)$$

This process is stochastic: the place of the emission of photons $s_i$ obeys the Poisson distribution and the energy of the emitted photons $u_i$ obeys the classical radiation law[2]. The stochastic property of this process obeys the Campbell-Rice law[3], (see below).
• < >: The second average is over the distribution in \( y'_0 \); it is implied that the first average \([ \cdot ]\) is already taken. The distribution of \( y'_0 \) is assumed to be Gaussian with the standard deviation \( \sigma'_0 \).

The Campbell-Rice law applied to the radiation spectrum, implies that

\[
\frac{\mu(s_1) \mu(s_2) \ldots \mu(s_n)}{E_0} = \frac{A_n}{\rho n+2} \delta(s_1 - s_2) \ldots \delta(s_{n-1} - s_n),
\]

(2.6)

where \([ \cdot ]_c\) is the cumulant average (or, simply, cumulant)\(^2\) and where

\[
A_n = \frac{\sqrt{3}}{2^m} \alpha \gamma (\frac{3}{2} \lambda_c \gamma^2)^n I_n, \quad I_n = \int_0^\infty \xi^n d\xi \int_0^\infty K_{\delta/3}(\xi) d\xi.
\]

Here \( \lambda_c \) is the Compton wave length of the electron, \( \alpha \) the fine structure constant, \( \gamma \) the relativistic Lorentz factor, \( \rho \) the radius of curvature of the electron, and \( K_{\delta/3} \) is the modified Bessel function of the second kind.

From these equations, we can evaluate \([Y^n]_c\) at the II', if we use \( \rho^{-1} = |K g(s)y'_0| \):

\[
[Y^n]_c = y'_0^n |y'_0|^{n+1} A_n F_n(\sqrt{k l}, \sqrt{K l}^n),
\]

(2.7)

so that

\[
[e^{i k Y}] = \exp\left\{ \sum_{n=1} \frac{[Y^n]_c}{n!} (ik)^n \right\}.
\]

(2.8)

\(^2\)Suppose there are \( N \) stochastic variables, \( X_1, X_2, \ldots X_N \). The cumulant, \( < X_{i_1} \cdots X_{i_k} >_c \), is defined and related to the usual averages (moments) \( < X_{i_1} \cdots X_{i_k} > \) as follows (1 \( \leq i_k \leq N \)):

\[
< \exp(k_{i_1} X_{i_1} + \cdots + k_{i_k} X_{i_k}) > = \exp \left[ \sum_{n=1}^N \sum_{(i_1, \ldots, i_n) = 1} \frac{1}{n!} < X_{i_1} \cdots X_{i_n} >_c (ik_{i_1}) \cdots (ik_{i_n}) \right]
\]

\[
= \sum_{n=0}^N \sum_{(i_1, \ldots, i_n) = 1} \frac{1}{n!} < X_{i_1} \cdots X_{i_n} > (ik_{i_1}) \cdots (ik_{i_n}).
\]

For example,

\[
< X_1 >_c = < X_1 >,
\]

\[
< X_1 X_2 >_c = < X_1 X_2 > - < X_1 > < X_2 >,
\]

\[
< X_1 X_2 X_3 >_c = < X_1 X_2 X_3 >
\]

\[
- < X_1 > < X_2 X_3 >_c - < X_2 > < X_3 X_1 >_c - < X_3 > < X_1 X_2 >_c
\]

\[
- < X_1 > < X_2 > < X_3 >,
\]

\[
< X_1 X_2 X_3 X_4 >_c = < X_1 X_2 X_3 X_4 >
\]

\[
- < X_1 > < X_2 X_3 X_4 >_c - \cdots (4 \text{ terms})
\]

\[
- < X_1 X_2 >_c < X_3 X_4 >_c - \cdots (3 \text{ terms})
\]

\[
- < X_1 X_2 >_c < X_3 > < X_4 > - \cdots (6 \text{ terms})
\]

\[
- < X_1 > < X_2 > < X_3 > < X_4 >.
\]

When \( N = 1 \), the cumulants have a simpler form [see Eqs. (2.8) and (2.10)]. For a more detailed discussion, refer to any textbook on noise theory.
Here \( F_n \)'s are "Oide functions" defined by

\[
F_n(x, y) = \int_0^x d\varphi \sin \varphi + y \cos \varphi |^{n+1} [\int_0^{y'} (\sin \varphi' + y \cos \varphi')^2 d\varphi']^n.
\]

The function \( F \) in Ref.[1] is \( F_2 \) here.

