SOME CONNECTIONS BETWEEN CLASSICAL AND QUANTUM ANHOLONOMY

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ABSTRACT

In this paper we study the interplay between the classical and quantum anholonomy effects (Hannay's angle and Berry's phase). When a finite-dimensional quantum system has a Berry phase, it has a non-zero Hannay angle. We show how infinite-dimensional systems can evade this correspondence, and find some necessary conditions for a system with a Berry phase to have no Hannay angle.

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Introduction

Over the last four years there has been a lot of interest in a new quantum effect known as Berry's phase\(^1\). This phenomenon occurs in systems which depend on a number of external parameters \(B_i\) and consists of an additional phase \(\gamma_n\) acquired by the wavefunction of the system during adiabatic transport around a closed loop in parameter space.

In a subsequent development of this idea, Hannay\(^2\) and Berry\(^2\) found a classical counterpart of the quantum effect. This classical phenomenon consists, for integrable systems, of an extra shift \(\Delta \Theta_j\) picked up by the angle variables \(\Theta_j\) as the parameters undergo a closed adiabatic excursion. These \(\Delta \Theta_j\) are now called Hannay's angles. For a review and more details see refs. 3 and 4.

Berry\(^2\) brought the analogy between these two effects a step further realizing that the semiclassical limit of Berry's phase \(\gamma_n\) is related to Hannay's angle, the precise relationship being

\[
\frac{\partial \gamma_n}{\partial n_j} = \Delta \Theta_j + D(I, C) \hbar^2 + E(I, C) \hbar^4 + \ldots, \tag{1}
\]

where \(C\) is the loop in parameter space while \(n\) is the set of quantum numbers associated to the wave function \(\psi_n\) and related to the classical action variables through the Bohr-Sommerfeld condition

\[
I_j = (n_j + \sigma_j) \hbar
\]

where the index \(j\) labels the classical degrees of freedom.

Equation (1) is quite instructive because it tells us that if the classical effect is present \((\Delta \Theta_j \neq 0)\), then the quantum effect is also there. It tells us nothing about the converse question: If Berry's phase is different from zero should we expect a nonzero Hannay angle? This is the kind of question we address in this paper.
In Section I we show that if one considers the dynamics generated by a time varying Hamiltonian it is necessary to embed the starting Hamiltonian into a wider class, in order to identify the effective parameter space where the Berry phase two-form has its singularities. Section II is devoted to the proof of some theorems concerning the dependence of the phase two-form on energy level. We also show that a quantum system with a finite number of levels has a straightforward classical counterpart in terms of Grassmann variables. In Section III we discuss the displaced harmonic oscillator as an example of the $N \to \infty$ limit of a system with $N$ levels and show how it evades the theorems pertaining to finite level systems. In the appendix we give the WKB expansion of Berry's phase in systems with one degree of freedom.
Section I

In his seminal paper\(^1\) Berry pointed out a connection between quantum anholonomy and degeneracies in the parameter space. Berry's phase can be written as a surface integral

\[ \gamma_n = -\iint_{\sigma} V_n \]  \hspace{1cm} (2)

where \(\sigma\) is any surface bordered by the loop \(C\), and

\[ V_n = -\iota \sum_{m \neq n} \frac{\langle n | dH | m \rangle \cdot \langle m | dH | n \rangle}{(E_n - E_m)^2} \]  \hspace{1cm} (3)

is called the phase two form, \(d\) being the exterior derivative in parameter space and \(|n\rangle\) the instantaneous eigenstates of the parameter-dependent Hamiltonian. From (2) we see that \(V_n\) develops singularities where levels cross, due to the vanishing energy denominator. Moreover, as Berry showed, the singularities act as monopole sources for the phase two-form.

