ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

PARTICLE BEAMS AND PLASMAS

J.D. Lawson

Edited by
A. Hofmann and E. Messerschmid

Lectures given in the
Academic Training Programme of CERN 1973-1974

GENEVA
1976
Propriété littéraire et scientifique réservée pour tous les pays du monde. Ce document ne peut être reproduit ou traduit en tout ou en partie sans l’autorisation écrite du Directeur général du CERN, titulaire du droit d’auteur. Dans les cas appropriés, et s’il s’agit d’utiliser le document à des fins non commerciales, cette autorisation sera volontiers accordée.

Le CERN ne revendique pas la propriété des inventions brevetables et dessins ou modèles susceptibles de dépôt qui pourraient être décrits dans le présent document; ceux-ci peuvent être librement utilisés par les instituts de recherche, les industriels et autres intéressés. Cependant, le CERN se réserve le droit de s’opposer à toute revendication qu’un usager pourrait faire de la propriété scientifique ou industrielle décrits dans le présent document.

© Copyright CERN, Genève, 1976

Literary and scientific copyrights reserved in all countries of the world. This report, or any part of it, may not be reprinted or translated without written permission of the copyright holder, the Director-General of CERN. However, permission will be freely granted for appropriate non-commercial use.

If any patentable invention or registrable design is described in thereport, CERN makes no claim to property rights in it but offers it for the free use of research institutions, manufacturers and others. CERN, however, may oppose any attempt by a user to claim any proprietary or patent rights in such inventions or designs as may be described in the present document.
PARTICLE BEAMS AND PLASMAS

J.D. Lawson

Edited by
A. Hofmann and E. Messerschmid

Lectures given in the
Academic Training Programme of CERN 1973-1974

GENEVA
1976
ABSTRACT

These lectures present a survey of some of the concepts of plasma physics and look at some situations familiar to particle-accelerator physicists from the point of view of a plasma physicist, with the intention of helping to link together the two fields.

At the outset, basic plasma concepts are presented, including definitions of a plasma, characteristic parameters, magnetic pressure and confinement. This is followed by a brief discussion on plasma kinetic theory, non-equilibrium plasma, and the temperature of moving plasmas. Examples deal with beams in the CERN Intersecting Storage Rings as well as with non-steady beams in cyclic accelerators and microwave tubes. In the final chapters, time-varying systems are considered: waves in free space and the effect of cylinder bounds, wave motion in cold stationary plasmas, and waves in plasmas with well-defined streams. The treatment throughout is informal, with emphasis on the essential physical properties of continuous beams in accelerators and storage rings in relation to the corresponding problems in plasma physics and microwave tubes.
FOREWORD

This report is based on six lectures given in February and March 1974 as part of the CERN Academic Training Programme. The aim was not to give a comprehensive and formal survey, but rather to introduce some of the phenomena and concepts of plasma physics, and then to look at problems of interest to accelerator physicists from a viewpoint that might be taken by a plasma physicist. At the same time, the basic properties of some space-charge flows, which have been studied mainly by microwave tube engineers, have also been included.

The choice of examples tends to reflect very much the lecturer's personal interests over a number of years, and in this sense the treatment is a very biased one.

The notes contain many points not in the original lectures; many of these have been contributed by the editors, whose role has greatly exceeded that of mere reporters. Besides clarifying a number of points they have corrected several errors and introduced new material to put sketchy and disjointed remarks into a more satisfactory context. The author would like to express his appreciation of their work, and acknowledge their contribution to the final report. He also wishes to thank Dr. C.J.H. Watson of the Culham Laboratory for the permission to use Fig. 1.10.

P.S. This postscript is being written 16 months after the lectures were delivered. Some topics are much clearer to me now than at the time of the lectures; I have however resisted the temptation to do more than correct mistakes. This remark applies particularly to Sub-Section 5.3 and much of Section 6 where the coverage is too great, and the particular viewpoint which I was striving to develop does emerge clearly enough. A more complete and careful account of the topics in these sections (together with some of the material from earlier chapters) will appear in my forthcoming book.
# CONTENTS

1. INTRODUCTION: SOME SIMPLE PLASMAS AND PLASMA CONCEPTS
   1.1 What is a plasma?
      1.1.1 Glow discharge
      1.1.2 RF discharge
      1.1.3 Caesium plasma
   1.2 Characteristic parameters of a plasma
      1.2.1 Maxwellian plasma
      1.2.2 Mean free path
      1.2.3 Debye shielding distance
      1.2.4 Plasma frequency
      1.2.5 Transverse electromagnetic waves in a plasma
   1.3 Plasma in a box, sheaths
   1.4 Magnetic pressure and confinement
      1.4.1 Magnetic pressure
      1.4.2 Pinch
   1.5 Concluding remarks

2. PLASMA KINETIC THEORY AND SOME SIMPLE MODELS
   2.1 Plasma kinetics, Vlasov equation, pressure and temperature
   2.2 Some simple plasma models
      2.2.1 Rotating layer of charged particles in a magnetic field
      2.2.2 Rotating coordinates
      2.2.3 Laminar pinch

3. THE TEMPERATURE OF MOVING PLASMAS
   3.1 Temperature, moving frames and effective mass
      3.1.1 Definitions of temperature
      3.1.2 Temperature in relativistic systems
      3.1.3 Longitudinal and transverse energies in moving frames
      3.1.4 Effective mass
      3.1.5 Negative temperatures
   3.2 Temperature in the ISR
      3.2.1 Transverse temperatures
      3.2.2 Longitudinal temperatures

4. SOME MORE BEAMS AND FLOWS
   4.1 Beam with Kapchinskij-Vladimirskij distribution
   4.2 Non-relativistic charged beam in an infinite $B_z$ field
   4.3 Flow in a uniform magnetic field
   4.4 Envelope equation of a beam with self fields and finite emittance
   4.5 Some solutions of the envelope equation
4.6 Non-uniform distribution and non-linear forces
4.7 Is the emittance invariant?
4.8 Entropy, emittance and filamentation
4.9 Application of the Vlasov equation to the determination of self-consistent equilibria
4.10 Concluding remarks on steady-state beams

5. WAVES IN PLASMAS
5.1 Electromagnetic waves in a lossless non-dispersive medium
5.2 Electromagnetic fields in an unbounded plasma
5.3 Instability and growth in plasma streams
   5.3.1 Equations for space-charge wave propagation
   5.3.2 Analysis of longitudinal space-charge waves
   5.3.3 Negative energy carried by slow space-charge waves
   5.3.4 Multiple plasma streams
   5.3.5 Convective and absolute instabilities
5.4 Continuous velocity distribution in plasmas and Landau damping
   5.4.1 Response of a continuum of lossless resonators

6. WAVES ON BEAMS OF FINITE DIAMETER
6.1 Dispersion relation for longitudinal waves on a beam of finite transverse dimensions
6.2 Plasma beam bounded by a resistive cylinder
6.3 Negative mass and longitudinal resistive instabilities of coasting beams in particle accelerators
6.4 Transverse resistive instabilities
6.5 Transverse cyclotron waves
6.6 Concluding remarks

REFERENCES
1. INTRODUCTION: SOME SIMPLE PLASMAS AND PLASMA CONCEPTS

Some of the problems encountered in particle accelerators and storage rings have much in common with those encountered in plasma physics and microwave tubes. To some extent the topics of interest in plasma physics and particle accelerators have converged. Originally, plasmas were considered as hot gases which have become ionized, and particle accelerators were concerned with the motion of particles in specified external electric and magnetic fields, as in electron optics. In recent years, however, plasma physicists have become more interested in collisionless systems in which the velocity distribution is not necessarily Maxwellian, whereas accelerator physicists have found it necessary to study not only collective collisionless effects, but also effects of collisions between particles in the beams of storage rings. An important example of a phenomenon widely discussed in both fields is Landau damping.

Since the region of common interest is approached from different directions, different points of view have developed. This has been very evident in the approaches to some problems, particularly the study of intense relativistic beams, and ring beams or layers such as those in the Electron Ring Accelerator and 'Astron' fusion device.

The object of these lectures is to present an informal review illustrating as far as possible these different viewpoints; we look at some problems familiar to accelerator physicists, and at some others which are closely related. The material covered represents a selection of topics rather than a systematic coverage of the field.

We start with a simple and elementary survey of some typical plasmas; however it is first pertinent to enquire what we mean by a plasma.

1.1 What is a plasma?

Webster's Dictionary gives the following definitions:

1) Green, faintly translucent quartz.
2) a) The fluid part of blood, lymph, or milk as distinguished from suspended material.
   b) The juice that can be expressed from muscles.
3) Protoplasma.
4) An ionized gas containing about equal numbers of positive ions and electrons.

The word plasma is derived from the Greek πλασμα meaning to mould, from which also words like plaster and plastic originate. The use of the word plasma to describe ionized gases goes back to Langmuir who wrote in 1929 1): “The word plasma will be used to designate that portion of an arc type discharge in which the densities of ions and electrons are high but substantially equal. It embraces the whole space not occupied by the sheaths”. Even more basic is the extended entry under πλασμα in Liddell and Scott's Greek Lexicon; among the meanings we find "conterfeit", "forgery"; Langmuir realized the similarity of discharge structures and was looking for a name to describe this state of matter. The equilibrium part of such a discharge acts as a sort of substructure carrying particles of special kinds, like high-velocity electrons, molecules and ions. This reminded Langmuir of the way blood plasma carries around red and white corpuscles and germs, and he took the word plasma over to describe ionized gases. For a long time medical journals were asking for reprints of articles written by Langmuir's group. The engineering world treated it first as a GE trade name (Langmuir worked at a General Electric laboratory). But later everybody started to use the word plasma for highly ionized gases.
Different books give quite different definitions of a plasma. Some state specifically that it has to be quasi-neutral, others do not find this to be necessary. Collective behaviour or a large number of particles in a Debye sphere are considered by some books to be essential features of a plasma. Another one calls a plasma: "Any state of matter which contains enough free charged particles for the dynamic behaviour to be dominated by electromagnetic forces".

Since it seems to be difficult to give a precise definition of a plasma it might be more instructive to just have a look at some old-fashioned plasmas which are widely known:

1.1.1 **Glow discharge**

The glow discharge consists of a gas-filled tube with a cathode (which can be hot), an anode, and a voltage $V$ between them. Electrons emitted by the cathode ionize the residual gas. An equilibrium between ionizing collisions and diffusion followed by recombinations at the wall is established. This discharge is quite complicated in detail.

![Glow discharge diagram](image)

**Fig. 1.1 Glow discharge**

1.1.2 **RF discharge**

An equilibrium discharge is set up in a gas-filled tube by an RF field. This plasma is also quite complicated because the field configuration is strongly affected by the discharge.

![RF discharge diagram](image)

**Fig. 1.2 RF discharge**

1.1.3 **Caesium plasma**

Caesium has a small ionization potential, which can be smaller than the work function of a hot surface. A Cs atom, which makes contact with such a surface, has a high probability of becoming ionized. In Cs gas, which is confined by a hot tungsten box, an equilibrium of electrons, ions and neutral atoms is established. This plasma is relatively simple and well understood and will be used later to illustrate the properties of a plasma sheath.
1.2 Characteristic parameters of a plasma

We now look very briefly at some characteristic properties of plasmas which are important for their understanding. In the following, MKSA units will be used.

1.2.1 Maxwellian plasma

We consider first a so-called Maxwellian plasma. This is a plasma which is in a thermodynamic equilibrium and the charges have a Maxwellian velocity distribution function

\[ f = n_0 \left( \frac{m_0}{2\pi kT} \right)^{3/2} \exp \left( -\frac{m_0}{2kT} \left( \frac{v_x^2 + v_y^2 + v_z^2}{2T} \right) \right), \]  

(1.1)

with

- \( m_0 \) = rest mass of the particles,
- \( k \) = Boltzmann's constant,
- \( v_x \) = velocity component measured in frame such that \( \langle v_x \rangle = 0 \),
- \( n_0 \) = number of particles per unit volume,
- \( T \) = temperature.

Most plasmas have more than one component, so that Eq. (1.1) represents a sum over species. For thermodynamic equilibrium \( T \) is the same for all species; in practical plasmas, however, it is quite common for electrons and ions both to have approximately a Maxwellian distribution, but for the electrons to be much hotter (several eV) than the ions (less than 1 eV). Note that 1 eV corresponds to 11,600°K.

The pressure \( p \) of such a system is given by \( p = nkT \), and the kinetic energy per unit volume by \( \frac{3}{2} nkT \). For a relativistic plasma the relation between pressure, energy and temperature is more complicated, as we see later.

1.2.2 Mean free path

For an uncharged gas the concept of "mean free path" between collisions of atoms or molecules is straightforward. When the particles are charged, on the other hand, they have no well-defined "size" and two isolated particles interact whatever their separation.

One characteristic length which can be readily defined is the "Landau length" \( \ell_L \); this is the distance between two particles for which the interaction energy \( q^2/4\pi\varepsilon_0 \ell_L \) is equal to their characteristic kinetic energy \( kT \),

\[ \ell_L = \frac{q^2}{4\pi\varepsilon_0 kT}. \]
At such a distance of approach a particle suffers a substantial deflection. One might expect, therefore, a mean free path of order \(1/(\text{mt}_{\perp}^2)\); more precisely, two equal particles with centre-of-mass velocities \(v\), approaching each other with directions such that, without interaction, their distance of closest approach would be \(b\), suffer a 90° deflection if \(b = q^2/(4\pi \varepsilon_0 kT)\). (The distance \(b\) is the "impact parameter" for 90° collisions.) In a plasma, however, as we see later, any particle interacts with a large number of other particles simultaneously, so that the total scattering is considerably greater, and the mean free path << 1/(mt_{\perp}^2).

A detailed calculation, based on the classical theory of Rutherford scattering in an inverse square law field, shows that the mean free path is reduced by a factor
\[
8 \ln (b_{\text{max}}/b_{\text{min}}),
\]
where \(b_{\text{min}}\) is the impact parameter for a 90° deflection and \(b_{\text{max}}\) is the "screening distance" beyond which particles no longer interact. This distance could be determined by the size of the vessel, or alternatively (and more commonly), by the "Debye shielding distance" \(\lambda_D\), which we derive later. Since \((b_{\text{max}}/b_{\text{min}})\) occurs in a logarithm, and we must average over all particle energies, the values \(b_{\text{max}}\) and \(b_{\text{min}}\) do not need to be specified precisely.

This subject is clearly related to multiple scattering theory, in which the logarithmic term arises in finding the r.m.s. scattering angle. For Rutherford scattering the cross-section \(\sigma(\theta) = \theta^{-4}\) (for \(\theta \ll \pi/2\)), and the mean squared angle is proportional to
\[
\frac{\int \sigma(\theta) \theta^2 2\pi d\theta}{\int \sigma(\theta) 2\pi d\theta} \approx \frac{\ln (\theta_{\text{max}}/\theta_{\text{min}})}{2\theta_{\text{min}}^2}.
\]

For scattering in solids, \(\theta_{\text{min}}\) is determined by the atomic screening distance. In general, it is necessary to check whether the classical or quantum mechanical value of \(\theta_{\text{max}}\) and \(\theta_{\text{min}}\) should be used. For high energies and close distances of approach the value of \(\theta\) given by diffraction \((\theta_{\text{diff}} = b/\lambda)\) may be larger than the classical value. If so, \(\theta_{\text{diff}}\) should be taken. A useful discussion of this field, extended to the calculation of diffusion and relaxation times for a non-Maxwellian distribution, is given in the book by Spitzer²).

1.2.3 Debye shielding distance

The Coulomb field of a single charge in a plasma is shielded by the surrounding charged particles. The Debye shielding distance indicates how far the potential \(\phi\) of this single charge penetrates the plasma. Let us assume a particle of charge \(q\) at the origin \((r = 0)\) in a neutral plasma. The neighbouring electrons and ions will be distributed according to the Boltzmann relation
\[
n_e = n_0 e^{+q/\varepsilon_0 kT} \quad \text{and} \quad n_i = n_0 e^{-q/\varepsilon_0 kT}
\]
(which we prove in Section 4).

The potential \(\phi\) can be calculated using the Poisson equation
\[
\nabla^2 \phi = \frac{q}{\varepsilon_0} (n_i - n_e) = \frac{2q}{\varepsilon_0} n_0 \sinh \left(\frac{q\phi}{kT}\right).
\]

For a practical case, where the potential energy is much smaller than the thermal energy, we can write (Debye-Hückel approximation)
\[ \phi = \frac{q}{4\pi \varepsilon_0 r} \exp \left[ -\left( \frac{2\pi q^2}{\epsilon_0 kT} \right)^{\frac{1}{2}} r \right] = \frac{q}{4\pi \varepsilon_0 r} \exp \left( -\frac{\sqrt{2} r}{\lambda_D} \right), \]

where

\[ \lambda_D = \left( \frac{\epsilon_0 kT}{nq^2} \right)^{\frac{1}{2}} = \text{Debye shielding distance} \]

This potential \( \phi \) of a single charge in a plasma is plotted in Fig. 1.4 against the distance \( r \) and compared with the potential of a charge in free space. The exponential decay removes the logarithmic infinity we found in Section 1.2.2. The shielding distance \( \lambda_D \) can be rewritten using the average distance \( \lambda \) of the particles \( \lambda = n^{-1/3} \) and the classical electron radius \( r_0 \):

\[ r_0 = \frac{q^2}{4\pi \varepsilon_0 m_0 c^2} = 2.82 \times 10^{-15} \text{ m}, \]

\[ \lambda_D = \left( \frac{\epsilon_0 kT}{nq^2} \right)^{\frac{1}{2}} \approx \left( \frac{8\lambda^3}{\pi r_0^3} \right)^{\frac{1}{2}} = 8\beta \left( \frac{\lambda}{4\pi r_0} \right)^{\frac{1}{2}}, \]

where \( \beta c \) is the r.m.s. velocity of the particles.