Now \( \tilde{\rho}_Y(k) \) can be obtained as

\[
\tilde{\rho}_Y(k) = \langle e^{ikY} \rangle = \int dy_0 \rho(y_0) [e^{ikY}],
\]

where \( \rho(y_0) \) is a Gaussian distribution for \( y_0 \). This quantity can also be expressed in the form:

\[
\langle e^{ikY} \rangle = \exp \{ \sum_{n=1}^{\infty} \frac{C_n}{(2n)!} (ik)^{2n} \},
\]

where \( C_n = \langle Y^n \rangle_c \) is the n-th cumulant of \( Y \). Note that \( C_n \neq \langle [Y^n]_c \rangle \).

Lower order cumulants can be evaluated by

1. expanding \( [e^{ikY}] \) in Eq.(2.9) into a power series of \( k \),

2. performing the integration of \( y_0 \) in Eq.(2.9),

3. and comparing the results with the expansion of Eq.(2.10).

Thus we get

\[
C_2 \equiv \sigma_Y^2 = A_1^2 \Sigma_1 + A_2 I_2 \Sigma_5,
\]

where

\[
\Sigma_n \equiv \langle y_0^n \rangle,
\]

that is,

\[
\Sigma_{2n} = \frac{(2n)!}{2^n n!} \sigma_0^{2n}, \quad \Sigma_{2n+1} = \frac{n(2n+1)}{\sqrt{2\pi}} \sigma_0^{2n+1}.
\]

The \( C_2 \) is the second cumulant of \( Y \). The second cumulant of \( y \) is the r.m.s. beam size \( \sigma \), which is

\[
\sigma^2 = \sigma_0^2 + C_2.
\]

This comes from the fact that the cumulants are additive. Thus \( C_2 \) represents the increase of the r.m.s. beam size due to the radiation in the quadrupole magnet. This effect is studied\(^3\) by Oide[1].

As will be shown later, \( \sigma \) is dominated by \( \sigma_0 \) for large \( \beta^* \) but by the second term of \( C_2 \) in Eq.(2.11) for realistic values of it. For extremely small \( \beta^* \), the first term of \( C_2 \) dominates.

\(^3\)The expression for \( C_2 \) contains an extra term compared to Ref. [1]. By a private communication, Oide informed the author that it should be present. This term was also found by Buon independently.
2.3 Fourth Cumulant

The higher cumulants are useful parameters of the deviation from a Gaussian. In particular, $C_4$ is called the “excess” (also “kurtosis”). When $Y$ is Gaussian, all $C_n$ for $n > 2$ vanishes. The higher cumulants of $y$ is identical to $C_n$, since cumulants are additive and $y_0$ has only second cumulant.

After some algebra, we get

$$C_4 = C_4^I + C_4^{II},$$

$$C_4^I = A_1^1 F_1^4 (\Sigma_{12} - 3 \Sigma_{02}^2) + 6 A_1^2 F_1^2 A_2 F_2 (\Sigma_{11} - \Sigma_{01} \Sigma_{01}) + 3 A_2^2 F_2^2 (\Sigma_{10} - \Sigma_{0}^2),$$

$$C_4^{II} = 4 A_1 F_1 A_3 F_3 \Sigma_{10} + A_4 F_4 \Sigma_9.$$

Let us see how a beam can differ from a Gaussian by using $C_4$ as the measure. We use two sets of rather realistic parameters. One is the 500 GeV collider used in Ref.[1] (A), the other is the 1 TeV collider considered as the CERN linear collider (CLIC)[4] (B). Their parameters are listed in Table 1.