On the other hand one can imagine many cases where the parameter space contains no degeneracy and yet some degeneracy outside the parameter space acts as a source for the phase two-form. For example consider the model

\[ H = B_1 \sigma_1 + B_2 \sigma_2 + \sigma_3 \]  \hspace{1cm} (4)

Here the parameter space is two-dimensional and for no value of \((B_1, B_2)\) does the system develop a degeneracy. Nevertheless a calculation of the phase two-form for the spin up and spin down states \(|\pm\rangle\) yields

\[ V_\pm = \mp \frac{dB_1 \cdot dB_2}{(1 + B_1^2 + B_2^2)^{3/2}} \]  \hspace{1cm} (5)

which is the restriction to the \(B_3 = 1\) plane of the monopole two-form arising from \(H = B_1 \sigma_1 + B_2 \sigma_2 + B_3 \sigma_3\). All the loops of (4)
lie in the plane $B_3 = 1$, but still the monopole sitting at the origin makes itself felt in (5).

The same thing happens with the system described by the Hamiltonian\textsuperscript{5}

$$ H = \frac{1}{2} \dot{\phi}^2 \frac{\omega}{r} (\cosh(2r) - \sinh(2r) \cos\theta) $$

$$ + \frac{1}{2} \omega \left( q'p + p'q \right) \sin(2r) \sin\theta $$

$$ + \frac{1}{2} p^2 \left( \cosh(2r) + \sinh(2r) \cos\theta \right) \quad (6) $$

The parameter space is a hyperboloid with coordinates $(r, \theta)$. The degeneracies are connected with the Hamiltonian

$$ H = \frac{1}{2} \left( x_1 q^2 + x_2 (qp + pq) + x_3 p^2 \right) $$

and again they lie outside the parameter space but nevertheless give rise to $V$.\textsuperscript{5}

It is possible for degeneracies lying outside the parameter space not to give rise to a phase two-form. This happens when the parameter space of the system is parallel to the flux generated by the degeneracies. An example of this is the model

$$ H = B_1 \sigma_1 + B_2 \sigma_2 \quad (B_1 > 0) $$

The flux $\vec{V}$ generated by the monopole at the origin is always parallel to the half plane in which the model lives.

From both examples (4) and (6) we see that the phase two-form felt by the system originates from degeneracies of an enlarged Hamiltonian living in an enlarged parameter space.\* What determines the effective parameter space and Hamiltonian of a system?

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\* We thank E. Kiritsis for suggesting this idea long ago in a private correspondence.
To answer this question, consider the exact evolution operator for a time-dependent Hamiltonian

$$U(T) = T \exp(-i \int_0^T H(t) dt).$$

It can be computed using the Magnus formula

$$U(T) = \exp \left\{ -i \int_0^T dt - \frac{1}{2} \int_0^T \int_0^T [H(t), H(t')] dt \, dt' \right\} + \ldots$$

where the higher order terms contain nested commutators like $[H(t), [H(t'), H(t'')]].$ So we see that new terms come into the evolution operator from the multiple commutators of $H$ with itself at different times.

Doing this for the Hamiltonian (4), we see that $U(T)$ contains terms like

$$[H(\vec{B}), H(\vec{B}')] = \sigma_1(-B_2'B_2) + \sigma_2(B_1'B_1 - B_2'B_2) + \sigma_3(B_1B_2'B_1B_2).$$

Thus the commutator has effectively generated a parameter multiplying $\sigma_3$, which was not present in the original Hamiltonian (4). The enlarged parameter space felt by the system is $\{B_1, B_2, B_3\}$ and contains a degeneracy of the effective Hamiltonian

$$H = B_1 \sigma_1 + B_2 \sigma_2 + B_3 \sigma_3$$

which acts as a source for the two-form (5).

Therefore we can prescribe a rule for generating the enlarged parameter space and Hamiltonian. We take nested commutators between terms of the original Hamiltonian until the Lie algebra closes, and then assign a parameter to each generator of the algebra. In general the starting Hamiltonian can be written as

$$H = \sum_{i=1}^g B_i T_i$$
where $T^i$ are generators of a Lie algebra of dimension $n$. Then the effective Hamiltonian felt by the system is

$$H'' = \sum_{i=1}^{g} B_i T^i + \sum_{j=g+1}^{n} B_j T^j$$

This means we have to add the generators $T^j$ ($j = g+1\ldots n$) which close the algebra and make all the parameters independent.