For a \( \beta \) which is not very close to zero, the Debye shielding distance is much larger than the average distance \( \lambda \) between the particles. The number of particles in the Debye sphere is called the "plasma parameter" \( h \), so that

\[ h = \frac{4}{3} \pi \lambda_D^3 n = \frac{1}{6\pi} \beta \left( \frac{\lambda}{r_0} \right)^{\frac{3}{2}}. \]
For a typical laboratory plasma with \( \beta = 0.02 \) (\( kT = 100 \text{ eV} \)), \( n = 10^{21} \text{ m}^{-3} \), hence \( \lambda = 10^{-7} \text{ m} \); with these numbers we find \( \lambda_D = 3.4 \times 10^{-6} \text{ m} \) and \( h = 1.6 \times 10^5 \).

1.2.4 Plasma frequency

Let us look at a neutral plasma in a box-type volume (Fig. 1.5). For simplicity, we can suppose that the ions are heavy and at rest. If we displace all the electrons by a distance \( x \), a surface charge \( \sigma \) of

\[
\sigma = \pm n q x
\]

is formed on the two opposite sides of the box, which results in an electric field

\[
E = \frac{\sigma}{\varepsilon_0} = \frac{n q x}{\varepsilon_0}
\]

inside the box. The restoring force of this field leads to plasma oscillations for the electrons

\[
\ddot{x} + \frac{n q^2}{m \varepsilon_0} x = 0,
\]

with the plasma frequency

\[
\omega_p = \left( \frac{n q^2}{m \varepsilon_0} \right)^{1/2}.
\]

(1.2)

We note the important fact that the frequency is independent of the size of the box. For the example in the last subsection we get

\[
\omega_p \approx 1.8 \times 10^{12} \text{ sec}^{-1}.
\]

Fig. 1.5 Plasma oscillation

1.2.5 Transverse electromagnetic waves in a plasma

We now look briefly at transverse electromagnetic waves in a neutral plasma. For such a plasma with the electron density \( n_e \) we can write Maxwell's equation:
\[ \hat{\mathbf{v}} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{\mathbf{J}}{c^2}, \]

with \( \mathbf{j} \) being the current density

\[ \mathbf{j} = q n_e \mathbf{v}, \quad \frac{d\mathbf{j}}{dt} = q n_e \mathbf{v} = \frac{n_e q^2 \mathbf{E}}{m}; \]

thus

\[ \hat{\mathbf{v}} \times \mathbf{B} = -\hat{\mathbf{v}} \times (\hat{\mathbf{v}} \times \mathbf{E}) = \nabla \times \mathbf{E} = \frac{\mu_0 n_e q^2 \mathbf{E}}{m} + \frac{\mathbf{J}}{c^2}. \]

For a transverse wave propagating in the x-direction we have:

\[ \frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = \frac{n_e q^2}{\varepsilon_0 m c^2} E_y = \frac{\omega_p^2}{c^2} E_y. \]

With \( E_y = E_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{x})} \) we get the dispersion relation

\[ \omega^2 - \omega_p^2 = c^2 k^2, \]

which is shown in Fig. 1.6.

![Fig. 1.6 Dispersion relation](image)

For \( \omega > \omega_p \), we have propagation of transverse electromagnetic waves with the phase velocity

\[ \mathbf{v} = \frac{\omega}{k} = \frac{\omega}{k} \left(1 - \frac{\omega^2}{\omega_p^2}\right)^{-\frac{1}{2}}. \]

For \( \omega < \omega_p \), there is no propagation, because \( k \) is imaginary, and the wave entering the plasma decays exponentially with \( x \). The skin depth \( \delta_c \) is

\[ \delta_c = \frac{1}{|k|} = c(\omega_p^2 - \omega^2)^{-\frac{1}{2}}. \]

When \( \omega \ll \omega_p \),

\[ \delta_c \approx \left(\frac{\varepsilon_0 m c^2}{n_e q^2}\right)^{-\frac{1}{2}} \approx \frac{\lambda_p}{B}. \]
1.3 Plasma in a box, sheaths

A characteristic feature of many plasmas is the sheath that forms between the body of the plasma and the walls of the containing vessel. Details of this are often complicated; here we illustrate a particularly simple system used as a basis for experimental "Q-machines".

The system consists of two infinite, parallel plates with temperature $T$ and work function $\Phi$ with Cs gas in between. If the ionization potential $V_i$ of the Cs atoms is smaller than the work function $\Phi$, surface ionization occurs; furthermore, electrons are produced by thermionic emission from the tungsten. Between the plates a plasma in thermodynamic equilibrium is formed. Since the electrons and ions are produced in a well-defined way, this plasma is well understood. We are going to study some phenomena close to the plate surface, where a so-called sheath is formed.

A few simple laws enable us to calculate what happens. First, between the plates the plasma is in thermodynamic equilibrium and the densities of ions, electrons and neutrals can be calculated using the "law of mass action". This law describes the equilibrium of a chemical reaction between the molecules $A$, $B$, ... and $M$, $P$, ..., of the form

$$a[A] + b[B] + ... \rightarrow m[M] + p[P] + ...$$

$a$, $b$, $m$, $p$, ... being the integer coefficients of the reaction. The concentrations $[A]$ of the molecules $A$, $[B]$ of the molecules $B$, etc. (in molecules per unit volume) are related by

$$\frac{[A]^a[B]^b ...}{[M]^m[P]^p ...} = f(T),$$

where $f(T)$ is a function of the temperature which can be calculated [see, for example, Fermi, "Thermodynamics"]. We can apply this law of mass action to our equilibrium reaction

$$\text{Cs}^+ + e^- \rightarrow \text{Cs}$$

to get the ion, electron and neutral concentrations inside the plasma

$$\frac{[\text{Cs}^+][e^-]}{[\text{Cs}]} = \frac{n_i n_e}{n_0} = f(T) = K(kT)^\frac{3}{2} \exp \left( -\frac{qV_i}{kT} \right),$$

with

$$K = \left( \frac{2\pi m_e}{\hbar^2} \right)^\frac{3}{2}$$

($m_e = $ electron mass, $\hbar = $ Planck's constant). When applied to ionized gas rather than chemical compounds this law is known as "Saha's equation". Since $f(T)$ is a function of the temperature $T$ only, the expression $n_i n_e / n_0$ has the same value everywhere inside the plasma.

Next we find that the electron density $n_e$ at the plate surface is given by Richardson's law:
\[ n_{e0} = 2k(kT)^{\frac{3}{2}} \exp \left( \frac{q\phi_e}{kT} \right). \]

These two laws allow us now to calculate the densities \( n_i \) and \( n_e \) of the plasma. The neutral density \( n_i \) is the same everywhere and determined by the vapour pressure of Cs. This can be controlled by providing a "cold spot" of appropriate temperature within the vessel containing the plates. For \( n_i \) and \( n_e \) we have two boundary conditions: firstly, far from the walls the plasma is neutral,
\[ n_e = n_i = n_p = \left[ n_{e0} n_{i0} \right]^{\frac{1}{2}} \]
and, secondly, at the plate surface \( n_{e0} \) is given by Richardson's law. The variation of \( n_i \) and \( n_e \) with the distance \( x \) from the wall may be found by solving the Boltzmann relation and the Poisson equation
\[ n_e = n_p \, e^{q\phi_e/kT}, \quad n_i = n_p \, e^{q\phi/kT}. \]

The electrostatic potential \( \phi \) has been chosen such that, far from the walls, \( \phi = 0 \),
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{q}{\varepsilon_0} \left( n_i - n_e \right) = \frac{2qn_p}{\varepsilon_0} \sinh \left( \frac{q\phi}{kT} \right). \]
The integration of this equation gives for the potential \( \phi \):
\[ \frac{q\phi}{4kT} = \tanh^{-1} \left\{ \exp \left[ -\frac{(x + x_0)/\lambda_0}{kT} \right] \right\}, \]
with
\[ \lambda_0 = \left( \frac{\varepsilon_0 kT}{n_p q^2} \right)^{\frac{1}{2}} = \text{Debye shielding distance}. \]
The boundary condition for \( \phi \) at the plate surface is satisfied by the right choice of the integration constant \( x_0 \). To find \( x_0 \), we use the Boltzmann relation quoted above with \( \phi \) set equal to the potential \( V_C \) at the wall. At the wall \( n_e = n_{e0}, n_i = n_{i0} \), and \( n_p = \left[ n_{e0} n_{i0} \right]^{\frac{1}{2}} \), so that
\[ \frac{qV_C}{kT} = \ln \left( \frac{n_{e0}}{n_p} \right) = \frac{1}{2} \ln \left( \frac{n_p}{n_{i0}} \right). \]  \hspace{1cm} (1.3)
The sign of \( V_C \) is positive or negative according to whether the electron or the ion density is higher at the wall. Because of the logarithmic dependence on \( n_{e0}/n_{i0} \), the expression \( qV_C/4kT \) [Eq. (1.3)] is never likely to be large. Near the wall, therefore, the potential varies roughly exponentially with decay length \( \lambda_0/\lambda \); this region of rapid variation of potential and charge density is known as a "sheath"; it is plotted in Fig. 1.7.

Similar sheaths, also with thickness of the order of a Debye length, are formed along the walls in gas discharge plasmas, although they are in general more complicated than the one treated above; in particular, the distribution of the ion velocities is often far from
Maxwellian and is instead directed towards the wall. This is true, for example, in an ordinary discharge tube; all the ions in the sheath move towards the wall, accompanied by an equal current of electrons. At the wall itself, the ions have low velocity and high density, whereas for the electrons the reverse is true.

In this subsection we have studied the behaviour of plasmas confined by material walls; in the next we introduce ideas relevant to "magnetic" confinement.

1.4 Magnetic pressure and confinement

An important concept appropriate to a plasma in a magnetic field is the "magnetic pressure". We look at this now in the context of a Maxwellian plasma, but return later to related ideas in beams. Before introducing magnetic pressure, we look at the physical idea of a pressure in an ordinary gas. This is often though: of in terms of the forces exerted on the walls by molecules which strike them, though collisions are not necessary for a pressure to exist.

The pressure is the force per unit area transmitted across a surface. One can, in principle, measure the pressure in a collisionless medium by inserting a pair of parallel plates and measuring the force between them with a spring (Fig. 1.8). For an isotropic distribution this poses no problems; for a distribution which is not isotropic, however, the meaning of pressure is somewhat more complicated, as we shall see later. In a medium where the pressure is transmitted continuously, one has a pressure gradient which is equal to the force density, as arises, for instance, in the earth's atmosphere for the gravitational force. Even if there were no collisions, such a pressure would evidently still exist.

We now look at the pressure exerted on a plasma by a magnetic field.
1.4.1 Magnetic pressure

In a plasma with the ion density \( n_i \) and the electron density \( n_e \), in a magnetic field \( \vec{B} \) and electric field \( \vec{E} \), one has a pressure gradient for the ions

\[
\nabla P_i = n_i |q| (\vec{E} + (\vec{v}_i \times \vec{B}))
\]

and one for the electrons

\[
\nabla P_e = -n_e |q| (\vec{E} + (\vec{v}_e \times \vec{B}))
\]

with \( \vec{v}_i \) and \( \vec{v}_e \) being the velocities of the ions and electrons. For a neutral plasma, where \( n_i = n_e = n \), the total pressure gradient is

\[
\nabla P = n |q| [(\vec{v}_i - \vec{v}_e) \times \vec{B}] = \vec{j} \times \vec{B},
\]

where \( \vec{j} \) = current density.

In a quasi-stationary field we have

\[
\vec{j} = \frac{1}{\mu_0} (\nabla \times \vec{B}),
\]

which gives

\[
\nabla P = \frac{1}{\mu_0} [(\nabla \times \vec{B}) \times \vec{B}] = \frac{1}{\mu_0} [-\frac{1}{2} \nabla \vec{B}^2 + (\vec{B} \cdot \nabla) \vec{B}].
\]

For many plasma geometries, for example straight cylindrical systems, the term \( (\vec{B} \cdot \nabla) \vec{B} \) vanishes so that

\[
\nabla \left( p + \frac{B^2}{2 \mu_0} \right) = 0 \quad \text{or} \quad p + \frac{1}{2} BH = \text{const.}
\]

In view of this result the magnetic energy density \( \frac{1}{2} BH \) is often called the magnetic pressure.
A quantity often used by plasma physicists in characterizing confinement systems is $\beta$, the ratio of kinetic to magnetic pressure. Obviously in fusion devices one wants "high $\beta$".

1.4.2 Pinch

As a simple example of a system exhibiting magnetic pressure, we study the pinch effect. Consider a cylindrical plasma of radius $R$ with a surface current $I$ (Fig. 1.9). This current produces a magnetic field $B$ which is 0 inside and $B = \mu_0 I / 2\pi r$ outside. The magnetic pressure at a radius just greater than $R$ is

$$p_m = \frac{1}{2} BH = \frac{1}{2} \frac{\mu_0 I^2}{4\pi R^2}$$

and this therefore is equal to the kinetic pressure where $r < R$

$$p = (n_i + n_e) kT = \frac{N_i + N_e}{\pi R^2} kT,$$

with $N$ being the linear particle density $N = n\pi R^2$. For the case of equilibrium between the magnetic and kinetic pressure, one has Bennett's pinch relationship

$$\frac{\mu_0 I^2}{4\pi} = 2(N_e + N_i) kT.$$

We relate this later to beam models.

![Fig. 1.9 Pinch effect](image)

1.5 Concluding remarks

In this section a number of the more important physical characteristics of plasma have been introduced in an elementary and descriptive way. Figure 1.10 shows the wide range of particle density and temperature spanned by plasmas; characteristic types of plasma in the different régimes are indicated. A journey along the S-curve leads one from the hot dense centre of the sun, through cooler outer layers, to the hot but tenuous solar corona, and then, blown by the stellar wind, to the interplanetary space. To the right of the "Saha 50%" line more than half of the atoms in a hydrogen plasma in thermodynamic equilibrium are ionized. The magnetic fields indicated are such that the magnetic pressure is equal to the kinetic pressure implied by the values of $n$ and $T$ for a hydrogen plasma.
Fig. 1.10 Classification of plasma
2. PLASMA KINETIC THEORY AND SOME SIMPLE MODELS

In this section, we look first in a general way at non-equilibrium plasmas, and then discuss two simple models designed to illustrate application of the general concepts of temperature and pressure. Accelerator and plasma viewpoints are compared.

2.1 Plasma kinetics, Vlasov equation, pressure and temperature

After the informal discussion of Section 1, we now look in a more formal way at plasma kinetic theory for a quite general distribution function for the particles. Collisions are sufficiently rare that thermodynamic equilibrium and the associated Maxwell distribution are not achieved. An approach which will be briefly outlined for the very special case of zero magnetic field, is the so-called "BBGKY theory", named after Bogoliubov, Born, Green, Kirkwood and Yvon. The treatment outlined in this section follows closely that of Watson. A full treatment, including magnetic field, is very lengthy. For further details see Chapter 12 of Ref. 4, or Montgomery and Tidman.

For most purposes one can take the classical Liouville theorem for a statistical description of a plasma. This theorem describes the evolution in time of the probability that the plasma is in a given state. Such a state at time \( t \) is completely defined by the set of positions and velocities \( X_i = (\tilde{x}_i, \tilde{v}_i) \) of all \( N \) particles \( (i = 1, 2, \ldots, N) \). In practice one does not know the initial coordinate-velocity pairs of the \( 6N \) dimensional phase space \((\Gamma\)-space\).