<table>
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<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
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<td>$\varepsilon (m)$</td>
<td>$2.5 \times 10^{-14}$</td>
<td>$5 \times 10^{-13}$</td>
</tr>
<tr>
<td>$\gamma$</td>
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<td>$2 \times 10^6$</td>
</tr>
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<tr>
<td>$K (m^{-2})$</td>
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Table 1: Typical parameters.

In Table 2, the normalized 4-th cumulant $C_4/C_2^2$ of $Y$ and the normalized 4-th cumulant $C_4/\sigma^4$ of $y$ are given as functions of $\beta^*$. It shows that the distribution of $y$ as well as that of $Y$ is far from a Gaussian.

In $C_4$, which represents the deviation from a Gaussian, we can classify two contributions.

1. $(C_4^I)$. If the classical radiation was Gaussian, we had only two terms in Eq.(2.8), $[Y]$ and $[Y]^2$. After the integration of $y_0$, they provide $C_4^I$. These terms represent the fact that the amount of the radiation depends on the amplitude of the particle [which is related to $y_0$ through $g(s)$].

2. $(C_4^{II})$. The other comes from non-Gaussian character of the radiation, $[Y^*], (n > 2)$. These terms are not so important for usual parameters of the storage ring[2] but are important in this case.

In Table 2, we showed the ratio $R \equiv C_4^{II}/C_4$. For realistic values of $\beta^*$, $C_4^{II}$ dominates $C_4$. In this respect, the non-Gaussian nature of the photon spectrum is important in this problem.

When $\beta^*$ becomes extremely small, $C_4^I$ becomes dominant. For this case, see Sect.4.1.
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<th>Type</th>
<th>No.</th>
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<th>$C_4/C_2^2$</th>
<th>$C_4/\sigma^4$</th>
<th>R(%)</th>
<th>$N_{\text{photon}}$</th>
<th>D</th>
<th>$\sigma$(nm)</th>
<th>$\sigma_{\text{eff}}$(nm)</th>
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<td></td>
<td>$\infty$</td>
<td>?</td>
<td>$\infty$</td>
<td>?</td>
</tr>
</tbody>
</table>

Table 2: Some typical values. The luminosity enhancement factor $D$ and $\sigma_{\text{eff}}$ are the simulation results with $N_e = 100$ (see Sect.3).
It is quite difficult to present general expressions for $C_n$ and the convergence of the exponent in Eq.(2.10) cannot be assured. (See Sect.4.2). Thus it seems better to regard the expansion as a symbolic expression. Of course, $C_n$ and $\tilde{\phi}_n(k)$ are meaningful,

$$C_2 = \langle Y^2 \rangle, \quad C_4 = \langle Y^4 \rangle - 3 \langle Y^2 \rangle^2,$$

apart from the relation between them.

3 Simulation Code SOIL

It seems quite difficult to properly treat a distribution which is so far from a Gaussian\(^4\). Thus, we need a computer simulation to evaluate the luminosity. For this problem, the multiparticle tracking does not seem to be the most appropriate one. (See Sect.4.3).

The method of SOIL is based on the frequency domain approach presented in Sect.2.1. We evaluate $\tilde{\phi}_n(k)$ to obtain $D$. To this end, SOIL solves the stochastic equation

$$\frac{dY(s)}{ds} = Ky_0 \frac{u(s)}{E_0} G(s), \quad Y(0) = 0, \quad (3.1)$$

to obtain the ensemble of $Y$, where $Y$ is $Y(L)$.

The rest consists of the numerical integration over Gaussian distributions of $y_0$ and $k$. By this, the calculation is much faster than the multiparticle tracking, which corresponds to the case where these integration are done by Monte Carlo method. (See Sect.4.3).

3.1 Method of SOIL

Preparation of Events First, it creates a set of events for a fixed value of $y_0$. An event is composed by a sequence of photon emission, Eq.(2.5), where the position of emission $s_i$ and the photon energy $u_i$ is determined in a stochastic manner.