For the Hamiltonian (4) the algebra is SU(2), for (5) it is SU(1,1) and for the displaced harmonic oscillator\textsuperscript{5}

$$H = \frac{1}{2} \{ (p-(\hbar\omega)^{\frac{1}{4}}x_2)^2 + \nu^2 (q-(2\hbar/\omega)^{\frac{1}{4}}x_1)^2 \}$$

it is E(2), the algebra of the Euclidean group in two dimensions. Of course it can happen that the algebra does not close and so the effective Hamiltonian contains an infinite number of parameters. An example of this is

$$H = \alpha p^2 + \beta (pq + qp) + \gamma q^2 + \delta q^4.$$
Section II

Using this definition of the effective parameter space we study the relationship between Berry's phase and Hannay's angle. We first note that Hannay's angle is not directly related to Berry's phase, but to its derivative with respect to \( n \). This means that the classical effect is only there when \( \gamma \) depends on \( n \). Berry's phase does not always depend on \( n \), in fact in the displaced harmonic oscillator\(^5\) it is nonzero and the same for all levels. Thus we are led to study the dependence of Berry's phase on energy level.

Within this context we formulate the following

Theorem 1: If a system has a finite number of levels and the phase two-form \( V_n \) is independent of \( n \), then \( V_n = 0 \).

Proof: We write the phase two-form \( V_n \) as

\[
V_n = \sum_{k \neq n} V_{nk} \tag{7}
\]

with

\[
V_{nk} = \frac{-i \langle n | dH | k \rangle \langle k | dH | n \rangle}{(E_n - E_k)^2}, \tag{8}
\]

and note that \( V_{nk} = -V_{kn} \). If we suppose that \( V_n \) is independent of \( n \), then

\[
V_n = V = \sum_{k \neq n} V_{nk} \tag{9}
\]

Now summing over \( n \) and supposing that the total number of levels is \( N \), we get
\[ \sum_{n=1}^{N} v_n = NV = \sum_{n,k=1}^{N} v_{nk} = 0, \quad (10) \]

from the antisymmetry of \( v_{nk} \), and therefore \( V = 0 \).

The condition that the number of levels be finite is essential because otherwise (10) is the sum of an infinite series and we cannot conclude that it gives zero. In Section III we shall show explicitly how the displaced harmonic oscillator, which has an infinite number of levels, gets around this theorem.

Theorem 1 tells us that for systems with a finite number of levels Berry's phase can only be nonzero if it depends on \( n \). From this follows

Theorem 2: If a system has a finite number of levels and if Berry's phase is nonzero, then Hannay's angle is also nonzero.

Proof: Theorem 1 implies that \( \frac{\partial \gamma_n}{\partial n} \neq 0 \). This does not yet mean that Hannay's angle exists since, as we have already mentioned,

\[ \frac{\partial \gamma_n}{\partial n} = \Delta \Theta + \text{higher order terms}, \]

and it might be that \( \Delta \Theta = 0 \) while some of the higher order terms are nonzero.

So we find it necessary to show explicitly how to construct Hannay's angle for the classical version of a finite-level system. We follow the method of the second paper in ref. 4. For a general \( N \)-level system the quantum Hamiltonian is an element of the algebra \( \text{su}(N) \) plus a multiple of the identity. The multiple of the identity is a (time-dependent) vacuum energy which does not affect Berry's phase, as can be seen by noting that in expression (3) for \( v_n \) only off-diagonal matrix elements of \( dH \) appear and therefore a multiple of the identity in \( H \) will not contribute to the phase two-form. Thus we can renormalize the quantum Hamiltonian by subtracting the vacuum energy and write it in the form
\[ \hat{H} = \hbar B^a T^a \] (11)

where \( T^a \) are generators of \( SU(N) \) in the fundamental representation, satisfying

\[ [T^a, T^b] = i f^{abc} T^c, \]

\( f^{abc} \) being the totally antisymmetric structure constants of \( su(N) \).