However, it may be possible to specify the probability \( F_N(\tilde{X})d\tilde{X} \) that the point in phase space representing the plasma lies in a small cell of hypervolume \( d\tilde{X} = d\tilde{x}_1 d\tilde{x}_2 \ldots d\tilde{x}_N \) around the point \( \tilde{X} = (X_1, X_2, \ldots, X_N) \). For simplicity, let us take the energy of interaction \( \psi(x_i) \) of the \( i \)th particle with any external field to be derivable from a scalar potential; then the Hamiltonian of the system may be written

\[
H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i \neq j} \phi_{ij}(\tilde{x}_i - \tilde{x}_j) + \psi(\tilde{x}_i),
\]

where \( p_i \) is the canonical momentum which, in the absence of magnetic fields, is equal to the mechanical momentum \( p_i \), and \( \phi_{ij} \) stands for the interaction energy of a pair of particles. For a plasma, where all components have charge \( q \), the interaction energy is given by the Coulomb potential

\[
|\phi_{ij}| = \frac{q^2}{|\tilde{x}_i - \tilde{x}_j|}.
\]

By Liouville's theorem, the probability distribution develops in time according to

\[
\frac{\partial F_N}{\partial t} + \{F_N, H_N\} = 0,
\]

(2.1)

where \( \{F_N, H_N\} \) signifies the Poisson bracket of \( F_N \) and \( H_N \), equal to

\[
\sum_{i=1}^{N} \left( \frac{\partial F_N}{\partial x_i} \frac{\partial H_N}{\partial p_i} - \frac{\partial F_N}{\partial p_i} \frac{\partial H_N}{\partial x_i} \right), \text{ or } \sum_{i=1}^{N} \left( \frac{\partial F_N}{\partial x_i} \frac{\partial}{\partial p_i} \tilde{x}_i + \frac{\partial F_N}{\partial p_i} \frac{\partial}{\partial \tilde{p}_i} \tilde{p}_i \right).
\]
For a system in thermodynamic equilibrium, \( F_N \) equals just the Boltzmann probability distribution \( e^{-\beta H/kT} \), but in plasma physics, as mentioned above, it is often necessary to seek non-equilibrium solutions of Eq. (2.1). The standard procedure involves constructing a hierarchy of equations by integrating \( F_N \) with respect to variables upon which it depends. Thus one can systematically eliminate the information which is not useful, for example the probability that any pair of particles have a given distance, any three particles a given mutual configuration, etc. By integrating with respect to one position-velocity pair \( x_1 \), a function \( F_{N-1} \) is obtained; an integration with respect to a second and a third pair yields the functions \( F_{N-2}, F_{N-3}, \) etc. Finally one arrives at the probability \( F_1(x_1,t) \) of finding a single particle at the position \( x_1 \) in the six-dimensional phase space (\( \mu \)-space). The function \( F_1 \) contains no information about any correlations between the particles. However, it contains information of practical interest and one would like to have an equation like (2.1), but only for the "one-particle distribution function" \( F_1(x_1,t) \).

In principle, it is not possible to derive an exact equation for \( F_1 \) without coupling to any distribution function of higher order. This is seen by integrating Liouville's equation over all variables except \( x_1 \). Then an equation

\[
\frac{\partial F}{\partial t} + (H_1, F_1) + \int (H_2, F_2) \, d^3x_2 = 0
\]

is obtained, where \( H_1 = (p_1^2/2m) + \psi(\hat{X}_1) \) and \( H_2 = H_1 + (p_2^2/2m) + \psi(\hat{X}_2) + \Phi_{12}(\hat{X}_1 - \hat{X}_2) \), in which \( F_1 \) is coupled to \( F_2 \). Correspondingly, equations which couple \( F_2 \) to \( F_3 \), \( F_3 \) to \( F_4 \), ..., and \( F_{N-1} \) to \( F_N \) can be found and, without approximation, we have just a hierarchy of equations completely equivalent to Eq. (2.1) both in physical content and mathematical intractability.

Fortunately, convenient approximations may be found for hot plasmas with low density, where the quantity

\[
\varepsilon = \frac{\text{interaction potential due to other particles}}{\text{kinetic energy of a typical particle}}
\]

is sufficiently small. In the limit in which there is no correlation between the particles (\( \varepsilon = 0 \)), the function \( F_k \) would be equal to \( \Pi_{i=1}^{K} F_1(x_i,t) \). For small \( \varepsilon \) we expect approximations like

\[
\begin{align*}
F_2(x_1,x_2,t) &= \prod_{i=1}^{2} F_1(x_i,t) + \varepsilon G(x_1,x_2,t) \\
F_3(x_1,x_2,x_3,t) &= \prod_{i=1}^{3} F_1(x_i,t) + \text{permutations of } \varepsilon F_1(x_i,t) \, G(x_2,x_3,x_4,t) ,
\end{align*}
\]

(2.2)

neglecting higher-order terms of \( \varepsilon \). The introduction of the lowest approximation (\( \varepsilon = 0 \)) into the equations of the BBGKY hierarchy results in an equation for \( f(\hat{x},\hat{p},t) = F_1(x_1,t) \):

\[
\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x_1} + \frac{q}{m} \left( E^\text{ext} + E^\text{in} \right) \frac{\partial f}{\partial v_1} = 0 ,
\]

(2.3a)
where we set

\[ E_{\text{ext}}^{\text{ext}} = -\frac{1}{q} \frac{\partial \psi}{\partial x_1} \]

and

\[ E_{\text{sc}}^{\text{ext}} = -\frac{1}{q} \frac{\partial}{\partial x_1} \int f(x_2, t) \delta(x - x_2) \, dx_2 \]

In this approximation \( f \) is determined by the external field \( E_{\text{ext}} \) and by the field \( E_{\text{sc}}^{\text{ext}} \), which is usually called the self-consistent field as it is itself determined by \( f \).

If Eqs. (2.2) are introduced into the BBGKY hierarchy of equations, then the first two equations are coupled with each other but decoupled from the remainder. The lowest of them equals Eq. (2.3a), but has, on the right-hand side, the term \(-e\langle H_2, G \rangle \, d^3X_2 \), in which the unknown function \( G \) can be found from the second equation. Equation (2.3a) is normally called the Vlasov equation or collisionless Boltzmann equation, as it does not contain the "collision term" of the next approximation, which is known as the Balescu-Lenard equation.

Equation (2.3a) is the non-relativistic Vlasov equation in the absence of magnetic field. There are various other forms, depending on what kind of plasmas are considered. Taking into account relativistic variations between force and acceleration, and the existence of a magnetic field, then the appropriate form can be shown to be

\[ \frac{\partial f}{\partial t} + \frac{p_1 c}{\sqrt{(\vec{p}_1^2 + m_1^2c^2)^{3/2}}} \frac{\partial f}{\partial x_1} + q \left[ E_1 + \frac{\left( p c \times \vec{B}_1 \right)}{\left( \vec{p}^2 + m_1^2c^2 \right)^{3/2}} \right] \frac{\partial f}{\partial p_1} = 0 \],

where \( E_1 \) and \( B_1 \) refer to the sum of the external fields and self fields.

Unfortunately, the Vlasov equation is still a partial differential equation for a function which contains still too much information for many purposes. A further simplification is obtained by multiplying Eq. (2.3a) by some power of the particle velocity \( \vec{v} \) and integrating with respect to \( \vec{v} \). Thus the Vlasov equation can be expressed in terms of average quantities like charge and current density and the plasma pressure tensor:

\[ \rho(\vec{x}, t) = q n(\vec{x}, t) = q \int f(\vec{x}, \vec{v}, t) \, d^3v \] \hspace{1cm} (2.4a)

\[ \vec{J}(\vec{x}, t) = q \langle \vec{v} \rangle = q \int \vec{v} f(\vec{x}, \vec{v}, t) \, d^3v \] \hspace{1cm} (2.4b)

\[ p_{ik} = m \int (v_i - \langle v_i \rangle)(v_k - \langle v_k \rangle) f(\vec{x}, \vec{v}, t) \, d^3v \] \hspace{1cm} (2.4c)

Integration of the Vlasov equation with respect to \( \vec{v} \) then yields the continuity equation

\[ \frac{\partial \rho}{\partial t} + \vec{J}^j \vec{v}_j = 0 \]

(2.5)

and integration after the multiplication with \( \vec{v} \) results in an equation of motion of the plasma

\[ \frac{a}{\delta t} (n(\vec{v}_i)) + \nabla_k p_{ik} = \rho E_1 + (\vec{J} \times \vec{B}_1) \]

(2.6)
which describes the change in momentum which the plasma experiences because of electric and magnetic self fields. It should be noted that Eqs. (2.5) and (2.6) plus Maxwell's equations do not form a closed set of equations describing the evolution in space and time of \( p, \dot{v}, E \) and \( \dot{B} \) because of the unknown pressure tensor \( p_{ik} \). However, if one multiplies the Vlasov equation by \( \dot{v} \) and integrates over \( \dot{v} \), an energy equation is obtained containing the tensor \( p_{ik} \), but only at expense of the unknown "heat flow tensor"

\[
\dot{Q} = \int \left( \dot{v} - \langle \dot{v} \rangle \right) \left( \dot{v} - \langle \dot{v} \rangle \right)^2 \sigma(x, \dot{v}, t) \, d\dot{v}.
\]

We may conclude already that continuation of the game of "forming moments of the Vlasov equation" by multiplying it by a successively increasing power of \( \dot{v} \) and integrating with respect to \( \dot{v} \) never closes the set of equations, as at each step a new independent variable will appear.

Fortunately in some cases, where the plasma is assumably close to thermodynamic equilibrium, hence \( f = \exp \left( \frac{1}{2} mv^2 / kT \right) \), the plasma pressure tensor \( p_{ik} \) is approximately scalar and related to the density \( n(x, t) \) by

\[
p_{ik} = p_{ik} \delta_{ik} = nkT.
\]

Equation (2.7) is a very useful approximation as it allows, through the vanishing off-diagonal elements of the tensor \( p_{ik} \), the introduction of temperature in a particular direction

\[
p_{ii} = nkT = nmv_i^2.
\]

For a system in complete thermodynamic equilibrium the temperature is the same in all directions, so that \( p = \frac{1}{2} nmv^2 \). With the last relationship the hierarchy of moment equations can be closed, and a consistent set of equations is available to attack many of the problems in plasma physics.

In this subsection we have seen that starting from the very general Liouville theorem and introducing several approximations, familiar relations such as Eqs. (2.5) and (2.6) are obtained. Sometimes a good understanding of simple plasma models already is possible just by beginning with these equations, as we shall do in the next subsections. However, for deeper insight into bulk plasma behaviour, the closed set of "fluid equations", including the Vlasov equation (an example is treated in Subsection 6.1) or the Balescu-Lenard equation or higher-order relations, may be appropriate.

2.2 Some simple plasma models

2.2.1 Rotating layer of charged particles in a magnetic field

The first example is not strictly a plasma, since the particles do not interact. Furthermore, it is certainly of no practical interest. It does, however, exhibit pressure and temperature, and allow contrasting viewpoints to be compared. The discussion is intended to bring out the features in a direct and simple way. We consider a layer with a density so low that there is no interaction between particles, and self fields are not important. Let us define the following characteristics:
i) Cylindrical symmetry.

ii) Uniform magnetic field $B_z$.

iii) No motion in z-direction.

iv) All particles have the non-relativistic energy $\frac{1}{2}m_0\beta^2c^2$.

v) All orbits touch two cylinders of radius $R + a$, $R - a$, where $R \gg a$ (see Fig. 2.1).

![Fig. 2.1 Model of low-density layer](image)

Implied in these conditions is that all particles have the same canonical angular momentum $P_\theta = p_\theta + qA_\theta r = p_\theta + \frac{1}{2}qaB_\theta r^2$, where $p_\theta = p_r \cos \theta$ is the mechanical angular momentum. The equilibrium between centrifugal force and Lorentz force acting on a particle yields

$$B_z = \frac{-p_\theta}{qR}.$$

Hence

$$p_\theta = \frac{p_\theta}{2R} \left( R^2 + r^2 - a^2 \right), \quad \frac{p_\theta r^2}{2R} = \frac{m_0\beta cR}{2} \left( 1 - \frac{a^2}{R^2} \right).$$

a) Description in terms of accelerator physics

To describe the model in the terminology of an accelerator physicist, we first identify an equilibrium orbit at radius $r = R$, where

$$R = \frac{8m_0c}{qB_\theta} = \frac{\beta c}{\omega_c}.$$

Furthermore, we note that the particles are all making a betatron oscillation of amplitude $a$, with $Q = 1$, where $Q$ is defined as the ratio of the betatron frequency to the orbital (or cyclotron) frequency $\omega_c$. In $x,x'$-space, where $x = r - R$ and $x' = dx/d(\theta_R)$, the beam at a fixed $R$ is represented by the boundary of an ellipse of semi-axes $a$ and $a/R$. The "emittance" is therefore $xx' = a^2/R$ (Fig. 2.2).

b) Fluid description

A plasma physicist calculates for the model a different set of parameters. He first notices that the density is not uniform; since the orbits are of the form
The density at a given value of \( x \) is inversely proportional to \( \frac{dx}{d\theta} \) for the orbits at that radius. Now \( \frac{dx}{d\theta} = (a^2 - x^2)^{1/2} \), and for a ring with \( N \) particles per unit length in the z-direction

\[
n(x) = \frac{N}{2\pi R(a^2 - x^2)^{3/2}}
\]

is found.

![Fig. 2.2 Emittance diagram for the layer. For a layer in which all orbits extend to \( x = a \), only the surface represents the beam.](image)

The azimuthal velocity of particles, at a given radius, is, to first order, \( \beta c \); this represents the fluid velocity there, and there is no azimuthal pressure. In the radial direction, however, there is a finite pressure, which is a maximum at \( x = 0 \) and decreasing to zero at \( x = a \). It is readily seen, by considering the orbits (Fig. 2.1), that, at a given radius \( R + x \), half the particles are moving outwards and the other half inwards with normalized velocity

\[
\frac{\beta_r}{\beta} = \pm \frac{\left( a^2 - x^2 \right)^{1/2}}{R}.
\]

Noting that \( \beta \) is the same for all the particles, we obtain with the equations for \( n(x) \) and \( \beta_r(x) \) for the only component of the pressure tensor \( P_{11} \)

\[
P_{rr} = n m c^2 \beta_r^2 = \frac{N m c^2 \beta^2}{2\pi R^3} \left( a^2 - x^2 \right)^{3/2}.
\]

The maximum pressure, at \( x = 0 \), is proportional to the width of the ring. Now the radial temperature may be found using Eq. (2.8) as

\[
kT = \frac{P_{rr}}{n} = m c^2 \beta \left( \frac{a^2 - x^2}{R^2} \right).
\]

Thus the ring is hot at the centre and cold at the edges (see Fig. 2.3).
The existence of a radial pressure gradient implies a radial force on a fluid element. The azimuthal velocity of a fluid element is independent of radius, giving rise to shear; the angular velocity varies as 1/r. It is readily verified that the pressure gradient term accounts for the difference between the \( \mathbf{j} \times \mathbf{B} \) and centrifugal forces on such an element. At \( x = 0 \) the latter two forces balance and the pressure gradient is zero. Later (Subsection 4.9) the layer is analysed in a more formal way from the distribution function, which is of the form \( f = \delta(H - H_0)\delta(P_0 - P_0) \), where \( H \) and \( P_0 \) represent the kinetic energy and canonical angular momentum.

This model can be extended rather easily to include self fields, although to maintain simplicity it is desirable to modify the distribution function in such a way that the charge and current density in the layer are uniform. This can be done by postulating a spread of orbit centres, rather than confining them to the circumference of a circle.

\[ n(x) \]

\[ P_{rr} \sim \beta_r(x) \]

\[ kT \]

\[ R-a \quad R+a \]

Fig. 2.3 Distribution functions of particle density \( n(x) \), radial pressure \( P_{rr}(x) \) and temperature \( kT(x) \).

3.2.2 Rotating coordinates

It is often convenient for non-relativistic systems to move into rotating coordinates; very often this simplifies the physical picture, or shows the equivalence of apparently different systems. It is appropriate to discuss this in connection with our simple model. To make the interpretation as simple as possible, we restrict our considerations to non-relativistic situations (relativistic rotating systems are notoriously complicated!).
Suppose a magnetic induction $B_z$ to be perpendicular to an azimuthally symmetric system. Then the radial and azimuthal forces on a moving particle in the laboratory frame are

$$F_{rt} = m_q \left( r - r^2 t \right) = q B_z r \theta_t$$

$$F_{e\theta} = m_q \left( r^2 \theta_t + 2r \theta_t \right) = -q B_z t \ .$$

The transformation into a system rotating with angular velocity $\omega_t$, $\theta_t = \theta_m + \omega_t t$, using the cyclotron frequency already defined in Eq. (2.9), yields

$$F_{rm} = -m_q r \omega_t (\omega_c - \omega_t) - m_q r \theta_m (\omega_c - 2\omega_t)$$

$$F_{e\theta} = m_q \left( \omega_c - 2\omega_t \right) .$$

We can interpret the first and the second term of the radial force as forces due to a new electric field and to a modified magnetic field, respectively, which are seen from the moving frame. The appearance of the term corresponding to an electric field becomes obvious from the transformation of the fields into the moving frame:

$$B_{zm} = \left( 1 - \frac{2 \omega_t}{\omega_c} \right) B_{zt}$$

$$E_{rm} = r \omega_t \left( 1 - \frac{\omega_t}{\omega_c} \right) B_{zt} .$$

(2.10)

The radial electric field is maximum and the magnetic field becomes zero for half the cyclotron frequency, the so-called Larmor frequency $\omega_L = \omega_c / 2$. This fact is used in electron optics to decouple the equations of motion (see Fig. 2.4).

![Fig. 2.4 Electric and magnetic fields in the rotating frame](attachment:image)

Looking at our ring model in the Larmor frame (rotating with Larmor frequency), we find from Eq. (2.10) an electric field which increases with the radius of the orbits:
The $n$-value, known from betatron theory and defined as the negative and normalized radial gradient of the electric field

$$E_{rn} = \frac{ru_m}{2} B_{zm}.$$ 

equals $-1$ and leads to a number of betatron revolutions around the mean orbit of

$$Q = \sqrt{3-n} = 2.$$ 

Figure 2.5 shows the elliptical trajectories of particles moving in the Larmor frame. This picture is completely equivalent to the circles ($Q = 1$) of the laboratory frame.