We divide the quadrupole magnet into many slices so that the expectation value of the number of the emitted photon

$$P = \frac{5\alpha\gamma}{2\sqrt{3}} \rho \text{(slice length)}.$$ 

be less than 0.1 at each slice. For each slice, we calculate $P$ and call a random variable $I$ (uniform between 0 and 1): if $I < P$ a photon is emitted. (Thus $s_i$ is determined). It is the approximation of Poisson distribution by a binomial distribution. Since $P$ is less than 0.1, this approximation holds well.

Once it is decided to emit a photon, we call another random variable called RNDSR made by Yokoya which gives the classical radiation spectrum[2] to decide the energy of the photon. (Thus $u_i$ is determined).

\(^{4}\)When the deviation is not so extreme, i.e., when the normalized cumulants are of order of unity, there is a way to handle it[5].
Figure 1: Effective $\sigma$: [A] and [B] corresponds to parameters listed in Table 1. The solid line is $\sigma$, dotted line $\sigma_0$ and (+) $\sigma_{eff}$.

**Ensemble of Y** For each event, the code integrates Eq.(3.1) to obtain an element of the ensemble of $Y$:

$$Y = \frac{K \gamma_0}{E_0} \sum u_i G(s_i).$$

Note that the ensemble is defined for each value of $y'_0$.

**Evaluate $\tilde{\rho}_Y$** First, it evaluates $[\cos kY]$ for various values of $k$ using the ensemble of $Y$ as the average:

$$[\cos kY] = \frac{1}{N_e} \sum_{\text{events}} \cos kY,$$

where $N_e$ stands for the number of events to each value of $y'_0$. After averaging again over $y'_0$ distribution by numerical integration, Eq.(2.9), we obtain $\tilde{\rho}_Y(k)$.

**Luminosity enhancement factor** We average it again according to Eq.(2.3) to obtain $D$.

### 3.2 Results of Simulation

**The Reduction of Luminosity**

In Fig.1 we showed the $\sigma_{eff}$ for cases of (A) and (B), shown in Table 2. (See also Table 2). In both cases, $\sigma_{eff}$ has a minimum at some values of $\beta^*$, quite consistent with the assertion of Oide[1].

To see how the luminosity is reduced, we show $<\cos kY>^2$ for various value of $\beta^*$ in Fig.2. The average of it over the $\tilde{\rho}_0(k)^2$, which is also shown in the figure by dotted lines,
Figure 2: The $<\cos kY>^2$. The horizontal coordinate is the value of $\sigma_0 k$. The $\tilde{\rho}_0(k)^2$ is also plotted by a dotted line. The numbers in the graphs correspond to the parameters listed in Table 2 with the same number.

gives $D$. From the figure, it seems that $D$ is monotonically decreasing when $\beta^*$ becomes small.

The Difference between $\sigma_{\text{eff}}$ and $\sigma$

The minimum of the $\sigma_{\text{eff}}$, however, is much less than that of $\sigma$ and occurs for the $\beta^*$ much smaller than that minimizes $\sigma$. From Fig.1, it seems that the difference between $\sigma$ and $\sigma_{\text{eff}}$ becomes larger and larger for smaller $\beta^*$. To evaluate the luminosity in terms of $\sigma$ corresponds to the Gaussian approximation of $\tilde{\rho}_Y(k)$; We approximate $<\cos kY>$ as

$$<\cos kY> \approx \exp\left(-\frac{C_2^2 k^2}{2}\right), \quad \text{[Gaussian Approximation]} \quad (3.2)$$

which is exact if the distribution of $Y$ is Gaussian. To see how this approximation works, let us compare $<\cos kY>^2$ to its Gaussian approximation. See Fig.3. This approximation becomes worse and worse for smaller $\beta^*$.

One reason is clear: due to the non-Gaussian character of $Y$, the distribution of $Y$ becomes thinner than a Gaussian with the same $C_2$ shown in Eq.(3.2) in $Y$ space. (In $k$ space, fatter and fatter.) There is, however, another reason. In Fig.3, for small $\beta^*$, it can be seen that the main contribution to $D$ comes from the long range 'back ground' of the spectrum $\tilde{\rho}_Y(k)$. This long tail cannot be expected from the Gaussian approximation. Physically, such a tail can be regarded as the effect of the delta-function like peak of the distribution of $Y$ at $Y \approx 0$. In fact, in the event generation in SOII, almost all $Y$ are zero or very small. It is quite plausible that such a soft photons contributes dominantly to increase the luminosity from $L_G$. 