To construct a classical version of (11) we introduce \( N^2 - 1 \) real Grassmann variables \( \eta^a \) which satisfy

\[ \{ \eta^a, \eta^b \} = \eta^a \eta^b + \eta^b \eta^a = 0 \]

and the Poisson brackets

\[ \{ \eta^a, \eta^b \}_{PB} = i \delta^{ab}. \] (12)

We build up from these Grassmann variables the classical Lie algebra generators

\[ T^a = \frac{i}{2} f^{bca} \eta^b \eta^c \]

which satisfy

\[ \{ T^a, T^b \}_{PB} = - f^{abc} T^c \]

When the theory is quantized these classical generators go over into the quantum generator \( \hbar T^a \) and \( \{ T^a, T^b \}_{PB} \rightarrow \frac{i}{\hbar} [\hbar T^a, \hbar T^b] \).

The classical version of (11) is then

\[ H = B^c T^c = - \frac{i}{2} \eta^a \eta^b f^{abc} B^c, \] (13)
corresponding to the classical Lagrangian

\[ L = \frac{i}{2} \dot{\psi}^a \dot{\psi}^a + \frac{i}{2} \dot{\psi}^a \dot{\psi}^b \Gamma^{abc} \psi^c, \]  

(14)

which yields the equations of motion

\[ \ddot{\psi}^a = -\Gamma^{abc} \psi^b \dot{\psi}^c, \]

and the method of ref. 4 can be used to derive Hannay's angles. We know that since the system (14) is quadratic the semiclassical approximation of Berry's phase is exact. Therefore if Berry's phase exists Hannay's angle will exist.

In the next theorem we relax the condition that the system have a finite total number of levels.

Theorem 3: If accidental degeneracies involving a finite number of levels are the sources for the phase two-form \( V_n \), then \( V_n \) and \( \gamma_n \) will depend on \( n \).

Proof: Suppose that at some point \( B^* \) in the effective parameter space the \( K \) levels \( \ket{n} \) (\( n = 1 \ldots K \)) become degenerate. Then in the neighborhood of \( B^* \) the dominant terms in \( V_n \) will contain the small energy denominators \( (E_n - E_m)^2 \) (\( n, m = 1 \ldots K \)). Therefore

\[ V_n (B \approx B^*) = \sum_{m \neq n}^K V_{nm} \]

up to small terms. Assume that the phase two-forms for all the near-degenerate levels are the same. Then summing them we get as in the proof of Theorem 1

\[ \sum_{n=1}^K V_n = KV = 0 \]
and hence $V = 0$. This contradicts the assumption that the degeneracy is a source for $V_n$. Therefore the phase two-forms of the near-degenerate levels cannot all be the same and $V_n$ has a dependence on $n$.

In this theorem it is necessary that the number of levels involved in the degeneracy be finite, otherwise we cannot conclude that the sum of the phase two-forms of the levels involved is zero.

Theorem 3 tells us that if $V_n$ does not depend on $n$ it cannot have its source in a degeneracy involving a finite number of levels. This statement also applies to the displaced harmonic oscillator where, as we shall see in the next section, the source of $V_n$ comes from a degeneracy involving an infinite number of levels.
Section III

In this section we study the displaced harmonic oscillator as an interesting example of how the theorems of the previous section can be evaded in an infinite-level system. The model is

$$H = \hbar \omega [(a^+ a^*) (a-a^*) + \frac{1}{2}]$$  \hspace{1cm} (15)

where $a = X_1 + iX_2$, and $X_1$, $X_2$ are slowly varying parameters. Chaturvedi, Sriram and Srinivasan\textsuperscript{5} calculate the Berry phase to be

$$\gamma_n(C) = -2\int_{\gamma_C} dx_1 dx_2$$ \hspace{1cm} (16)

$$\gamma_n(C) = -2 \text{(area of loop)}$$

independent of the energy level $n$. Therefore Hannay's angle is zero.

Here there is apparently no degeneracy and we have to enlarge the parameter space according to the rule of Section I and ref. 6. The algebra is $E(2)$, consisting of $\{a, a^+, a^a, 1\}$. Assigning an independent parameter to each of these terms we have

$$H = \hbar \omega a^+ a - \hbar \omega a^* a - \hbar \omega a^a + K.$$

The parameter space is now $(\omega, X_1, X_2, K)$ and the degeneracy is at $\omega = 0$. This degeneracy involves all of the infinite number of levels. So the system need not obey Theorem 3 about degeneracies involving a finite number of levels and, indeed, Berry's phase has no dependence on $n$.