In the cyclotron frame, at $\omega_f = \omega_c$, the electric field becomes zero as it is in the laboratory frame, but now the magnetic induction is reversed: $E_{zm} = 0$, $B_{zm} = \vec{B}_z$. As a consequence, in this frame the particles circulate backwards with the same frequency as in the original frame (Fig. 2.6). It is interesting to note that the transformations do not change the radial velocity and hence the temperature remains everywhere the same:

$$kT_{rn} = kT_{r}.$$ 

---

**Fig. 2.5** Rotating layer in the Larmor frame  
**Fig. 2.6** Rotating layer in the cyclotron frame

### 2.2.3 Laminar pinch

This model again illustrates the application of temperature and pressure to a simple beam model. Consider a neutral background of stationary ions of constant density in a cylinder of radius $a$ (Fig. 2.7). Assume electrons spiralling in the $z$-direction such that its centrifugal force is just balanced by the force due to the induced self field:

$$m_e \ddot{\vec{r}} = q(\vec{\omega} \times \vec{B})$$

or

$$\frac{m_e \omega_e^2 c^2}{r} = q\beta_e c B_0.$$  

(2.11)
Using Ampère's law

\[ B_0 = \frac{\mu_0}{4\pi} \frac{I_0}{r} \quad \text{and} \quad I_z = q n r^2 \beta_c, \]

with \( n = N / a^2 \), where \( N \) = number of electrons per unit length of the column, from Eq. (2.11) it follows that

\[ v = \frac{N q^2}{4 \pi e_0 m_e c^2} = \frac{\beta^2_c a^2}{2 \beta^2_c r^2}. \]

The parameter \( \nu = N r_0 \) (\( r_0 \) = classical electron radius) is named after Budker, who first discussed this quantity which is of importance in accelerator theory. By averaging \( \beta^2_0 = \left[ \left( r / a \right) (\hat{\beta}_0)_{r=a} \right]^2 \) over the number of particles, the relation above can be written as

\[ \nu = \langle \beta^2 \rangle \beta^2_c. \quad (2.12) \]

As we have a purely laminar flow in this highly artificial fluid description, pressure and temperature are zero everywhere. But if it is modified slightly, such that one half of the electrons flow clockwise and the other half anticlockwise, the fluid element is stationary. So a pressure gradient is obtained, which just balances the centrifugal force. This defines a temperature which is zero at \( r = 0 \) and maximum at \( r = a \), or, in other words, the beam is cold in the middle and hot at the edge. Defining an average temperature by

\[ kT = mc^2 \langle \beta^2 \rangle \]

relation (2.12), multiplied at both sides by the Budker parameter \( \nu \), leads to

\[ \frac{\mu_0}{4\pi} I^2 = N kT. \quad (2.13) \]

This is essentially the same as Bennett's pinch relationship we had in the first section. The factor 2 difference on the right-hand side arises because we only have a one-dimensional temperature in this example. If we write the right-hand side in terms of velocity it becomes \( \mu v^2 \) in both cases. So the results of our model are in full agreement with models assuming Maxwell distributions for both electrons and ions.
3. THE TEMPERATURE OF MOVING PLASMAS

3.1 Temperature, moving frames and effective mass

We now look again at the idea of temperature, including, this time, relativistic situations and moving frames. We inquire into the meaning of "temperature" as applied to the ISR.

3.1.1 Definitions of temperature

There is a classical, thermodynamic definition, which assumes that the idea of entropy has been developed first. The inverse temperature is given as the change of the entropy S with the internal energy:

$$\frac{1}{T} = \frac{\partial S}{\partial U}.$$  \hspace{1cm} (3.1)  

This definition is very general and applies to all systems in thermodynamic equilibrium.

A further definition appropriate to non-relativistic situations can be obtained from the distribution function f. The pressure tensor [Eq. (2.4c)] in a moving frame is

$$p_{ij} = m \int v_i v_j f(\vec{v}) \, d^3v,$$

where $v_i = v_x, v_y, v_z$ is the velocity of a particle with respect to the mean velocity of all particles. The diagonal terms $p_{ii}$ of this pressure tensor are related to temperature by Eq. (2.8):

$$p_{ii} = n k T \quad \text{($p_{ii}$ is not summed over i),}$$

with n being the number of particles per unit volume.

Finally, we can define temperature as a rough average kinetic energy per particle with respect to a moving frame. This definition is useful for calculating the temperature of a beam. In beams, the velocity distributions in the longitudinal and transverse directions are often different. In any direction

$$\frac{1}{2} k T_i \approx \frac{1}{2} m_0 \langle \beta_i^2 \rangle c^2;$$

in the longitudinal direction

$$\frac{1}{2} k T_i \approx \frac{1}{2} m_0 \langle \beta_i^2 \rangle c^2;$$

but in the transverse direction, if $\langle \beta_x^2 \rangle = \langle \beta_y^2 \rangle$ we have

$$\langle \beta_i^2 \rangle = \frac{1}{2} \langle \beta_i^2 \rangle,$$

$$\frac{1}{2} k T_i \approx \frac{1}{2} m_0 \langle \beta_i^2 \rangle c^2 = \frac{1}{4} m_0 \langle \beta_i^2 \rangle c^2,$$

so that

$$k T_i \approx \frac{1}{2} m_0 \langle \beta_i^2 \rangle c^2.$$
3.1.2 Temperature in relativistic systems

A non-relativistic plasma in thermodynamic equilibrium has a Maxwellian distribution function:

\[ f(v) = \text{const} \ e^{-m_v v^2/2kT} = \text{const} \ e^{-E_k/kT} . \]

In the relativistic case, the kinetic energy \( E_k \) is given by

\[ E_k = (\gamma - 1)m_0c^2 \]

and we have a Gibbs distribution

\[ f(v) = \text{const} \ e^{-(\gamma - 1)m_0c^2/kT} . \]

We still can give a pressure tensor, which is defined in terms of momentum flux

\[ p_{ij} = \int \left( p_i - \langle p_i \rangle \right) \left( v_j - \langle v_j \rangle \right) f(\mathbf{x}, \mathbf{v}, t) \, d^3v . \]

Using \( \beta_i = v_i/c \) we can rewrite this

\[ p_{ij} = m_0c^2 \int \left( \beta_i \gamma - \langle \beta_i \rangle \gamma \right) \left( \beta_j - \langle \beta_j \rangle \right) f(\mathbf{x}, \mathbf{v}, t) \, d^3v . \]

However, because of the relativistic relation between momentum and energy, the momentum flux is no longer simply proportional to the kinetic energy and we can no longer connect pressure and temperature in the usual way

\[ p_{ii} \propto nkT . \]

A more detailed treatment of relativistic temperatures, using a four-dimensional formalism, is given by Landau and Lifshitz\(^8\).

3.1.3 Longitudinal and transverse energies in moving frames

We consider now a paraxial, relativistic beam of particles. The average, total longitudinal energy of a particle seen in the laboratory frame \( S \) is:

\[ E = m_0c^2\gamma . \]

Its longitudinal velocity is

\[ v_\parallel = c\beta_\parallel \]

and its longitudinal momentum

\[ p_\parallel = m_0c\beta_\parallel \gamma . \]

Another particle may have a longitudinal energy which differs from the average by

\[ \Delta E_\parallel = m_0c^2\Delta \gamma . \]

Its velocity difference in units of \( c \) is

\[ \Delta \beta_\parallel = \frac{\Delta \gamma}{\beta_\parallel \gamma^3} . \]
The difference in momentum is

\[ \Delta p_y = m_0 c \Delta \beta \gamma \]

and

\[ \frac{\Delta p_y}{p_y} = \Delta \gamma \frac{\Delta \beta}{\beta \gamma} . \]

We make now a Lorentz transformation into a frame \( S' \), which moves with the average velocity of the particles. The velocity (in units of \( c \)) in this frame is

\[ \Delta \beta_y' = \beta_y' = \frac{\Delta \beta_y}{1 - \beta_y (\beta + \Delta \beta_y)} = \frac{\gamma \Delta \beta_y}{\gamma - \Delta \gamma} \]

\[ \beta_y' = \gamma \Delta \beta_y \approx \frac{\Delta \gamma}{\beta \gamma} . \]

In most practical cases \( \Delta \gamma / \gamma \) is much smaller than 1 and the longitudinal motion with respect to the moving frame can be treated non-relativistically. We assume now \( \beta_y' \ll 1 \) and get for the momentum in the moving frame

\[ \Delta p_y' = p_y' = m_0 c \beta_y' = m_0 c \frac{\Delta \gamma}{\beta \gamma} \]

and the longitudinal kinetic energy

\[ \Delta E_y' = E_y' = \frac{1}{2} m_0 c^2 \beta_y'^2 = \frac{1}{2} m_0 c^2 \left( \frac{\Delta \gamma}{\beta \gamma} \right)^2 . \]

We consider now the transverse motion, which for a paraxial beam is much smaller than the longitudinal motion:

\[ \beta_{\perp} \ll \beta_y \approx 1 . \]

In this approximation the transverse momentum is

\[ p_{\perp} = m_0 c \gamma \beta_{\perp} \]

and the kinetic energy

\[ E_{\perp} = \frac{1}{2} m_0 c^2 \gamma \beta_{\perp}^2 . \]

In the moving frame \( S' \) the transverse momentum is the same

\[ p_{\perp}' = p_{\perp} \]

and the transverse velocity in units of \( c \) becomes

\[ \beta_{\perp}' = \gamma \beta_{\perp} . \]

In many practical cases also

\[ \beta_{\perp}' \ll 1 \]
and we can treat the transverse motion in the moving frame non-relativistically. We then get for the kinetic energy

\[ E'_\perp = \frac{1}{2} m_c c^2 \beta^2_\perp = \frac{1}{2} m_c c^2 \gamma^2 \beta^2_\perp = \gamma E_\perp. \]

It is interesting to write down the ratio between the transverse and longitudinal energy in the moving frame

\[ \frac{E'_\perp}{E'_\parallel} = \frac{\gamma \beta^2_\perp}{(\Delta \gamma)^2}. \]

For a beam in a circular accelerator of radius \( R \), the transverse motion is mainly given by the betatron oscillations. A particle with a horizontal betatron oscillation amplitude \( x_0 \) crosses the beam axis with an angle

\[ \theta = \frac{\beta_\perp}{R} \approx \beta_\perp \approx x_0 \frac{Q}{R}. \]

Here \( Q \) is the number of betatron oscillations per revolution, and the ratio \( R/Q \) is an approximation for the average amplitude function. In many cases the beam width \( x_0 \) due to betatron oscillations is comparable to the width due to momentum spread

\[ x_0 \approx \alpha_p \frac{\Delta p}{p_0}. \]

Here \( \alpha_p \) is the average dispersion function which itself can be approximated by

\[ \alpha_p \approx \frac{R}{Q^2}. \]

With these assumptions we get now

\[ \beta_\perp \approx \frac{\Delta p}{p} \frac{1}{Q} \approx \frac{\Delta \gamma}{\gamma} \frac{1}{Q}, \]

and the ratio between transverse and longitudinal energy becomes

\[ \frac{E'_\perp}{E'_\parallel} \approx \frac{\gamma^2}{Q^2}. \]

3.1.4 **Effective mass**

In connection with the transformations into a moving frame, it is useful to develop the idea of the effective mass. It is defined as the change of momentum with velocity

\[ m^* = \text{grad} \ \tilde{p}. \]

Here "\( \text{grad} \ \tilde{p} \)" is the gradient with respect to the velocity coordinates. The effective mass is in general a tensor. In a relativistic linear system we get for the effective mass in the longitudinal direction
\[ m_1^* = \frac{\partial \mathbf{p}_1}{\partial \mathbf{v}_1} = m_0 \frac{\partial (\mathbf{A} \cdot \mathbf{r})}{\partial \mathbf{r}} = \gamma m_0 \]

and in the transverse direction

\[ m_1^* = \frac{\partial \mathbf{p}_1}{\partial \mathbf{v}_1} = \gamma m_0 . \]

For accelerators, circular systems are very important. If we use such a system and \( r \) and \( \theta \) as coordinates (with \( \delta = \omega \)), we get for the \( r \)- and \( \theta \)-component of the effective mass

\[ m_r^* = \frac{\partial \mathbf{p}_r}{\partial \mathbf{r}} = \gamma m_0, \quad m_\theta^* = \frac{1}{r} \frac{\partial \mathbf{p}_\theta}{\partial \mathbf{\omega}} . \]

For small deviations from the nominal orbit of radius \( R \) and momentum \( p_\theta \), the derivative \( \partial p_\theta / \partial \omega \) is approximately constant and is usually expressed with the parameter \( \eta \), which is defined as

\[ \eta = -\frac{p_\theta}{\omega} \frac{\partial \omega}{\partial p_\theta} = \frac{1}{\gamma_T} - \frac{1}{\gamma^2} . \]

Here \( \gamma_T \) is \( \gamma \) at transition energy, where \( \omega \) is to first order independent of \( p_\theta \). On the normal orbit we further have

\[ \omega = \frac{p_\theta}{\gamma m_0 R} . \]

We finally get for the \( \theta \)-component of the effective mass

\[ m_\theta^* = -\frac{\gamma m_0}{\eta} = m_0 \frac{\gamma^2}{1 - (\gamma / \gamma_T)^2} . \]

The effective masses \( m^*_r \) in the moving frame are obtained in the same way, but by using the momentum and velocity coordinates in the moving frame. For a rectilinear system we get the obvious result

\[ m_1^* = m_1^* = m_0 . \]

For the circular case we have a rotating frame of reference which is not an inertial system and the exact relativistic transformation is complicated. For small deviations from the nominal orbit, however, we get approximately

\[ m_r^* = m_0 \quad \text{and} \quad m_\theta^* = \frac{m_0}{1 - (\gamma / \gamma_T)^2} . \]

For \( \gamma > \gamma_T \), the last expression can be approximated by

\[ m_\theta^* \approx m_0 \frac{\gamma_T^2}{\gamma^2} . \]
Below transition energy the effective mass $m^*_{0'''}$ (and $m^*_{0'}$) is positive, which means that a larger momentum corresponds to a larger $\omega$. Above transition energy $m^*_{0'''}$ is negative and a larger momentum has a smaller $\omega$. At transition energy itself the effective mass is infinite, which expresses the fact that here the angular velocity $\omega$ is to first order independent of the momentum $p_0$.

The concept of the effective mass can be used to express a difference in momentum in a simple way:

$$\Delta p = m^* \Delta \vec{k} \quad \text{or} \quad \Delta p' = m''' \Delta \vec{k}' ,$$

where $\vec{k}$ stands for any velocity coordinate.

3.1.6 Negative temperatures

We can now use the effective mass in our expression for temperature

$$\frac{1}{2}kT = \frac{1}{2}m^* \nu^2 .$$

When the effective mass is negative we can use the concept of negative temperature. For the $\phi$-component of the temperature in a circular accelerator we get

$$\frac{1}{2}kT_\phi = \frac{1}{2}m^*_e c^2 \left( \frac{\Delta \gamma}{\gamma} \right)^2 = \frac{1}{2} \frac{m_e c^2}{1 - \nu^2 / c^2} \left( \frac{\Delta \gamma}{\gamma} \right)^2 . \quad (3.1)$$

Above transition energy $T_\phi$ will be negative. Negative temperatures are well known in thermodynamics. They usually occur in systems which have a limited number of degrees of freedom, like a set of spins. Negative temperature involves population inversion. The example in Fig. 3.1 represents a system with a limited number of energy levels each particle can occupy.

If all particles are in the lowest level (Fig. 3.1a), we have obviously $T = 0$. By heating the system up slightly, some higher levels will be partially occupied (Fig. 3.1b). If all levels have equal population we have maximum entropy and according to Eq. (3.1) an infinite temperature $T = \infty$. But we can also have a situation where some higher levels are more populated than the lower ones (Fig. 3.1d). The entropy is now smaller than in the previous case, but the internal energy is larger and we have a negative temperature. This situation is actually "hotter", because it can give energy away and turn into the previous case shown.
in Fig. 3.1c. If only the highest level is populated the temperature is $T = -0$. In an accelerator the longitudinal temperature is negative above transition energy. If there is a mechanism which couples longitudinal motion into transverse motion, like intra-beam scattering, the relative longitudinal motion will always feed energy into transverse motion. There will be a diffusion going on forever. Due to the negative mass we will never get equipartition of the energy between the modes. Below transition energy, however, an equilibrium distribution will be reached and longitudinal and transverse temperatures will be the same. These effects have been calculated in detail by Piwinski\(^3\).

3.2 Temperature in the ISR

We apply now the concept of temperature, just developed, to the beam in the ISR.