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Figure 3: The $<\cos kY>^2$ and its Gaussian approximation (dotted line). Parameters: (a) A-2, (b) A-4, (c) A-5, (d) A-7, where A-2, for example, refers to the parameter listed in Table 2 with the same number.
Figure 4: The $<\cos kY >^2$ with different number of events; $N_e = 1000$ (solid line), $N_e = 100$ (dashed line), $N_e = 10$ (-----), $N_e = 1$ (dotted line). Parameters are B-5 in Table 2.

3.3 Credibility of Results

Our ensemble of events becomes more and more accurate when $N_e$ becomes large. Thus, the credibility depends on whether our $N_e$ is enough or not. A criterion of the sufficiency of $N_e$ is that the ensemble should agree with the theoretical evaluation of $C_2$, $C_4$ and $N_{\text{photon}}$, the average number of emitted photons ($A_0 F_0 \Sigma_1$).

The agreement, however, is merely the necessary condition and not sufficient. These values refer to the behavior of $\tilde{\rho}_Y$ for $k$ close to the origin and do not assure the ensemble offers good estimate of $\tilde{\rho}_Y(k)$ for large values of $k$.

As stated before, the distribution of $Y$ has a sharp peak at the origin and at the same time it has a long tail. In this case, it is quite difficult to evaluate $<\cos kY >$ for large value of $k$ by the simulation. Since the number of events $N_e$ is limited, our ensemble is rather coarse for large value of $Y$; Large $Y$ appears in a random manner. When we reconstruct the distribution function of $Y$ from the simulation, it will have rather chaotic tail instead of the real continuous tail. On the other hand, in general, there occurs a large cancellation between terms with different realization of $Y$. This cancellation cannot be expected for large values of $k$, because of the unphysical coarse tail. As a result, the resulting $\tilde{\rho}_Y(k)$ would be chaotic for the large value of $k$.

Actually, however, to evaluate $D$, all we need is the $\tilde{\rho}_Y(k)$ for $k$ up to twice or three times of $\sigma_0$. It seems, thus, a good criterion that $\tilde{\rho}_Y(k)$ should be smooth up to $k = 2\sigma_0$; If $\tilde{\rho}_Y(k)$ becomes chaotic, or fluctuating for $K \leq 2\sigma_0$, it implies $N_e$ is not enough. In Fig. 4, we showed $\tilde{\rho}_Y$ for different number of events. In this example, $N_e = 100$ is almost enough.

The corresponding $N_{\text{photon}}$, $C_2$ and $C_4$ are shown below.
<table>
<thead>
<tr>
<th>$N_e$</th>
<th>$N_{\text{photon}}$</th>
<th>$C_2$</th>
<th>$C_4$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.82</td>
<td>2.9D-15</td>
<td>1.1D-27</td>
<td>0.546</td>
</tr>
<tr>
<td>10</td>
<td>1.98</td>
<td>5.4D-15</td>
<td>3.8D-27</td>
<td>0.596</td>
</tr>
<tr>
<td>100</td>
<td>1.901</td>
<td>5.2D-15</td>
<td>3.9D-27</td>
<td>0.568</td>
</tr>
<tr>
<td>1000</td>
<td>1.910</td>
<td>5.1D-15</td>
<td>3.9D-27</td>
<td>0.563</td>
</tr>
<tr>
<td>5000</td>
<td>1.914</td>
<td>5.18D-15</td>
<td>4.0D-27</td>
<td>0.568</td>
</tr>
<tr>
<td>exact value</td>
<td>1.913</td>
<td>5.2D-15</td>
<td>4.1D-27</td>
<td></td>
</tr>
</tbody>
</table>

For smaller value of $\beta^*$, the larger $N_e$ is required.

In summary, the user should check following items before he believes the results.