In order to gain some more insight into this system, let us view it as the Lie algebra contraction of the $SU(2)$ Hamiltonian\textsuperscript{5}

$$H = \hbar \vec{J} \cdot \vec{B} = \hbar (B_2 J_z + B_+ J_+ + B_- J_-)$$ \hspace{1cm} (17)
with

\[ J_\pm = \frac{1}{\sqrt{2}} (J_x \pm iJ_y), \quad B_\pm = \frac{1}{\sqrt{2}} (B_x \pm iB_y) \]

which has matrix elements \( |jm\rangle \)

\[
H_{nm} = \hbar B_z n \delta_{nm} + \frac{\hbar}{2} \left( B_- [j(j+1) - m(m+1)]^{1/2} \delta_{n,m+1} + B_+ [j(j+1) - m(m-1)]^{1/2} \delta_{n,m-1} \right).
\]

\( H_{nm} \) is a finite dimensional matrix, so we can relabel the states

\[ n \to n' = n + j \]
\[ m \to m' = m + j \]

\[
H_{n',m'} = (n' - j) \hbar B_z \delta_{n',m'} + \hbar \left[ \frac{j}{2} \right]^{1/2} (B_- [m'+1]^{1/2} \delta_{n',m'+1} + B_+ \sqrt{m'} \delta_{n',m'-1}) + O \left( \frac{\hbar}{\sqrt{j}} \right) \tag{18}
\]

Now we take the limit as \( j \to \infty \) in such a way that \( \sqrt{j} B_\pm \to \text{const.} \). We drop the term \( j B_z \delta_{n',m'} \), an infinite renormalization of the energy. In this limit the matrix element \( n' \delta_{n',m'} \) becomes \( a^+ a \) while \( [m'+1]^{1/2} \delta_{n',m'+1} \) and \( \sqrt{m'} \delta_{n',m'-1} \) become, respectively, \( a^+ \) and \( a \). Thus we reproduce the displaced harmonic oscillator (up to a constant)

\[
H = \hbar \omega (a^+ a + \sigma a^+ + \sigma^* a)
\]

with

\[
\omega = B_z \propto B, \quad \sigma = \lim_{j \to \infty} \left[ \frac{j}{2} \right]^{1/2} \frac{B_-}{B_+},
\]

appearing as the \( j \to \infty \) limit of \( \hbar J \cdot \vec{B} \).

For (17), the element \( V_{nm} \) of the phase two-form is (in the
neighbourhood of \( \frac{B_x}{B} = \frac{B_y}{B} = 0 \))
\[ V_{nm} = \sum \frac{2}{B^2(m-n)^2} \langle n | \frac{\partial H}{\partial B_x} | m \rangle \langle m | \frac{\partial H}{\partial B_y} | n \rangle \right) dB_x dB_y \]

\[ = -\frac{1}{2B^2} \left\{ [j(j+1)-m(m+1)] \delta_{n,m+1} + [j(j+1)-m(m-1)] \delta_{n,m-1} \right\} \]

\[ = -\frac{j}{B^2} \left\{ (m'+1) \delta_{n',m'+1} - m' \delta_{n',m'-1} + \mathcal{O}(\frac{B}{J}) \right\} \]

The dependence on \( n \) is now suppressed by a factor of \( j \) and in this limit we get

\[ V_{n'} = V_{n',n'+1} + V_{n',n'-1} = \text{const.} \]

it does not vanish, yet it does not depend on \( n' \).

To summarize, we have related the displaced harmonic oscillator Berry phase to a degeneracy. The degeneracy is the one causing all levels to collapse, i.e. \( \omega = 0 \), because relative to the \( \mathcal{J} \cdot \mathcal{B} \) problem - this means \( B_z = B_x = B_y = 0 \). But, since \( B_z \approx B \) is scaled up from \( B_x \) and \( B_y \) by a factor of \( \sqrt{j} \), this degeneracy is infinitely far away in the limit.
Conclusion