3.2.1 Transverse temperatures

The horizontal and vertical emittances of a single pulse of $p = 22.5$ GeV/c in the ISR are about

$$
e_x \approx 2 \times 10^{-6} \text{ rad m, } \quad e_y \approx \frac{1}{2} 10^{-4} \text{ rad m}.
$$

Taking the average horizontal and vertical amplitude functions (17.5 m and 25.4 m) of the ISR and assuming an r.m.s. betatron oscillation amplitude which is about half of the maximum amplitude, we get:

$$
\langle x'^2 \rangle^{\frac{1}{2}} = 3.0 \times 10^{-3} \text{ m, } \quad \langle x'^2 \rangle^{\frac{1}{2}} = 1.7 \times 10^{-4},
$$

$$
\langle y'^2 \rangle^{\frac{1}{2}} = 1.8 \times 10^{-3} \text{ m, } \quad \langle y'^2 \rangle^{\frac{1}{2}} = 7.0 \times 10^{-5}.
$$

Here $x'$ and $y'$ are the angles between the particle trajectory and the beam axis $z$

$$
x' = \frac{3x}{\beta z} = \frac{\beta x}{\beta z} \approx \beta_x, \quad y' = \frac{3y}{\beta z} = \frac{\beta y}{\beta z} \approx \beta_y.
$$

The transverse r.m.s. velocities in units of $c$ are, in the moving frame,

$$
\langle v_x'^2 \rangle^{\frac{1}{2}} = \gamma \langle v_x'^2 \rangle^{\frac{1}{2}} = 4 \times 10^{-3}, \quad \langle v_y'^2 \rangle^{\frac{1}{2}} = \gamma \langle v_y'^2 \rangle^{\frac{1}{2}} = 1.7 \times 10^{-3}.
$$

These velocities are much smaller than the speed of light; a non-relativistic treatment is therefore justified. The temperatures can now be determined with

$$
\frac{1}{2}kT_x = \frac{1}{2}m_c c^2 \langle v_x'^2 \rangle.
$$

Therefore

$$
T_x \approx 170 \times 10^6 \text{ K, } \quad T_y \approx 30 \times 10^6 \text{ K}.
$$
3.2.2 Longitudinal temperatures

A single pulse of $p = 22.5$ GeV/c has, after de-bunching, a total momentum spread of

$$\frac{\Delta p}{p} \approx 7 \times 10^{-5}$$

and the r.m.s. deviation from the nominal value is

$$\frac{\langle \Delta p^2 \rangle^{1/2}}{p} \approx 1.8 \times 10^{-5}.$$ We calculate now first the temperature of a "straightened out" ISR, neglecting curvature. The temperature is then determined by

$$\frac{1}{2} k T_s = E_s' = \frac{1}{2} m_c c^2 \langle \delta t_s'^2 \rangle \approx \frac{1}{2} m_c c^2 \frac{\langle \Delta p^2 \rangle}{p^2},$$

which gives

$$T_s \approx 3500^\circ K \quad \text{for a single pulse}.$$ If we stack 80 pulses in the ISR we obtain a beam with a total momentum spread $\Delta p/p \approx 5.6 \times 10^{-3}$. Since the average dispersion function $\sigma_p$ is about 2 m, this beam has a width due to momentum spread of $\approx 11$ mm, which is comparable to the width due to betatron oscillations. The temperature is now

$$T_s \approx 22 \times 10^6^\circ K \quad \text{for 80 pulses}.$$ If we stack many pulses, the width due to momentum spread will become much larger than the betatron amplitudes. Particles with considerably different momenta will occupy different locations and never meet each other. The concept of temperature assumes, however, that the particles occupy the same volume. The temperature therefore, does not increase much, if more pulses are stacked.

We now consider the fact that the ISR is a circular machine and use the concept of negative temperature. Equation (3.1) gives with $\gamma_T = 9$:

$$T_0 = -570^\circ K \quad \text{for a single pulse}$$

and

$$T_0 = -3.7 \times 10^6^\circ K \quad \text{for 80 pulses}.$$
4. SOME MORE BEAMS AND FLOWS

In this section we study a number of further idealized beams and flows; some are more appropriate to microwave tubes, others to accelerators.

4.1 BEAM WITH KAPCHINSKIJ-VLADIMIRSKIJ DISTRIBUTION

We now take a brief look at the properties of an idealized distribution, which has been used frequently in studies of beams with finite emittance and space charge. Introduced by Kapchinskij and Vladimirskij it is generally known as the "K-V distribution". It is discussed in detail in Kapchinskij's book.\(^{10}\)

We consider a paraxial beam moving in the \(z\)-direction with momentum \(p_z = m_0c\beta z\). At a waist -- where the envelope curve is parallel to \(z\) -- this distribution is given in four-dimensional phase space \((x,y,p_x,p_y)\) by

\[
n = n_0\delta\left(\frac{x^2 + y^2}{a^2} + \frac{p_x^2 + p_y^2}{p_0^2} - 1\right).
\]

Here we have assumed a circular beam with radius \(a\) and maximum transverse momentum \(p_0\). The particle density is everywhere zero except on the three-dimensional surface volume of the four-dimensional ellipsoid. By replacing \(p_x\) and \(p_y\) by the derivatives

\[
x' = \frac{\partial x}{\partial z} \quad \text{and} \quad y' = \frac{\partial y}{\partial z}
\]

we get for the surface

\[
\frac{x^2 + y^2}{a^2} + \frac{(x'^2 + y'^2)a^2}{\epsilon^2} = 1,
\]

with

\[
\epsilon = e_x = e_y = x_{\text{max}}'x_{\text{max}} = y_{\text{max}}'y_{\text{max}} = \frac{ap_0}{m_0c\beta z}
\]

being the transverse emittance of the beam. We can further simplify the expression and obtain spherical symmetry by introducing the variables \(\tilde{x} = x/a, \tilde{x}' = x'/a/\epsilon\), etc., and get

\[
x^2 + y^2 + x'^2 + y'^2 = 1.
\]

We investigate now the projections of this distribution on two-dimensional spaces \((\tilde{x},\tilde{y})\); \((\tilde{x},\tilde{x}')\); etc. They represent circles of radius 1 with uniform charge distribution. This is a generalization of a theorem by Archimedes, which states that the projection of a surface of a sphere, down by two dimensions, on the one-dimensional axis gives uniform density.

The projections of single particle trajectories are shown in Fig. 4.1. For the \(\tilde{x},\tilde{x}'\)-projection one gets circles of different radii between 0 and 1.

The \(\tilde{x},\tilde{y}\)-projections are ellipses, circles and straight lines. Any particle which reaches the edge of the beam will afterwards move through the centre to the opposite edge, giving in the \(\tilde{x},\tilde{y}\)-projection a straight line of total length 2. Particles with equal \(\tilde{x}\) and \(\tilde{y}\) motion, but being 90° out of phase, give circles of radius \(1/\sqrt{2}\). All other particles give ellipses with different eccentricities such that the sum of the squares of the semi-axes is 1.
Fig. 4.1 Projections of particle trajectories in a K-V distribution

All other projections are equivalent to one of the two projections we just investigated.

We now have a closer look at the momentum distribution within the beam. At a distance $r = (x^2 + y^2)^{1/2}$ from the centre, the absolute value of the transverse momentum $p_A$ has a fixed value for a given $r$:

$$p_A = \left( p_x^2 + p_y^2 \right)^{1/2} = p_0 \left( 1 - \frac{r^2}{a^2} \right)^{1/2} = \frac{\delta y m_0 c}{a} \left( 1 - \frac{r^2}{a^2} \right)^{1/2}.$$

We can now calculate the transverse temperature:

$$k T_1 = \frac{1}{2} \frac{p_A^2}{\gamma m_0} = \frac{\gamma m_0 c^2 \beta^2 \varepsilon^2}{2a^2} \left( 1 - \frac{r^2}{a^2} \right).$$

It has a maximum in the centre of the beam and is zero at the edges (see Fig. 4.2). It is interesting to compare this with the temperature found for the example in Subsection 2.2.1.

Fig. 4.2 Transverse temperature in a beam of radius $a$ with a K-V distribution

The centre temperature is inversely proportional to $a^2$ and therefore to the cross-section of the beam, which we might call its two-dimensional volume $V_2$. For an adiabatic change of the beam size (due to changing of focusing) we have the relation

$$TV_2 = \text{const}.$$

It is interesting to note that this relation is identical to the adiabatic law in thermodynamics for a monatomic gas in two-dimensional space, where $\gamma' = (2 + 2)/2 = 2$, so that
\( T V_2^{X_1} = TV_2 \)

\( \gamma' = \text{ratio of specific heats} = \text{Cp/Cv} \).

The K-V distribution gives uniform distributions in the two-dimensional projections. Such uniform densities create linear space-charge fields which are easy to understand. For this reason the K-V distribution is important and often used.

There is another, simpler, distribution which has projections with the same density and area as the K-V distribution. It consists of a beam with helical orbits, and is only of interest in systems with axial symmetry. The distribution occupies a two-dimensional shell in four-dimensional phase space. Such a distribution occurs in the laminar pinch (Fig. 2.7) and also in a uniform magnetic field (Fig. 4.4). This distribution has finite emittance, but zero temperature.

4.2 Non-relativistic charged beam in an infinite \( B_z \) field

We now study a very simple idealized flow, which is probably more relevant to microwave tubes than to accelerators. We do not impose the restriction that the spread in longitudinal momentum is small.

We consider a round beam of radius \( a \) and uniform density \( n \) moving in the \( z \)-direction. An external, infinitely strong magnetic field \( B = B_z \) suppresses any transverse motion. The particles, in practical cases usually electrons, are non-relativistic, and magnetic fields created by the beam are neglected. The electric field due to the space charge produces a potential difference inside the beam (a depression for electrons, a rise for protons). This field, far away from the beam source, is

\[
E = E_r = \frac{nq_m r^2}{2 e_o r} = \frac{nq}{2 e_o} r,
\]

and the difference in potential energy between the edge of the beam and a point at a distance \( r \) from the axis is

\[
qU = \int_a^r \frac{n q^2}{2 e_o} r \, dr = \frac{n q^2}{4 e_o} \left[ a^2 - r^2 \right]. \tag{4.1}
\]

Introducing the longitudinal particle density \( N = n \pi a^2 \), we get

\[
qU = \frac{N q^2}{4 e_o} \left( 1 - \frac{r^2}{a^2} \right) = \nu m_o c^2 \left( 1 - \frac{r^2}{a^2} \right),
\]

where \( \nu \) is the Budker parameter (Subsection 2.2.3). The electrons are emitted by a cathode and accelerated by the field between the negative cathode \((-\theta)\) and the grid at ground potential (see Fig. 4.3). The beam is surrounded by a metallic tube also at ground potential. At the grid, the potential is zero and independent of \( r \), but far away from the grid we have the potential depression given by Eq. (4.1). Since the field integral

\[
\oint E \, d\vec{s} = 0
\]
over the path indicated in Fig. 4.3 is zero, there must be a longitudinal field inside the beam. This produces a deceleration of the particles, which is different for different $r$ so that some shearing will be present. The longitudinal energy potential must be smaller than the kinetic energy of the particles to make a flow possible. At the axis $r = 0$ this condition gives

$$qU_0 = \nu m_0 c^2 \ll \frac{1}{2} m_0 c^2 b_0^2 = q \phi,$$

or

$$\nu \ll \frac{\beta_0^2}{2}.$$

![Fig. 4.3 Potential depression](image)

The current of such a beam is

$$I = qN\bar{v}c = \frac{4\pi e_0 m_0 c^3 \nu \phi}{q},$$

which is for electrons

$$I = 17,000\nu \phi^3 \text{ A}.$$

If the potential depression is small, we can replace the average $\bar{v}$ by the maximum $\nu \phi$ and get a relation between the current of a beam and the voltage $\phi$ by which the particles were accelerated

$$\frac{I}{|\phi|^3} \ll \frac{4\pi e_0 (2|q|)}{m_0} = 66 \times 10^{-6} \text{ A/V}^3$$

(for electrons). The expression on the left-hand side of this equation is known as the permeance of the beam. For typical klystron beams it is of the order of $10^{-6} \text{ A/V}^3$. If one tries to put more current in, the potential depression gets larger and the particles close to the centre will be slowed down. More charge is then accumulated and the depression increases more. The current breaks down catastrophically. Electrons in the centre are actually reflected by the potential and one gets a so-called "virtual cathode".
4.3 Flow in a uniform magnetic field

Beams moving in longitudinal, axially symmetric magnetic fields are studied in connection with electron tube technology. First we wish to demonstrate the usefulness of the Larmor frame and the canonical angular momentum $P_\theta$ on a simple example:

$$P_\theta = p_\theta + qA_\phi r$$

($p_\theta$ is the mechanical angular momentum and $A_\phi$ the vector potential).

We consider now a single particle moving in a field-free region parallel to the $z$-axis at a distance $R$ with velocity $v_z$. At one point the particle enters a solenoid with uniform magnetic field $B_z$. At the entrance the particle goes through some radial field $B_r$ which produces an azimuthal force (Fig. 4.4). The particle will gain an azimuthal velocity

$$v_\theta = \frac{qV_z}{m_0} \int B_r \, dt = \frac{q}{m_0} \int B_r \, dz, \quad v_\theta \ll v_z.$$

Using $\text{div} \, \mathbf{B} = 0$ and the Gaussian theorem, one finds

$$\int B_r \, dz = -\frac{1}{2} B_z R$$

and

$$v_\theta = -\frac{qB_z R}{2m_0} = \omega_L R.$$

The particle will move inside the solenoid on a helix of radius $R/2$. It will rotate around the axis of this helix with the cyclotron frequency $\omega_c$ and at the same time around the solenoid axis with the Larmor frequency $\omega_L = \frac{1}{2} \omega_c$. We can deal with this problem in a more elegant way if we use the canonical angular momentum $P_\theta$ with respect to the $z$-axis, which is a constant of motion for these axially symmetric fields. Outside the field the mechanical momentum is zero and so is the vector potential $A_\phi$. In the field

$$A_\phi = \frac{1}{2} B_z r$$

and the mechanical angular momentum must be

$$p_\theta = m_0 r^2 \omega = -\frac{qB_z r^2}{2m_0}$$

to preserve $P_\theta$. This gives the frequency with which the particle moves around the axis

$$\omega = \frac{-qB_z}{2m_0} = \omega_L.$$

Often the motion is easier to understand if one goes into the Larmor frame, which rotates with the Larmor frequency around the axis. In this frame the magnetic field is zero and we have a radial electric field (see Fig. 2.4)

$$E_r = \frac{\omega L B_z}{2} r.$$
Observing our particle from the Larmor frame we find \( P_\theta = 0 \), and since \( A_\theta = 0 \) also the mechanical angular momentum must be zero. We have only a motion in the radial direction

\[
\dot{r} = \frac{q \omega B_z}{2m_0} r = -\omega_L^2 r .
\]

The particle will oscillate through the axis with the Larmor frequency.

---

**Fig. 4.4** Particle going through a solenoid

After this introduction we consider now a round beam of uniform density \( n \) which moves with the velocity \( v_z \) in a uniform longitudinal magnetic field. At the same time the beam rotates with the angular velocity \( \omega \) around the axis, such that there is an equilibrium between centrifugal force, Lorentz force, and space-charge force

\[
m_0 r \omega^2 + q B r \omega + \frac{q n q^2}{2e_0} r = 0 .
\]

Using the plasma frequency \( \omega_p \) [Eq. (1.2)] and the Larmor frequency \( \omega_L \) we get

\[
\omega^2 - 2i \omega_L + \frac{i \omega_p^2}{2} = 0
\]

\[
\omega = \omega_L \pm \sqrt{\omega_L^2 - \frac{i \omega_p^2}{2}} .
\]

This solution is plotted in Fig. 4.5 versus the plasma frequency. For \( \omega_p = 0 \) (zero density) \( \omega \) is either zero or \( 2\omega_L = \omega_c \). In the first case the particle moves only in the \( z \)-direction and in the second case it rotates in addition with the cyclotron frequency. With increasing

---

**Fig. 4.5** Solutions of Eq. (4.2)
\( \omega_p \) the two solutions approach each other and in the limiting case \( \omega_p = \sqrt{2} \omega_L \) there is only one solution \( \omega = \omega_L \). This is the solution with the largest plasma frequency, i.e. the largest current density, and is called the Brillouin flow. For this flow one has the following relations:

\[
\omega^2 = \omega_L^2 = \frac{1}{2} \omega_p^2 = \frac{1}{4} \omega_c^2 = \frac{2e^2}{a^2},
\]

\( a \) being the beam radius. For Brillouin flow the canonical angular momentum is zero. In a frame which rotates with \( \omega \) we have according to Eqs. (2.10) a magnetic field

\[
B_{rm} = \left(1 - \frac{\omega}{\omega_L}\right) B_z
\]

and an electric field

\[
E_{rm} = ru \left(1 - \frac{\omega}{2\omega_L}\right) B_z.
\]

Expressing \( B \) with the Larmor frequency we get

\[
E = \frac{r m_B}{q} (-2I \omega_L + \omega^2) = -\frac{1}{2} \omega_L^2 \frac{r m_B}{q}
\]

\[
E = -\frac{n q}{2e} r = -E_{sc}.
\]

The electric field in the moving frame balances the field due to space charge. For Brillouin flow \( \omega = \omega_L \), the moving frame is the Larmor frame where the magnetic field is zero. Figure 4.6 shows the realization of such flows. A beam is emitted by a cathode and accelerated by the potential difference between the cathode and the grid. The electron source can be in a longitudinal magnetic field of different strength, but in all cases the beam, just after passing the grid, will enter a region of magnetic field \( B_z \). In the case (a) the cathode is already in the field \( B_z \) and the beam just moves straight. This corresponds to the solution a in Fig. 4.5, where \( \omega = 0 \). To have equilibrium the plasma frequency must be zero. Seen in the Larmor frame the beam rotates with \( \omega' = -\omega_L \). In the second case (b), the cathode is in a field of strength \( B_z/2 \). At the entrance into the full field the beam will start to rotate with \( \omega = \omega_L/2 \), which corresponds to the solution b in Fig. 4.5. The plasma frequency is \( \omega_p = \sqrt{5/2} \omega_L \). This corresponds roughly to the type of partially-immersed gun used in klystrons; in practice, however, the field lines are gently converging, and the cathode area exceeds the area of the beam (Fig. 4.7). In the case (c'), the beam comes from a field-free region and will rotate with the Larmor frequency inside the full field. This is the basis of the "Wang gun". The plasma frequency is \( \omega_p = \sqrt{2} \omega_L \); we have Brillouin flow. In the case (d), the cathode is in a field of strength \(-1/2 B_z\) and the beam will rotate finally with \( \omega = 1.5 \omega_L \). Finally, in the case (e), the beam comes from a region with \( B = -B_z \) and will rotate with \( \omega = 2 \omega_L \). The plasma frequency is zero. This type of motion through a cusp is used in the Maryland ERA (Electron Ring Accelerators) experiment.
4.4 Envelope equation of a beam with self fields and finite emittance

Several characteristic types of flow will now be illustrated by taking various terms in the standard envelope equation, which is based on the paraxial equation.