1. The average number of emitted photons, $N_{\text{photon}}$, cumulants, $C_2$ and $C_4$ should agree well with the theory.

2. The $\hat{p}(k)$ should be smooth for $k$ up to $k\sigma_0 \leq 2$.

The $N_{\text{photon}}$, $C_2$ and $C_4/C_2^2$ are given by SOIL for both theoretical and simulation results.

4 Discussion

From considerations given thus far, we may conclude that

1. A reduction of $\beta^*$ may still be useful in order to reach a larger luminosity, even if the r.m.s. beam size is not reduced or even enhanced.

2. An "Oide limit" still exists. That is, the highest luminosity is limited by the radiation effect.

3. The optimum parameter is, however, quite different from what is evaluated by the Gaussian approximation.

4. The energetic photons are irrelevant for the luminosity. Rather, the very soft photons play an important role. (This will be a good hint for the future effort to establish an analytical theory).

For realistic discussion, however, we should add considerations on the following subjects;

1. Other Sources of Deviation from Gaussian. We assumed that the nominal distribution is Gaussian. Actually, the nominal distribution can be already different from a Gaussian due to other nonlinear elements such as sextupole magnets and the radiation effect in other quadrupole magnets\cite{6}. The Lemma given in Sect.2.1 holds also in these cases.

2. Beam-Beam Interaction. We have not considered the disruption of the beam due to the very strong beam-beam interaction\cite{7} at the IP. It is usually assumed that the incoming beam is Gaussian in transverse direction. The effect of the non-Gaussian distribution of the incoming beam, e.g. due to the radiation, may be important. The
enhancement factor due to the disruption should thus be corrected. The parameters should be optimized with consideration of both the radiation effect at the final quadrupole and the beam-beam effect.

In below, we will give additional discussions to clarify some characteristic points of the problem.

4.1 Deviation from a Gaussian

The case $\beta^* \to 0$. It is useful to consider such an extreme case. In this case, since $\sigma'_0 \to \infty$, $C_2$ and $C_4$ are dominated by the first terms in Eq.(2.11) and $C_4'$, respectively:

$$C_2 \to A^2_1 \sigma^2_6, \quad C_4 \to A^4_1 \sigma^4_6 (\Sigma_{12} - 3 \Sigma^2_6).$$

Therefore, $C_4/C_2^2$ converges into an universal constant

$$C_4/C_2^2 \to \Sigma_{12}/\Sigma^2_6 - 3.$$

It is also easy to see that other higher normalized cumulants approach universal constants. Thus the distribution of $Y$ approaches a unique distribution regardless of other parameters.

It is of great theoretical interest to find $D$ and $\sigma_{eff}$ in this limiting case. It is not clear whether $\sigma_{eff}$ becomes $0$, $\infty$ or some finite value.

Central Limit Theorem. As is clear from the value of $C_4/C_2^2$ at $\beta^* = 0_+$, this limiting distribution is still far from Gaussian. Since $N_{\text{photon}}$ approaches $\infty$, it might be expected that the distribution should become Gaussian from the central limit theorem. This is, however, not the case. In order for the theorem to be applicable, all the cumulants should increase at the same rate. The number of photons is not the criterion.

If we increase the length $L$ of the quadrupole magnet, the same will happen. Since $F_n$ is roughly proportional to $L^{n+1}$, (see Sect.4.2), the first terms will dominate $C_2$ and $C_4$ again.

Difference between $[Y^n]_c$ and $C_n$. The cumulants of $[Y^n]_c$ behave more peacefully than $C_n$. They will approach a Gaussian when $y'_0 \to \infty$ and $L \to \infty$. This difference comes from the averaging over the Gaussian distribution of $y'_0$. The fact that $\Sigma_n$ increases faster than $n!$ causes such an effect. Physically, a particle with large $y'_0$ is affected much more than that with small $y'_0$. This does not affect $N_{\text{photon}}$ much, but more $C_2$ and still more $C_4$. From the simulation results, it seems that the luminosity is not so affected by this effect.