To understand the relationship between classical and quantum anholonomy, we have studied quantum systems of finite dimension. For these systems the correlation between Hannay's angle and Berry's phase is exact: if one is nonzero so is the other. But there are examples of infinite-dimensional systems which break this correspondence. Thus the remaining questions have to do with the special properties of infinite-dimensional quantum systems. If the anholonomy arises because a finite number of levels becomes degenerate somewhere in parameter space, the Hamiltonian can be treated as finite-dimensional in the neighbourhood of the degeneracy. Thus only degeneracies among infinitely many levels concern us. The displaced harmonic oscillator is an explicit example of how an infinite system breaks the link between classical and quantum anholonomy and shows an infinite degeneracy. It would be very useful to understand the dependence of Berry's phase on energy level \( n \) in the higher orders of \( \hbar \), since Hannay's angle depends only on the semiclassical term. To go beyond the semiclassical limit we need a complete \( \hbar \) expansion of the Berry phase. Such an expansion has not been published before, and we provide one in the appendix. This expansion and the theorems presented above are a step towards understanding the special relationship between classical and quantum anholonomy for systems with infinitely many energy levels.

Acknowledgment

E.G. would like to thank members of TPN in Orsay, where this work was begun, for their hospitality and especially A. Comtet for valuable discussions. G.G. thanks members of the Theory Division at CERN for their kind hospitality.
Appendix: WKB Expansion of Berry's Phase

In this appendix we consider the WKB expansion of Berry’s phase one-form

\[ A = \text{Im}(\langle n;B | d | n;B \rangle) \]

for which the integral around a closed loop yields Berry’s phase. The phase one-form is related to the phase two-form \( V \) by \( dA = V \).

For simplicity we restrict our considerations to systems with one degree of freedom and Hamiltonians of the form

\[ H(p,q;B) = \frac{B_1}{2} p^2 + \frac{1}{2} (f(q;B)p + pf(q;B)) + V(q;B) \]
\[ = \frac{B_1}{2} p^2 + f(q;B)p + V(q;B) + \frac{\epsilon}{2} f'(q;B) \]

where \( B = \{ B_k \} \) is a set of external parameters, the linear term in \( p \) is there to ensure that the wave function be complex and Berry’s phase exists, and the prime means derivative with respect to \( q \).

It is convenient to use the WKB expansion of the wave function in the form given by Maslov and Fedoruk\(^9\)

\[
\text{(A1)} \quad \psi(q) = e^{\frac{\epsilon}{\hbar}} \sum_{n=0}^{\infty} \epsilon^n \psi_n(q), \quad \epsilon = -i\hbar.
\]

Substituting (A1) into the Schrödinger equation

\[ \frac{B_1}{2} \epsilon^2 \psi''(q) + \epsilon f(q,B) \psi'(q) + (V(q;B) + \frac{\epsilon}{2} f'(q;B) - E) \psi(q) = 0 \]

we get

\[
\text{(A2)} \quad \sum_{n=0}^{\infty} \epsilon^n \left[ (\frac{B_1}{2} S'' + f S' + V - E) \psi_n \right.
\]
\[ + \epsilon \left( (\frac{B_1}{2} S'' + \frac{f}{2}) \psi_n + (B_1 S' + f) \psi_n' \right) + \frac{\epsilon^2 B_1}{2} \frac{\psi_n''}{\psi_n'} = 0 \]
The $O(\epsilon^0)$ term in (A2) gives the Hamilton-Jacobi equation for $S$

$$\frac{B_1}{2} S'' + f S' + V - E = 0 \quad (A3)$$

which has the solution

$$S(q,I;B) = \int_{q_0}^{q} dq' \ p(q',I;B) \quad (A4)$$

where

$$p(q,I;B) = - \frac{f(q;B)}{B_1} \pm \left[ \frac{f(q;B)}{B_1^2} + \frac{2(E(I) - V(q;B))}{B_1} \right]^{\frac{1}{2}} \quad (A5)$$

and (A4) is understood to be a sum of integrals along different branches of the torus of action

$$I(E;B) = \int p(q,E;B) \ dq \quad (A6)$$

(see Berry\textsuperscript{10}). Eq. (A6) can in principle be inverted to give $E(I)$ in (A5).