As an example of the kind of behaviour we expect, we select a rather simple situation, in which we have

i) Axial symmetry, so that the focusing in x- and y-planes is identical.

ii) No magnetic or electric field along the orbit. The first condition can always be
achieved by working in the Larmor frame; the second is a genuine restriction, which can be removed by working in "reduced variables" -- for simplicity, however, we do not do this here.

iii) Uniform current and charge density across the beam.

If we wish to write down the equation of motion of a particle, this is straightforward; we just have to balance the inertial force against the sum of the focusing forces arising from the external field and the self field. We may write this as

$$x'' + \frac{x}{\lambda^2} - \frac{Kx}{a^2} = 0,$$

where $\lambda(z)$ is a parameter representing the external focusing force, and $K/a^2$, where $a$ is the beam radius, represents the self force.

We assume $\lambda(z)$ is specified; if $\lambda$ is independent of $z$, then it represents $1/2\pi$ times the wavelength of the oscillation which the particle makes. To determine $K$, we need to know the force on a particle arising from the self fields

$$F_x = q(E_x - \gamma c B_y).$$

In terms of the beam density $n$, and the fraction $f$ of neutralizing charges present in the beam, the fields are

$$E_x = \frac{nq}{2\varepsilon_0}(1-f)x, \quad E_y = \frac{nq\beta x}{2\varepsilon_0 c},$$

so that

$$F_x = \frac{nq^2x}{2\varepsilon_0}(1-f - \beta^2) = \frac{nq^2}{2\varepsilon_0} \left( \frac{1}{\gamma^2} - f \right).$$

In terms of the number of particles per unit length in the beam, $N = \pi a^2 n$, the Budker parameter $\nu = Nq^2/(4\pi\varepsilon_0 n_0 c^2)$, and the relation

$$\gamma m_V x = \gamma m_O x' \beta^2 = F,$$

we can make the identification in Eq. (4.3)

$$K = \frac{2\nu}{\beta^2 \gamma} \left( \frac{1}{\gamma^2} - f \right);$$

we call $K$ the "generalized permeance". For an unneutralized, non-relativistic beam this is proportional to the quantity conventionally known as the permeance

$$K = \frac{1}{\rho^2} \text{ amperes/(volt)}^2.$$

For a fully neutralized, relativistic beam, on the other hand, $K = -2\nu/\gamma$. The negative sign implies a tendency for the beam to constrict rather than spread. When $f = 1/\gamma^2$ the magnetic and electric contributions to the force cancel, and $K = 0$. 
Equation (4.3) now represents the equation of a single particle in the beam; we do not know how \( a \) varies, however, unless we have further information. For a laminar beam, all trajectories are similar, so that we can consider, for example, the edge particle of the beam. If all the orbits are planar and pass through the axis, then we set \( x = a \), and the boundary equation becomes

\[
x'' + \frac{x}{x^2} - \frac{K}{x} = 0 .
\]

If the particles have no angular velocity about the axis, then this can equally well be written

\[
x'' + \frac{x}{x^2} - \frac{K}{r} = 0 ,
\]

where \( r \) is the beam radius.

If the particles do have a finite angular velocity about the axis, then there is an additional outward centrifugal force. If the angular momentum of the particle at radius \( r \) is \( p_0 \), then this term may readily be shown to be

\[
\frac{p_0^2}{(\beta \gamma \mu_c)^2} \frac{1}{r^3}.
\]

Alternatively, if the beam is not rotating but has a finite emittance \( \epsilon \), and a K-V distribution, there is a pressure gradient which acts on a small volume element at the beam edge. This can be shown to lead again to a term of similar form, \( \epsilon^2 / r^3 \). Indeed, the rotating laminar beam has an emittance \( \epsilon = p_0 / \beta \gamma \mu_c \), as may be readily seen by noting that the spiral angle is \( p_0 / \beta \gamma \mu_c r \). The projection of such orbits on the \( x, x' \) plane forms ellipses with axes equal to \( x = r \), and \( x' = p_0 / \beta \gamma \mu_c r \).

Although explained here as an outward "pressure", the emittance term is normally obtained by purely optical arguments using phase-amplitude variables. See, for example, Courant and Snyder\(^{12}\), or, more explicitly, Garren\(^{13}\).

4.5 Some solutions of the envelope equation

Our equation now stands as

\[
x'' + \frac{x}{x^2} - \frac{K}{r} - \frac{\epsilon^2}{r^3} = 0 .
\]

To appreciate further the physical content we study some simple cases where only two of the four terms are different from zero.

a) \( 1/x^2 = 0, \ K = 0 \). We are left with

\[
x'' - \frac{\epsilon^2}{r^3} = 0 ,
\]
which describes the spreading of a beam with finite emittance. The solution

\[ r = \pm \left( x^2 + \frac{\epsilon^2 z^2}{\lambda^2} \right)^{\frac{1}{2}} \]

is the equation of a hyperboloid and gives the waist of a beam. For a laminar spiralling beam all the trajectories are straight lines. (The outer layer of the beam is thus similar to an old-style waste-paper basket!).

\[ R \]

b) \( \frac{1}{\lambda^2} = 0, \epsilon^2 = 0 \). We have a zero-emittance beam with space-charge force

\[ rr'' = +K \quad (|K| \ll 1) \]

For \( K > 0 \) we get a spreading out of a beam, while for \( K < 0 \) we have a contraction (pinch).

\[ R \]

c) \( r'' = 0, K = 0 \). We have a matched beam with finite emittance

\[ r^2 = \kappa \epsilon = \text{const} \]

\[ R \]

d) \( r'' = 0, \epsilon = 0 \). This is a beam with balance between space-charge forces and focusing forces; we have laminar flow

\[ \kappa \lambda^2 = r^2 \]

Solutions are only possible for \( K > 0 \). Brillouin flow seen in the Larmor frame is an example of this case.
e) \( r'' = 0, l/\chi^2 = 0 \). We have a beam with balance of the self force against the emittance; this is only possible for \( K < 0 \):

\[
-K\varepsilon^2 = \varepsilon^2 = r^2x_{\text{max}}^2,
\]

\[
x_{\text{max}}^2 = \frac{\beta_{\perp}^2}{\beta_{\parallel}^2} = -K = -\frac{2\nu}{\beta_{\parallel}^2} \left( \frac{1}{\gamma^2} - f \right).
\]

We assume now the case of perfect neutralization \( (f = 1) \) and where \( \langle \beta_{\perp}^2 \rangle = \frac{1}{6}\beta_{\perp,\text{max}}^2 \), which gives

\[
\langle \beta_{\perp}^2 \rangle = \frac{\nu}{\gamma} \beta_{\parallel}^2.
\]

By multiplying both sides with \( \gamma m_e c^2 \) we get

\[
m_e c^2 \gamma \langle \beta_{\perp}^2 \rangle = 2kT = \frac{2Nq^2 \beta_{\parallel}^2 c^2}{4\pi e^2 c^2} = \frac{I_{\nu_0}^2}{4\pi}\nu_0,
\]

\[
\frac{I_{\nu_0}^2}{4\pi} = 2NkT.
\]

This is the Bennett pinch relation (2.13) [the above equation differs by a factor of 2 from (2.13), because of the definition of \( T_{\perp} \) given in Subsection 3.1.1]. A more complete discussion along the lines of the present section including \( E_z \) fields, is given by Lawson\textsuperscript{14}.

4.6 Non-uniform distribution and non-linear forces

The analysis of the previous section is simplified in that it relates only to beams with a distribution uniform with radius and which have a sharp edge. Practical beams are not like this. An important class of such beams have linear external focusing, but non-uniform density. Examples are accelerator beams, and Brillouin flow (observed in the Larmor frame) where the electrons have a transverse thermal velocity arising from the finite temperature of the cathode.

For a beam uniform in the \( z \)-direction, self-consistent solutions appropriate to different distribution functions of the appropriate constants of motion are not difficult to find. When the focusing (external, self, or both) varies with \( z \), however, the problem is considerably more difficult.

One steady-state problem, which has been solved independently in several contexts, is a beam with uniform external focusing, but a transverse Maxwellian velocity distribution\textsuperscript{16,15}. Extreme solutions are obvious; if \( kT = 0 \), the beam just fills up the well, so that the space-charge force balances the external force. Such a system is represented by Brillouin flow viewed in the Larmor frame; the distribution is stationary, with the external force balancing the space-charge force. As \( kT \) now increases to a small value, the edges become less sharp, until ultimately when \( kT \) greatly exceeds the potential difference between beam centre and edge, the distribution becomes Gaussian. This is the situation in accelerators.
The intermediate cases can be found by the methods of Subsection 1.2.3; the Boltzmann relation is solved simultaneously with Poisson's equation to give an integral equation for \( n(r) \). The result is sketched in Fig. 4.8; the transition between the roughly uniform and roughly Gaussian solutions occurs, as might be guessed, when the beam radius is equal to the Debye shielding distance.

![Graph of distribution for different \( \lambda_D/a \)](image)

**Fig. 4.8** Distribution for different \( \lambda_D/a \)

For beams in which the focusing varies with \( z \), the situation is complicated, and solutions are probably best obtained by iterative computational methods.

One special type of system is susceptible to analysis; however, this is a linear optical system with axial symmetry, fed from a beam emerging with a thermal velocity distribution from a source of given radius. The analysis is only valid in the absence of space charge, but the effect of this can be added iteratively. Details are given in a book by Kirstein, Kino and Waters\(^{16}\). The envelope equation with non-uniform space charge is discussed by Lapostolle\(^{17}\) and Sacherer\(^{18}\) in an accelerator content, and by Weber\(^{19}\) in connection with cathode-ray tube design.

### 4.7 Is the emittance invariant?

One of the difficulties with non-linear systems is that the emittance does not seem to be invariant. Particles launched from a point with zero emittance in a non-linear focusing channel, which is uniform in the \( z \)-direction, ultimately appear to be randomly phased, and thus to have finite emittance. This, of course, is the familiar phenomenon of filamentation, which is difficult to quantify. The growth of emittance can, however, be expressed precisely, by introducing the closely related concept of entropy\(^{20}\). We look briefly at this in the next subsection.

### 4.8 Entropy, emittance and filamentation

The concept of entropy can be applied to a particle distribution in a beam. We divide the two-dimensional phase-space \( x, x' \) into cells of area \( A \). If \( N \) is the total number of particles and \( n_i \) their number in the \( i^{th} \) cell, we can distribute these particles in \( W \) different ways among the cells:
\[ W = \frac{N!}{n_1! \ n_2! \ldots \ n_M!} . \]

The entropy \( S \) of this system is, according to Boltzmann:

\[ S = k \ln W . \]

For large \( n_1 \) this can be approximated by

\[ S = k \left( N \ln N - \sum_{i=1}^{M} n_i \ln n_i \right) . \]

We now take a distribution which is uniform in the \( x,x' \)-phase-space, for example, a Kapchinskij-Vladimirskij distribution. The total area occupied by the beam is \( \pi \varepsilon \) (\( \varepsilon = \) emittance) and the number of particles in one cell is just \( \forall \varepsilon / \pi \varepsilon \). The entropy \( S \) is

\[ S = k \left\{ N \ln N - \frac{\pi \varepsilon}{A} \left[ \frac{NA}{\pi \varepsilon} \ln \left( \frac{NA}{\pi \varepsilon} \right) \right] \right\} = kN \ln \left( \frac{\pi \varepsilon}{A} \right) - k \ln A \]

\[ S = k \left( N \ln \left( \frac{\pi \varepsilon}{A} \right) \right) . \]

The entropy is, therefore, closely related to the emittance and the size of a cell.

This connection can be used to gain a more quantitative understanding of the effect of filamentation. Let us choose the cell size \( A \) such that it expresses the resolution of the device or experiment which measures the particle distribution. We assume a beam which is first uniform and occupies just four cells lying on a line (see Fig. 4.9). The entropy of this distribution is

\[ S = k \ln \left\{ \frac{N!}{[(N/4)!]^4} \right\} \approx kN \ln 4 . \]

After going through a non-linear focusing system, our original line distribution may have become longer and bent into a spiral. The phase space occupied by the beam would still be the same if measured with perfect resolution, i.e. infinitely small cell size. However, because of the finite size \( A \), i.e. finite resolution, the beam apparently occupies a larger area or a larger number of cells, let us say \( 20 \). The entropy is now larger:

\[ S = k \ln \left\{ \frac{N!}{[(N/20)!]^20} \right\} \approx kN \ln 20 . \]

The increase of the entropy or of the emittance is now expressed in a more quantitative way.
4.9 Application of the Vlasov equation to the determination of self-consistent equilibria

In Section 2 we talked about the Liouville theorem (Subsection 2.1) and the Vlasov equation (2.3a), but never actually applied the latter. In plasma physics the Vlasov equation is used to find self-consistent equilibrium distributions of particles in the external plus self-field. In accelerators the self fields are small and one knows the distribution to good approximation. The Vlasov equation is used in a second stage to calculate time-varying perturbations.

From the Liouville theorem one can see that any function which depends only on constants of motion is a solution of the Vlasov equation. We demonstrate that by a simple example. For a particle in a potential well $\Phi(x)$ the Hamilton function is

$$H = \frac{1}{2}mv^2 + q\Phi(x).$$

The distribution function for this problem is

$$f = f_0 e^{-H/kT} = f_0 e^{-[\frac{1}{2}mv^2 + q\Phi(x)]/kT},$$

which is only a function of the constant of motion $H$. Inserting $f$ into the Vlasov equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{q}{m} \frac{\partial \Phi}{\partial x} = 0,$$

one confirms easily that $f$ satisfies it. We get the density $n(x)$ of the distribution by integrating over $v$

$$n(x) = \int_{-\infty}^{\infty} f dv = f_0 e^{-q\Phi/kT} \int_{-\infty}^{\infty} e^{-m v^2/2kT} dv,$$

$$n(x) = n_0 e^{-q\Phi/kT},$$

which is the Boltzmann distribution we used before. Usually one finds a distribution by postulating some suitable function of constants of motion and tries to express it in terms of space $x$ to get a physical idea of the configuration. One has to solve simultaneously the Vlasov equation and the Maxwell equation. This leads to integral-differential equations which are difficult to solve. One puts in some assumptions about how the solution is expected to look and leaves some free parameters which can be matched later.
This technique has been extensively used to find self-consistent equilibria with axial symmetry, in connection with for example the "Astron" thermonuclear device, and ring currents which are not paraxial. The algebra is often extensive; to illustrate the method we use a very simple example; starting with a knowledge only of the constants of motion, we arrive at the rotating layer studied in Subsection 2.2.1. Other calculations using the method are mentioned at the end of this section.

For our example we take a system in which the particles all have the same energy and the same canonical angular momentum (self-fields can be neglected) and they move in a uniform magnetic field $B$. The motion is confined to planes perpendicular to $B$, so that $P_z = 0$.

In terms of the constants of motion we can write formally

$$F(P, x) = \text{const} \delta(H - H_0) \delta(P_{\theta_0} - P_0).$$

Now $H$ can be expressed by

$$H = \frac{1}{2m} \left[ P_r^2 + \left( \frac{P_0}{r} \right)^2 \right] = \frac{1}{2m} \left[ P_r^2 + \left( \frac{P_0}{r} - qA \right)^2 \right],$$

where $P$ denotes the canonical momentum and $p$ the mechanical momentum. Because of the axial symmetry the density is independent of $\theta$ and it is an arbitrary function of $z$. We choose a density independent of $z$, and, using the fact that $A = \frac{1}{2} Br$, obtain

$$n(r) = C \int \delta \left[ P_0 - \frac{1}{2m} \left(P_r^2 + \left( \frac{P_0}{r} - \frac{1}{2} qBr \right)^2 \right)\right] \, dp_r,$$

where $C$ is a normalizing constant to be found later. Integrating with respect to $\theta$,

$$n(r) = C \int \delta[f(p_r)] \, dp_r, \quad C \left( \frac{df}{dp_r} \right)_{p_r = 0} = C \left( \frac{P_r}{m} \right)_{p_r = 0}$$

$$= C \sqrt{2mH_0 - \left( \frac{P_0}{r} - \frac{1}{2} qBr \right)^2}.$$  \hspace{1cm} (4.4)

To proceed, we substitute for $B$ the radius of curvature of the particle

$$B = -m\bar{B}/qR$$

(the minus sign expresses the fact that in a positive field, the particles rotate with negative $\omega$).