4.2 Convergence of the Cumulant expansion

The fact that the expansion of the exponent of $[e^{iY}]$, Eq.(2.8), has a finite convergence radius can be shown from the following considerations. Since

$$|\sin \phi + y \cos \phi| \leq \sqrt{1 + y^2} \equiv B,$$
Figure 5: Bad (or non) convergence of cumulant expansion. The solid line is the simulation result, the dotted line the Gaussian approximation and the dashed line the cumulant approximation.

we have

\[ |F_n| \leq (B \sqrt{K} L)^{2n+1}. \]

And since,

\[ I_n = \frac{2^n}{n+1} (3n+1)! (\frac{3n+1}{6})!, \]

we have

\[ \frac{|Y^n|}{n!} \leq \text{const.} \frac{Z^n}{\sqrt{n}}, \]

for large \( n \), where

\[ Z = |y_0^2|(B \sqrt{K} L)^2 \left( \frac{3}{2} \lambda_\gamma \gamma^2 \right). \]

Thus, the expansion converges for \( k \) such that

\[ |k| < 1/Z. \]

The expansion Eq.(2.8) is reliable within this radius.

On the other hand, the expansion of \( \langle e^{i K X} \rangle \), Eq.(2.10), seems to converge very slowly. Further, if the expansion is truncated at a low order, the resulting \( \tilde{\rho}_Y \) can easily breaks the upper bound of \( |\tilde{\rho}_Y| \), Eq.(2.4).

Since it is quite difficult to calculate higher order \( C_n \)'s in a form like Eq.(2.13), let us see numerically what happens to \( \tilde{\rho}_Y(k) \) if we approximate it as

\[ \tilde{\rho}_Y(k) \simeq \exp(-\frac{C^2}{2}k^2 + \frac{C^4}{4!}k^4). \]

(Let us call it cumulant approximation). In Fig.5, we show a comparison for the parameter type A-4 in Table 2. It is clearly seen in the figure that the cumulant expansion is worse
that the Gaussian approximation and also that the cumulant expansion easily break the upper-bound condition.

The former fact implies that the convergence radius of the cumulant expansion is very small or even zero. It even seems to have vanishing convergence radius. Such a thing can happen because we integrated each terms of Eq.(2.8) where the convergence radius depends on $y_0$. This seems to be the most characteristic point of the present problem.

It seems that we should treat the exponent as it is, i.e., without expanding into a series.

### 4.3 Comparison between Multiparticle Tracking

The frequency domain approach employed in SOIL has several merits over the multiparticle tracking. Roughly speaking, the latter is equivalent to performing the integrations Eqs.(2.3) and (2.9) by three dimensional (integral over $y_0$ is twofold) Monte Carlo integration. When the integrand is a Gaussian, the Monte Carlo integration is much slower to obtain the same accuracy as the numerical integration. (We used Simpson's method).

Related to this is the fact that it is quite difficult to evaluate $C_4$ accurately in the multiparticle tracking[8] with appropriate number of test particles. On the other hand, in our case, it is rather easy to achieve the right value of $C_4$ by increasing $N_e$ a little. (See Sect 3.2).

As for the luminosity or $\sigma_{eff}$, the present program gives consistent answers with the multiparticle tracking[8] with 8000 particles, although $C_4$ is almost completely misevaluated in the tracking. This fact seems to imply that the luminosity is less sensitive to the detailed structure of the distribution function than $C_4$ and even than $C_2$. 

Finally, it should be noted that the frequency domain approach cannot easily be used if other nonlinear effect are considered. In this case, it is usually difficult to know the nominal distribution function $\rho_0(y)$. Perhaps, the multiparticle tracking is the unique way to simulate the effects in this case. If, however, $\rho_0(k)$ can be approximated properly, the same technique can be applied.

### Summary

A computer code 'SOIL' [Simulation of Olde Limit] was made and some sets of parameters of TdV linear collider were investigated by it. The code relies on the frequency domain analysis and is faster and more accurate than the multiparticle tracking. We have found that there is a large deviation of the final particle distribution from a Gaussian, due to the non-Gaussian nature of the radiation. Although the r.m.s. beam size does not become any smaller, the luminosity can still be increased by making $\beta^*$ smaller. One should, therefore, optimize the system by minimizing the effective beam size rather than the r.m.s. one.