The coefficient of $\epsilon$ in (A2) gives

$$\left[ \frac{B_1}{2} S'' + f' \right] \varphi_0 + (B_1 S' + f) \varphi'_0 = 0$$

or

$$\frac{\varphi'_0}{\varphi_0} = - \frac{1}{2} \frac{\pi'}{\pi} \quad (A7)$$

where we have introduced

$$\pi(q,I;B) = p(q,I;B) + \frac{f(q;B)}{B_1}$$
which is like a kinetic momentum for the "gauge field" \( \frac{f(q;B)}{B_1} \).

The solution of (A7) has the form

\[
\varphi_0(q,I;B) = c(\pi(q,I;B))^{\frac{1}{2}}
\]

which is proportional to the square root of the Jacobian for the transformation from configuration space to angle variables, as one can see by substituting (A.5) and (A.6) into

\[
\frac{\partial \Theta(q,I;B)}{\partial q} = \frac{\partial^2 S(q,I;B)}{\partial q \partial I} = \frac{\partial p(q,I;B)}{\partial I} = \frac{\partial p(q,E;B)}{\partial I(E;B)}
\]

We leave open the renormalization of the wave function and fix the constant \( c \) so that

(A.8) \quad \varphi_0 = \left[ \frac{1}{2\pi} \frac{\partial \Theta}{\partial q} \right]^{\frac{1}{2}}

For \( n \geq 1 \) we obtain from (A.2)

(A.9) \quad \varphi_n' + \frac{1}{2} \frac{\pi'(q)}{\pi(q)} \varphi_n = - \frac{\varphi_{n-1}''}{2\pi(q)}

which has the solution

\[
\varphi_n(q) = - [\pi(q)]^{-\frac{1}{2}} \int_{q_0}^q \frac{\varphi_{n-1}(q')}{2 \pi(q')} dq'
\]

\[
= \left[ \frac{1}{2\pi} \frac{\partial \Theta}{\partial q} \right]^{\frac{1}{2}} \Lambda_n(q)
\]

This allows us to find \( \varphi_n \) recursively to all orders. We thus obtain

(A.10) \quad \varphi(q,I;B) = \exp\left( \frac{1}{\epsilon} S(q,I;B) \right) \left[ \frac{1}{2\pi} \frac{\partial \Theta}{\partial q} \right]^{\frac{1}{2}} \left( 1 + \sum_{K=1}^{\infty} A_K \epsilon^K \right)

where
\( A_K(q) = -\int_{q_0} dq' \frac{[A_{K-1}(q') \pi(q')^{-\frac{1}{2}}]''}{2 [\pi(q')]^{\frac{3}{2}}} , \quad A_0 = 1 \)

and the action satisfies the quantization condition \( I = (n + \sigma)\hbar \).

Since (A.10) is an asymptotic series which does not converge\(^1\), we would in actual practice cut off the series at some \( K = M \). Realizing this, we retain the infinite series and consider them as formal expressions.

Now we are ready to write down the expansion for the phase one form.

\[
A_{\text{Berry}} = \frac{\text{Im}[\int dq^* (q; B) \ d_B \ d_B^* (q; B)]}{\int dq^* (q; B) \ d_B^* (q; B)}
\]

Using (A.10) and keeping only odd powers of \( \epsilon \), which are pure imaginary, we get

\[
A_{\text{Berry}} = \frac{1}{\hbar} \left\{ \frac{d\Theta}{2\pi} \left[ \sum_{k=1}^{\infty} 2A_{2k} \epsilon^{2k} + \sum_{k=1}^{\infty} A_k \ A_{-1} (-1)^l \epsilon^{k+l} \right] \right. \\
+ \sum_{k=1}^{\infty} \epsilon^{2k-1} d_B A_{2k-1} + \sum_{\ell, k=1}^{\infty} (-1)^{\ell+k} A_{\ell} d_B A_k \left. \right\}
\]