We now write $r = R + x$, with $r_{\text{max}} = R + a$ where $p_r$ is zero. At this radius $P_0 = P_{\theta_0}$, so

$$P_{\theta_0} = m\bar{B}(R + a) + \frac{1}{2} qB(R + a)^2 = m\bar{B}(R + a) - \frac{1}{2} m\bar{B} \left( \frac{R + a}{R} \right)^2$$

$$= \frac{1}{2} m\bar{B} R \left( 1 - \frac{a^2}{R^2} \right).$$
To find \( n \) from Eq. (4.4), we need \((P_0/r - \frac{1}{2}qBr)^2\). From the expression just calculated, and the relation between \( B \) and \( R \)

\[
\frac{P_0}{r} - \frac{1}{2}qBr = \frac{\frac{1}{2}m\beta c(1 - a^2/R^2)}{R + x} + \frac{\frac{1}{2}q}{qR} \left( R + x \right).
\]

To simplify the problem, we now assume that \( a \ll R \), and expand. This assumption is not necessary, but its implication and justification can be examined later. Doing the algebra we get

\[
\frac{P_0}{r} - \frac{1}{2}qBr = m\beta c \left( 1 - \frac{a^2 - x^2}{2R^2} \right)
\]

and, again with \( a \ll R \), find that the denominator of Eq. (4.4) is simply

\[
\frac{m\beta c}{R} (a^2 - x^2)^\frac{3}{2}
\]

whence

\[
n(r) = \frac{RC}{m\beta c (a^2 - x^2)^\frac{3}{2}}.
\]

We can find \( C \) by integrating and putting the number of particles per unit length in the \( z \)-direction equal to \( N \),

\[
N = \int_{R-a}^{R+a} n(r) 2\pi r dr \approx \frac{2\pi^2 R^2 C}{m\beta c},
\]

whence, finally

\[
n(r) = \frac{N}{2\pi^2 R(a^2 - x^2)^\frac{3}{2}}.
\]

We have found again the rotating layer discussed in Subsection 2.2.1. We can proceed in an analogous way to find the velocity distribution and temperature.

We have arrived at a simple result by rather a lengthy and non-physical route. For more complicated problems, where self fields are included and the distribution functions are more elaborate, the algebra often becomes excessively difficult; it is more rewarding to approach such problems in a more physical and less rigorous way. The derivation of a beam with K-V distribution in a betatron field by this method is given by Davidson and Lawson\textsuperscript{21}. The paper contains 18 pages! A very interesting and instructive example, where this technique works, is the relativistic self-constricted beam of Hammer and Rostoker\textsuperscript{22}; this has constants of motion \( H \) and \( P_z \), but includes self-fields. In the paraxial limit, \( P_P \gg P_L \), or \( K \ll 1 \), it reduces to a K-V distribution. Non-paraxial toroidal beams have been studied by several authors, for example Ott and Sudan\textsuperscript{23}. A further simple example is given in the book by Longmire\textsuperscript{24}. 
4.10 Concluding remarks on steady-state beams

A representative survey of steady-state beams, and some different ways of looking at them, have been presented. At one extreme there is a purely optical approach; at the other the beam may be considered as a hot gas confined by the focusing forces. So we see, for example, the emittance as either representing an invariant arising from constraints placed by dynamical principles on the orbit geometry, or as a manifestation of an outward force associated either with a pressure gradient or, in a rotating beam, the centrifugal force.

Which viewpoint is best depends of course partly on one's experience, and partly on the nature of the problem. Some beams are between the region traditionally studied by plasma physicists, on the one hand, and accelerator physicists, on the other. It is here that some acquaintance with both viewpoints is helpful.
5. WAVES IN PLASMAS

We now take a look at waves in plasmas. In order to set the scene, and bring out some of the physical features of importance, we first study waves in free space, and the effect of boundaries. Next we examine wave motion in stationary plasmas, and then point out the new effects associated with moving plasmas, or plasmas in which there are well-defined streams.

We then treat instabilities from the point of view of a moving plasma with finite size, in which boundary conditions are important. The effective mass concept, discussed in Subsection 3.1.4, is invoked to convert form rectilinear to circular geometry.

5.1 Electromagnetic waves in a lossless non-dispersive medium

Before we describe the properties of waves in plasmas, the plane-wave solution of Maxwell's equation in a lossless medium without dispersion will be discussed. Coordinates are chosen such that the magnetic field is in the z-direction; then if the direction of propagation of the waves makes an angle \( \theta \) with the x-axis, the field components are given by the real part of the expressions

\[
H_z = H_0 \exp \left[ i(\omega t + k(-x \cos \theta - y \sin \theta)) \right]
\]

\[
E_x = -Z_0 H_z \sin \theta
\]

\[
E_y = Z_0 H_z \cos \theta
\]

(5.1)

\( Z_0 \) is the intrinsic impedance of the medium, and the wavelength and the frequency are related to the velocity of propagation \( v \) by the equation

\[
v = \frac{\omega}{k}, \quad k = \frac{2\pi}{\lambda} = \text{wave number.}
\]

(5.2)

A diagrammatical representation of such a plane wave is given in Fig. 5.1a. It will be seen that the apparent wavelength measured along the x- and y-axis are \( \lambda / \cos \theta \) and \( \lambda / \sin \theta \), respectively. The phase velocity of the disturbance is therefore \( v / \cos \theta \) along the x-axis and \( v / \sin \theta \) along the y-axis. These velocities vary between \( v \) and \( \omega / k \) as \( \theta \) varies, so that in such a wave they are always greater than \( v \).

It is of interest also to consider the impedance of the wave system defined by Eq. (5.1). In a wave system referred to Cartesian coordinates, the impedance \( Z_x \) measured in the x-direction is given by \( Z_x = E_y / H_z \), provided that there are no other components of field in the yz-plane. The significance of this quantity is that it is continuous across boundaries parallel to the yz-plane. For the plane wave under consideration (Fig. 5.1a) \( Z_x = E_y / H_z = Z_0 \cos \theta \). Similarly \( Z_y = -E_x / H_z = Z_0 \sin \theta \).

In deriving the solution of the wave equation given in Eq. (5.1), the only restriction placed on \( \cos \theta \) and \( \sin \theta \) is that \( \cos^2 \theta + \sin^2 \theta = 1 \). It is, therefore, necessary to investigate solutions in which \( \cos \theta \) is greater than unity and \( \sin \theta = i(\cos^2 - 1)^{1/2} \) is pure imaginary. Writing the solution in terms of \( \cos \theta \) and choosing the negative imaginary value of \( \sin \theta \) yields
\[ H_x = H_0 \exp \left\{ -ky(\cos^2 \theta - 1)^{1/2} \right\} \exp \left\{ i(\omega t - kx \cos \theta) \right\} \]

\[ E_x = iz_H H_y (\cos^2 \theta - 1)^{1/2} \]

\[ E_y = z_H H_x \cos \theta . \]

In such a wave the fields decrease exponentially away from the x-axis and, since \( \cos \theta > 1 \), the phase velocity along the x-axis is less than \( v \). The x-component of the electric field is pure imaginary; this means that it is \( \pi/2 \) out of phase with the other fields. Consequently, the impedance \( Z_y \) is also imaginary, \( Z_y = -iz_H (\cos^2 \theta - 1)^{1/2} \). A negative imaginary value denotes that the magnetic field "leads" the electric field, and hence that the impedance is capacitive. Such a wave is shown diagrammatically in Fig. 5.1b. Waves of this type are known as surface waves, or sometimes as slow waves or inhomogeneous plane waves. They occur, for example, at the surface of a slab of optically dense material at which total internal reflection is occurring; under these circumstances, Snell’s law gives a value greater than unity for \( \cos \theta \), the angle between the "refracted" ray in the less dense medium and the surface. Evidently the phase velocity along the interface is less than the velocity of light in the less dense medium (Fig. 5.2).

Such surface waves may also be produced on corrugated surfaces. A surface structure such as that shown in Fig. 5.3 is purely inductive, since it appears as a system of short-circuited transmission lines of length less than \( \lambda/4 \). Provided that the fine structure of the surface can be neglected, its impedance may be matched to that of a capacitive surface wave. The impedance \( Z_y \) looking into the surface is \( iz_i \tan \left( 2\pi d/\lambda \right) \), where \( d \) is the depth of the slot, and the impedance looking away from the surface into the the surface wave \( Z_y \).
is \( -iZ_0(\cos^2 \theta - 1)^{1/2} \). Now the condition for a match is that these should be equal numerically, but opposite in sign, so that

\[
Z_0 \tan \left( \frac{2\pi d}{\lambda} \right) = Z_0(\cos^2 \theta - 1)^{1/2},
\]

whence

\[
\cos \theta = \frac{v}{v_x} = \frac{1}{\cos \left( \frac{2\pi d}{\lambda} \right)}.
\]

As \( d \) increases from 0 to \( \lambda/4 \), the phase velocity drops from \( v \) to zero.

---

**Fig. 5.2** Reflection at an interface. The wave in medium 2 is a slow wave if \( \lambda_1/\cos \theta_1 < \lambda_2 \).

**Fig. 5.3** Inductive corrugated surface

Having described the properties of the two forms of plane wave, we now consider the effect of combining two or more waves travelling in different directions. As a simple example, we add two waves propagating symmetrically with respect to the x-axis at the angle \( \pm \theta \) (Fig. 5.4); we get by superposition of the plane waves [expressions (5.1)]:

\[
H_z = 2H_0 \cos \left( ky \sin \theta \right) \exp \left\{ i(ut - kx \cos \theta) \right\}
\]

\[
E_x = 2iZ_0H_0 \sin \left( ky \sin \theta \right) \sin \theta \exp \left\{ i(ut - kx \cos \theta) \right\}
\]

\[
E_y = Z_0H_z \cos \theta.
\]
With these fields we can satisfy already the boundary conditions of a perfectly conducting waveguide. Putting two plates parallel to the xz-plane at y = 0 and y = a then we have to demand there $E_x = 0$. For this mode the dimension in the other transverse direction has no effect. Thus the simplest mode consists of two interfering plane waves at the angles

$$\theta = \pm \arcsin \left( \frac{\lambda}{2a} \right)$$ (5.3)

to the direction of propagation. As time goes on the fields in the guide appear to be travelling along the x-direction with a wavelength $\lambda_x$. That distance is related to the angle $\theta$ by

$$\cos \theta = \frac{\lambda}{\lambda_x}$$

and with Eq. (5.3) it follows that

$$\lambda_x = \frac{2\pi}{k_x} = \frac{\lambda}{\sqrt{1 - (\lambda/2a)^2}}$$
or

$$\omega^2 - \omega_c^2 = k_x^2 c^2.$$ (5.4)

There is only wave propagation above the cut-off frequency $\omega_c = \pi c / a$, as there is only interference if the free-space wavelength is shorter than 2a. According to Eq. (5.2) the phase velocity is

$$v_{ph} = \frac{\omega}{k_x} = \frac{c}{\sqrt{1 - (\omega_c/\omega)^2}}.$$  

For frequencies above cut-off, $v_{ph}$ is real and greater than the speed of light. Note that it is just the nodes of the wave which are moving with velocities greater than the velocity c of light and not energy or information which travel with the group velocity $v_g = d\omega / dk$. Taking the derivative of Eq. (5.4) with respect to $\omega$ and inverting the differential yields

$$v_g = c \sqrt{1 - (\omega_c/\omega)^2}.$$  

Fig. 5.4 Superposition of two plane waves
Hence we arrive at the well-known result \( v_{ph} = c^2 \).

We can satisfy boundary conditions even for more complicated waveguides just by adding up a set of plane waves. This set becomes infinite for a circular waveguide; the sum becomes an integral, which leads to a description in terms of Bessel functions.

### 5.2 Electromagnetic fields in an unbounded plasma

We now inquire into the nature of the fields of unbounded plasma. Before we can write down Maxwell's equation for the plasma we must be aware that, in the presence of a magnetic field which defines a preferred direction, the linear relation between the electric displacement \( \vec{D} \) and the electric field \( \vec{E} \) is a tensor rather than a scalar:

\[
D_i = \epsilon_0 \sigma_{ij} E_j .
\]  

(5.5)

Let us calculate the conductivity tensor by assuming that there are no collisions in the plasma and that the motion of the positive ions can be neglected. For convenience, the \( z \)-axis is chosen to be parallel to the magnetostatic field: \( |\vec{B}_0| = B_0 \). Then from the equation of motion for an electron, assuming \( \vec{v} \) to be harmonic in time, in the complex notation

\[
i \omega m \vec{v} = -q \left[ \vec{E} + \vec{\nabla} \times \vec{B}_0 \right],
\]

follows after a little algebra

\[
x = \frac{iq}{m \omega} \frac{1}{1 - \omega^2 / \omega^2} \left( E_x + \frac{\omega}{\Omega} E_y \right) \]

\[
y = \frac{iq}{m \omega} \frac{1}{1 - \omega^2 / \omega^2} \left( -i \frac{\Omega}{\omega} E_x + E_y \right) \]

\[
z = \frac{iq}{m \omega} E_z .
\]

where the cyclotron frequency \( \Omega = qB_0 / m \) has been used. With the proportionality between current density and velocity [see Eq. (2.4b)] we find

\[
j_i = \sigma_{ij} E_j ,
\]

where

\[
\sigma_{ij} = -\frac{n q^2}{m \omega} \frac{1}{1 - \omega^2 / \omega^2} \left( \begin{array}{ccc} 1 & i \Omega / \omega & 0 \\ -i \Omega / \omega & 1 & 0 \\ 0 & 0 & 1 - \Omega^2 / \omega^2 \end{array} \right) .
\]

From that follows the tensor relation (5.5) between the electrical displacement vector \( \vec{D} \) and the electric field \( \vec{E} \), where the dielectric tensor is

\[
\epsilon_{ij} = \delta_{ij} - i \frac{\sigma_{ij}}{\epsilon_0 \omega} .
\]

By means of the dielectric tensor \( \epsilon \) we are able to express Maxwell's equations for an electromagnetic field in the plasma:
\[ \nabla \times \vec{E} = -i\omega \vec{B} \]
\[ \nabla \times \vec{B} = \frac{1}{c^2} \varepsilon \vec{E} \ . \]

Supposing the fields have the space variation exp (-i\vec{k}r) of a plane wave, then from the above equations we find by elimination of \( \vec{B} \)
\[ -\mu^2 \vec{k} \times (\vec{k} \times \vec{E}) = \varepsilon \vec{E} \ , \quad (5.6) \]
where \( \vec{k} = \frac{\vec{k}}{|\vec{k}|} \) and \( \mu = c|\vec{k}|/\omega \) is the refractive index. In suffix form Eq. (5.6) may be written as
\[ [\varepsilon_{ij} - \mu^2(\delta_{ij} - \vec{k}_i\vec{k}_j)] E_j = 0 \ . \quad (5.7) \]

This dispersion relation gives the possible values of the refractive index \( \mu \) for non-vanishing field \( \vec{E} \) and thus the components of \( \vec{E} \) specifying the polarization.

Simple expressions for the refractive index are found for \( \vec{k} \) parallel and perpendicular to the magnetostatic field \( \vec{B}_0 \):
\[ \vec{k} \parallel \vec{B}_0 : \quad \mu^2 = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_p)} \ . \quad (5.8) \]
\[ \vec{k} \perp \vec{B}_0 : \quad \mu^2 = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{or} \quad \mu^2 = 1 - \frac{\omega_p^2(\omega^2 - \omega_p^2)}{\omega^2(\omega^2 - \omega_p^2 - \omega_0^2)} \ . \]

\( \omega_p = \sqrt{n q^2/(m \varepsilon_0)} \) is the plasma frequency already defined in the first section. In the first of Eqs. (5.8) there is no dependence on the magnetostatic field, since a purely transverse plane wave, linearly polarized with the electric vector parallel to \( \vec{B}_0 \), does not affect the electron motion with respect to the static field \( \vec{B}_0 \).