### Acknowledgement

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A User’s Guide of the Code SOIL

To use SOIL, one needs some files as follows;

**FT15F001** Input data.
**FT16F001** Written Output data.
**FT17F001** Output data for Graphics in TOPDRAWER format.

In CERN VM system, an example of the exec file will be as follows;
‘EXEC CERNLIB GENLIB’
’FILEDEF 15 DISK CLIC SOILDAT A (PERM’
’FILEDEF 16 DISK SOIL RESULTS A (PERM’
’FILEDEF 17 DISK SOIL TOPDRAW A (PERM’
’VFORT SOIL(GO’

A.1 Input Data File

Input data should be given as namelists as follows;
**NAMELIST DATA**

**QK (M⁻²)** Focusing gradient of the Quadrupole magnet, K.
**QL (M)** Length of the Quadrupole magnet, L.
**DL (M)** Length of the Drift space, l*
**GAMMA** Relativistic Lorentz factor, γ.
**EMIT (M)** Emittance, ε.
**BETA (M)** Beta function at the interaction point, β*. BETA can be given in sequence.
**NBETA** Number of data of BETA. The first NBETA values is used. Default is 10. Maximum is 30.

**NAMELIST PARM**

**ESIGK** End of the k integration in unit of σ₀. Default is 2.
**ESIGY** End of the y₀ integration in unit of σ₀. Default is 6.
**NE** Number of events per each y₀. Default is 100. Maximum is 10000.
**NK** Number of meshes for k integration. Default is 50. Maximum is 1000.
**NY** Number of meshes for y₀ integration. Default is 50. Maximum is 1000.
**IG** Seed of random numbers. Default is 584287.
NAMELIST COND

ICAL  
• If 0, only theoretical results are given.
• If 1, Simulation is done.
• If 2, \(< \cos kY >^2\)'s are also written in DISK FT16F001.
Default is 1.

CTITLE Title of run (CHARACTER*48).

An example for CLIC data is given below;
&DATA
QK=0.819581,QL=1.3,DL=1.1878,GAMMA=2.0D6,EMIT=0.5D-12,
BE=2.4D-3,6.D-4,5.D-4,4.D-4,2.D-4,1.D-4,0.5D-4,0.2D-4,
NBETA=5
&END
&PARAM
ES=5.0,ESIGY=6.0,NE=100,NK=50,NY=50
&END
&COND
ICAL=1,CTITLE='I LOVE CLIC'
&END

A.2 Output File
The graphic outputs does not need to be discussed. The written outputs (FT16F001) contains the following;

1. Title.
2. List of input data.
3. Values for \(I_n, A_n, P_n\) (FOIDE) and \(\Sigma_n\) (\(\sigma_0'\) being factored out).
4. Theoretical values for each \(\beta^*\), including
   (a) \(\sigma_0\), nominal r.m.s. beam size.
   (b) \(\sigma\), r.m.s. beam size.
   (c) Ratio \(\sigma/\sigma_0\).
   (d) \(\sigma_0'\).
   (e) \(N_{\text{photon}}\), average number of emitted photons.
   (f) \(C_2\), second cumulant of \(Y\) and its square root.
   (g) \(C_4\), fourth cumulant of \(Y\).
   (h) \(C_4/C_2^2\), normalized fourth cumulant of \(Y\).
(i) \( C_4/\sigma^4 \), normalized fourth cumulant of \( y \).

5. Simulation results for each \( \beta^* \), including

(a) The luminosity enhancement factor, \( D \).
(b) \( \sigma_{eff} \).
(c) \( N_{\text{photon}} \).
(d) \( C_2 \) of \( Y \).
(e) \( C_4 \) of \( Y \).
(f) \( C_4/C_2^2 \) of \( Y \).
(g) Table of \( <\cos kY>^2 \) if ICAL is 2 or more.
(h) Final seed of the random number.

References