The first four terms of this series are

\[
A_{\text{Berry}} = \frac{1}{\hbar} \int \frac{d\Theta}{2\pi} d_B S \\
+ \hbar \left[ \int \frac{d\Theta}{2\pi} ((-2A_2 + A_1^2) d_B S - d_B A_1) \\
- \left( \int \frac{d\Theta}{2\pi} d_B S \right) \left( \int \frac{d\Theta}{2\pi} (A_1^2 - 2A_2) \right) \right] \\
+ \hbar^2 \left[ \int \frac{d\Theta}{2\pi} ((2A_4 - 2A_1A_3 + A_2^2) d_B S + d_B A_3 \\
+ A_2 d_B A_1 - A_1 d_B A_2 \right]
\]
\[ + \int \frac{d\Theta}{2\pi} (-2A_2 + A_1^2) \int \frac{d\Theta}{2\pi} (d_B S (2A_2 - A_1^2) + d_B A_1) \]
\[ + \int \frac{d\Theta}{2\pi} (-2A_4 + 2A_1 A_3 - A_3^2 + (2A_2 - A_3^2)^2) \int \frac{d\Theta}{2\pi} d_B S \]
\[ + \ldots . . . \]

In (A.12) and (A.13) the integrand should be understood as a function of \((\Theta, I; B)\). Following Berry\(^2\) we can introduce the single-valued functions \((0 \leq \Theta \leq 2\pi)\)

\[ S(\Theta, I; B) = S(q(\Theta, I; B), I; B) , \]
\[ \mathcal{A}_K(\Theta, I; B) = \mathcal{A}_K(q(\Theta, I; B), I; B) , \]

and write

\[ d_B S(q, I; B) = d_B S(\Theta, I; B) - p \int d_B q(\Theta, I; B) , \]
\[ d_B \mathcal{A}_K(q, I; B) = d_B \mathcal{A}_K(\Theta, I; B) - \frac{\partial \mathcal{A}_K(q, I; B)}{\partial q} d_B q(\Theta, I; B) \]

Alternatively, we could replace \(\frac{d\Theta}{2\pi}\) by \(\frac{cdq}{\pi(q)}\) in these expressions and consider them as sums of integrals over branches of the torus, the integrands being functions of \((q, I; B)\).

Now we can use (A.12) to find out how the various terms in \(\lambda_{\text{Berry}}\) change under a Hannay angle gauge transformation, i.e., under a shift in the origin of the angle variable

\[ (\Theta(I; B) \rightarrow \Theta(I; B) + \alpha(I; B) \]

In the expression (A.4) for \(S(q, I; B)\) the point \(q_0\) (or better \((q_0, p_0)\)) is a reference point for which \(S(q_0, I; B) = 0\) and \(\Theta(q_0, p_0, I; B) = 0.\) Changing the origin of the angle variable corresponds to shifting the reference point

\[ (q_0, p_0) \rightarrow (\bar{q}_0, \bar{p}_0) \]

so that

\[ S(q, I; B) \rightarrow S(q, I; B) + \int p(q', I; B) dq' = S(q, I; B) + f(I; B) \]

\[ \bar{q}_0 \]
where

$$\frac{\partial f(I;B)}{\partial I} = a(I;B)$$

and therefore

(A.16) \quad d_B S(q,I;B) \to d_B S(q,I;B) + d_B f(I;B)

Since obviously \( f(I;B) \) is independent of \( q \)

$$\frac{\partial S(q,I;B)}{\partial q} \to \frac{\partial S(q,I;B)}{\partial q}$$

and since all the amplitudes \( A_K \) depend on \( q \)-derivatives of \( S \), we have

(A.17) \quad A_K(q,I;B) \to A_K(q,I;B)

\( d_B A_K(q,I;B) \to d_B A_K(q,I;B) \)

Substituting (A.17) and (A.16) into (A.12), we find that under a Hannay angle gauge transformation

(A.18) \quad A_{\text{Berry}} \to A_{\text{Berry}} + \frac{1}{n} d_B f(I;B)

and only the \( O(\frac{1}{n}) \) term in \( A_{\text{Berry}} \) gets transformed. The reader may satisfy himself by looking at (A.13) that the higher order terms in \( A_{\text{Berry}} \) remain unaffected by (A.14).

It would also be interesting to investigate the dependence on \( I \) of the higher order terms in \( A_{\text{Berry}} \). Is it possible to show that if Hannay's angle is zero then Berry's phase is independent of \( I \), and hence of \( n \), to all orders? We leave this question open.
References