If the magnetostatic field \( \vec{B}_0 \) is zero, then \( \mu^2 = 1 - \omega_p^2/\omega^2 \) and Eq. 5.7, for \( \vec{k} \) along the \( z \)-axis, is
\[
\begin{pmatrix}
1 - \frac{\omega_p^2}{\omega^2} - \frac{k^2c^2}{\omega^2} & 0 & 0 \\
0 & 1 - \frac{\omega_p^2}{\omega^2} - \frac{k^2c^2}{\omega^2} & 0 \\
0 & 0 & 1 - \frac{\omega_p^2}{\omega^2}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
= 0 .
\]

We have rediscovered the dispersion relation for transverse electromagnetic waves already discussed in Subsection 1.2.5:
\[ \omega^2 - \omega_p^2 = k^2c^2 \ , \quad (5.9) \]
and the trivial dispersion relation for longitudinal electromagnetic waves in a plasma as already derived in Subsection 1.2.4.
\[ \omega^2 - \omega_p^2 = 0. \]  \hspace{1cm} (5.10)

Figure 5.5 shows the frequency \( \omega \) as a function of the wave number \( k \) for both longitudinal and transverse electromagnetic waves in a plasma at rest.

\[ \omega \]

\[ \text{transverse waves} \]
\[ \text{longitudinal waves} \]
\[ +\omega_p \]
\[ k \]
\[ -\omega_p \]

Fig. 5.5  Plot of the dispersion relations (5.9) and (5.10)

It is interesting to note that from Eq. (5.9) the phase velocity and group velocity, respectively, follow:

\[ v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} > c \]

\[ v_g = \frac{d\omega}{dk} = c \sqrt{1 - \omega_p^2/\omega^2} < c, \]

hence the geometric mean of both equals just the velocity \( c \) of light. This result we had already in Subsection 5.1, where the geometry-dependent frequency \( \omega_c \) appears instead of the density-dependent plasma frequency \( \omega_p \). So we see a general correspondence between electromagnetic waves in a plasma and in a bounded free space.

At this point another interesting feature may be exhibited, which relates the dispersion relations above even to quantum physics. For a particle of rest mass \( m \), the momentum \( p \) and total energy \( E \) are related by \( E^2 = p^2c^2 + m^2c^4 \). By the usual substitution \( p = \hbar k \) (\( \hbar \) = Planck's constant \( h \) divided by \( 2\pi \)), and \( E = \hbar \omega \), this can be written as another dispersion relation

\[ \omega^2 - \omega_q^2 = k^2c^2, \]

where \( \hbar \omega_q = mc^2 \). The waves in the plasma can thus be thought of as corresponding to photons with rest mass equal to \( \hbar \omega_q/c^2 \). Because these photons have mass, a longitudinal polarization exists.

The subject matter of this section will be found in most texts on plasma physics (for example, Ref. 4) and no detailed references are therefore given.
5.3 Instability and growth in plasma streams

In the previous subsections we considered plasmas which were in the unperturbed state at rest. So we might ask what happens if the charged particles of a plasma stream move with velocity \( v \) with respect to the laboratory frame? If there is only a single stream with no velocity spread, nothing is different from what has been discussed up to now, as this situation is simply equivalent to a uniform motion of the observer past a stationary plasma. But if there is more than one stream, say two, three, four, etc., or an infinite number of streams each having a particular speed, then new things come out of the problem.

These effects arise because, in circuit terms, a moving plasma can be considered as an "active" medium. If we compute the total kinetic and electromagnetic energy of a stationary plasma in the absence of a wave, and again when a wave is present, we always find that the energy is greater when the wave is present. When the plasma is moving, however, this is not necessarily true. Waves with less energy than quiescent plasma are known as "negative energy waves"; in the presence of a dissipative mechanism their amplitude increases.

Another possibility, which occurs when the plasma contains well-defined streams of particles, is that the negative energy waves in one stream interact with the positive waves in another, to give resonant growth. A further type of effect is typified by Landau damping, where there is resonant interaction between a wave and particles moving at the same velocity. All these effects are discussed in the following subsections.

5.3.1 Equations for space-charge wave propagation

The motion of space-charge waves in an electron stream can be described in terms of the perturbation of charge density, current density, and velocity. Both perturbed charge density and perturbed current density in the beam generate electromagnetic fields. To calculate the interaction of these fields with the beam, Maxwell's equations and the equations of motion are used as before, the latter being modified by the existence of the stream velocity:

\[
\nabla \times \vec{B} = \mu_0 \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \\
\n\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\n\frac{\partial \vec{j}}{\partial t} + \frac{\partial \vec{v}}{\partial t} = 0, \quad \vec{j} = \rho \vec{v} \quad \text{charge density} \\
\n\frac{m}{\partial t} \frac{\partial \vec{v}}{\partial t} = -q \vec{E} \times \vec{B}.
\]

In the following we will perform the analysis in the Eulerian picture, which means that variables like velocity, charge, and current density are computed at a fixed position within the stream. If we assume the perturbation parameters to be small compared with the unperturbed quantities, the theoretical approach may be that the non-linear equations above are linearized and one seeks solutions which are harmonic in space and time. In the case of instability this procedure can only lead to the prediction of its existence and some indications of its nature; but it gives no answer to what will happen after the initial build-up. The variables above are split into an unperturbed (d.c.) term and a small perturbed (RF) term.
\[ \vec{A}(r,t) = \vec{A}_0(t) + \vec{A}_1(t) e^{i\omega t - \vec{k} \cdot \vec{r}}, \] 
where \( |\vec{A}_1| \ll |\vec{A}_0| \).

Substituting this into Eqs. (5.11a) to (5.11d), and neglecting products of perturbed quantities, one obtains apart from the unperturbed equations the following relations between perturbed variables:

\[ \vec{\nabla} \times \vec{B}_1 = \mu_0 \frac{\partial \vec{E}_1}{\partial t} + i \frac{\omega}{c^2} \vec{E}_1 \]  
(5.12a)

\[ \vec{\nabla} \times \vec{E}_1 = -i \omega \vec{B}_1 \]  
(5.12b)

\[ \vec{J}_1 = -\rho_e \vec{V}_1 + \rho_i \vec{V}_0 \]  
(5.12c)

\[ i \omega \vec{V}_1 + (\vec{\nabla} \times \vec{B}_0) \vec{V}_1 + (\vec{\nabla} \times \vec{E}_0) \vec{V}_0 = -\frac{q}{m} \left( \vec{E}_0 + \vec{\nabla} \times \vec{B}_1 + \vec{V}_1 \times \vec{B}_1 \right) \]  
(5.12d)

Note that in the Eulerian picture the \( d/dt \) operation becomes \( \partial/\partial t + \vec{V}_0 \cdot \vec{\nabla} \). Together with the velocity \( \vec{V}_0 \) which can be computed from the non-linear equations of d.c. variables, the set of equations (5.12a) to (5.12c) allows one to describe the longitudinal space-charge waves involving longitudinal RF electric fields and longitudinal electron oscillations.

5.3.2 Analysis of longitudinal space-charge waves

In the following, longitudinal oscillations of a large number of electron streams are considered. The streams are infinite in extent and have a definite velocity, i.e. there is no random motion within a stream. We also assume implicitly in the subsections to come that an equal number of ions is present, so that the system is neutral from the macroscopic point of view. Consequently, if some electrons are displaced from their equilibrium position, restoring Coulomb forces are set up between the displaced electrons and the surrounding ions; but the disturbance is localized to the region in which the electrons oscillate.

Consider the motion of electron streams constrained to the \( z \)-direction by a very large static magnetic focusing field \( |\vec{B}_0| = B_z \). The restriction to one space coordinate is not a limitation, as it contains the essential features of space-charge waves. As in particle accelerators and storage rings, there may also be a guiding mechanism with magnetic fields perpendicular to the stream motions which do not affect the oscillations in the longitudinal direction. Then if we assume the unperturbed velocity to be \( V_z \) and the perturbed one to be \( V_z \), Eqs. (5.12) become

\[ \sum \nu \ j_\nu + i\omega e_\nu E = 0 \]  
(5.13a)

\[ \frac{\partial}{\partial t} j_\nu = -\frac{\partial}{\partial t} \left( -\rho_0 V_\nu + \rho_i V_\nu \right) = -i\omega e_\nu \]  
(5.13b)

\[ i\omega e_\nu + V_\nu \frac{\partial V_\nu}{\partial t} = -\frac{q}{m} E, \]  
(5.13c)
where the subscripts \( z \), for the current density \( j \) and the electric field \( E \), and \( 1 \), for the perturbed quantities, have been omitted. We also drop the suffix \( \mu \), bearing in mind that, due to Eq. (5.13a), only the summation over the RF current of all \( \mu \) beams yields the electric field. From the conservation of energy:

\[
q(U_0 + U) = \frac{1}{2} m (v^2 + 2v V)
\]

we may define a perturbed "kinetic voltage",

\[
U = \frac{m}{q} V V .
\]

With the definitions

\[
\begin{align*}
  j_0 &= -\rho_0 V, \\
  k_e &= \omega/V, \\
  k_p &= \omega_p/V,
\end{align*}
\]

propagation constant of the electromagnetic waves of the \( \mu \)th stream,

plasma propagation constant of the \( \mu \)th stream, where

\[
\omega_p = \frac{q}{m} \frac{e_0}{\varepsilon_0}
\]

electron plasma frequency,

the equations above can be reduced to two equations relating the voltage \( U \) to the current density \( j \):

\[
\begin{align*}
  (v_z + ik_e)U &= 2iU_0 \frac{k_p^2}{k_e} j/j_0, \\
  (v_z + ik_e)j &= \frac{1}{2i} \frac{U}{U_0} k_e j_0 .
\end{align*}
\] (5.15)

With the definitions

\[
\alpha^2 = \frac{1}{8Z_0} \left[ U \pm Z_0 (v_z j) \right], \text{ with}
\]

\[
Z_0 = \frac{2\omega_p}{\omega} \frac{U_0}{j_0} \text{ characteristic impedance ,}
\] (5.16)

Eqs. (5.15) can be written in normal mode form:

\[
[v_z + i(k_e \pm k_p)]\alpha^2 = 0 .
\] (5.17)

From the solution to Eq. (5.17),

\[
\alpha^2(z) = \alpha^2(0) e^{-i(k_e \pm k_p)z},
\] (5.18)

it becomes obvious that there are two waves present with the propagation constants

\[
k^2 = k_e \pm k_p = \frac{\omega \mp \omega_p}{V} .
\] (5.19)
These two dispersion relations for fast \((k^+)\) and slow \((k^-)\) space-charge waves are plotted in Fig. 5.6.

Either of the two waves in Fig. 5.6 can exist on its own. If two waves of equal amplitude with the same value of \(\omega\) are present, then a standing wave pattern is obtained in the reference frame \((z = Vt)\) of the moving electrons.

5.3.3 Negative energy carried by slow space-charge waves

It is possible to ascribe to the particles of a modulated electron beam "kinetic power", the formula for which involves only terms which appear in the linearized equations for the system, and which, when added to the Poynting flux of the associated electromagnetic field, is properly conserved. The analysis of an arbitrary disturbance of a free beam leads into a fast wave of positive kinetic power and a slow wave of negative kinetic power. This may become obvious from the relation between the kinetic RF power and the time-averaged stored energy per unit length

\[
\langle P \rangle = \nu_g \langle W \rangle ,
\]

where \(\nu_g\) is the group velocity of the waves. On the other hand, from transmission-line concepts we know that \(\langle P \rangle\) equals the real part of the quantity \(-j \omega \eta /2\), or in terms of the normal-mode amplitudes (5.18)

\[
\langle P \rangle = |\alpha^+|^2 - |\alpha^-|^2 .
\]

The group velocity of these waves is equal to the stream velocity, which is always taken as positive. Hence if the beam carries only the \(\alpha^-\)-mode, then this slow-wave mode contains negative RF energy.
This topic has given rise to a certain amount of confusion and controversy because of the difficulty of identifying in a unique way the precise physical nature of the negative energy and its magnitude. This problem is discussed, and resolved, by Sturrock \(^{25,24}\). The difficulty arises especially with transverse waves; for longitudinal waves the physical picture is clearer. We may say that negative RF energy is stored in the \(\alpha^-\)-mode by the mechanism of bunching of electrons in regions of decelerating electric field forces and, in the case of \(\alpha^+\)-modes, bunches are created by accelerating phases of the electric field. Thus in the \(\alpha^-\)-case the bunches are slowed down in comparison with the d.c. beam and the beam contains less total energy, whereas in \(\alpha^+\)-modes the net energy of the beam increases.

5.3.4 Multiple plasma streams

By inserting the definitions (5.14) into Eq. (5.19) we obtain the dispersion relation for one-dimensional space-charge waves in the form

\[
1 = \frac{\omega_p^2}{(\omega + \kappa V)^2} .
\]

Trivially for \(V = 0\) the longitudinal dispersion relation (5.10) of a beam at rest is obtained. Remembering that in Eq. (5.13a) we studied initially a set of streams with plasma frequencies \(\omega_{\text{p}u}\) and of discrete velocities \(V_u\), the dispersion equation for a multiple plasma stream is

\[
1 = \sum_u \frac{\omega_{\text{p}u}^2}{(\omega + \kappa V_u)^2} .
\]

For two electron streams this may be written explicitly

\[
f(w) = \frac{\omega_{\text{pl}}^2}{(w - V_1)^2} + \frac{\omega_{\text{pl}}^2}{(w - V_2)^2} = k^2 ,
\]

(5.20)

with \(w = \omega/k\). The plot of \(f(w)\) in Fig. 5.7 may offer perhaps the most direct way to see that the dispersion relation (5.20) predicts an instability. Assuming \(V_1 < V_2\), it is easily confirmed that there is a pair of conjugate complex values of \(\omega\), for all real values of \(k\), for which \(k^2\) is less than the minimum value, \((\omega_{\text{pl}}^2 + \omega_{\text{pl}}^2) / (V_2 - V_1)^2\), reached by \(f(w)\) in the interval \(V_1 < w < V_2\).

![Fig. 5.7 Plot of Eq. (5.20) as a function of \(w\)](attachment)
Now because of the homogeneity of variables in time and in -- at least -- one spatial
dimension, we assumed by the form \( \exp \left[ i(\omega t - kz) \right] \), a wave is unstable if for some real wave
number \( k \) a complex \( \omega \) with

\[
\text{Im } \omega < 0
\]

is obtained from the dispersion equation. Hence the two plasma streams can be unstable in
both of the two cases to be distinguished according to whether the streams travel in the same
or opposite directions. If the streams travel in the same direction, the plot of \( \omega \) against
\( k \) then appears as shown in Fig. 5.8a and, if they travel in opposite directions, the plot of
the dispersion relation is as shown in Fig. 5.8b. The instabilities that can occur are dif-
ferent in both cases, as will be discussed briefly.

![Dispersion curves for longitudinal waves travelling parallel to two streams that flow a) in
the same direction, and b) in opposite directions](image)

**5.3.5 Conveective and absolute instabilities**

It was first indicated by Twiss and very clearly pointed out by Sturrock\(^{27}\) that two
types of instabilities can be distinguished physically: "convective" instabilities, and
"absolute" instabilities. If in an infinite system, a pulse disturbance that is initially
of finite spatial extent, grows in time without limit at every point in space, there is an
absolute instability. In a physical system, of course, the amplitude of the oscillation is
limited by non-linear effects, so with the adjective "absolute" it is only indicated that
the linearized analysis has a response tending to infinity. In contrast, if in an infinite
system a pulse disturbance that is initially of finite spatial extent propagates along the
system, so that its amplitude eventually decreases with time at any fixed point in space,
there is a convective instability.

Clearly the labelling of an instability is always with respect to a particular reference
frame, since for a moving observer a convective instability can appear as an absolute one.
To determine from the form of the dispersion relation whether a particular instability is
convective or absolute is in general rather complicated. A detailed discussion is given by
Briggs\(^{28}\).
One situation where the behaviour is clear, however, is when the coupling is weak. Weak coupling between two waves implies that the behaviour is only significantly modified in the immediate vicinity of the point where the two dispersion curves appropriate to the two waves in the absence of coupling intersect. The simplest system, which we consider here, involves the coupling between two waves with dispersion curves \( \omega - kV_1 = 0 \), and \( \omega - kV_2 = 0 \). In the absence of coupling the system is represented by two straight lines through the origin; in the presence of coupling the dispersion curves become hyperbolae asymptotic to the straight lines.

Four cases need to be considered, since the waves can carry either positive or negative energy, and the group and phase velocities can be in either the same or opposite directions. (These situations are more appropriate for beams of finite cross-section, as we see in the next section. One wave is on the beam, and another on a neighbouring structure.) Nevertheless, we present the essential physical results now in the form of a table. The behaviour summarized in the last column of Table 5.1 can be inferred from the first two columns, by simple physical reasoning, which takes into account the direction of the energy flow. It may also be deduced more formally from the dispersion relations; for more details see Ref. 28, or the excellent summary in Ref. 4.

Examples of convective and absolute instability are exhibited by the travelling-wave tube (TWT) and backward-wave oscillator (BWO). In the TWT the phase velocity of the waves on the beam and in the circuit are in the same direction; in the BWO the phase velocities are opposite\(^5\). This is shown in Fig. 5.9.

![Diagram](image)

**Fig. 5.9** By adjusting the beam to condition C or D (see also Table 5.1), either a travelling-wave tube or a backward-wave oscillator may be obtained with the same circuit.

5.4 Continuous velocity distribution in plasmas and Landau damping

In the case of a continuous distribution \( f(V) \) of the velocities, the summation in Poisson's equation (5.13a) has to be replaced by an integration over all velocities \( V \). The derivation of the dispersion function is completely analogous to the case of discrete velocities we had before; it results in

\[
1 = \frac{\omega_p^2}{n_0} \int \frac{f(V) \, dV}{(\omega - kV)^2},
\]
### Table 5.1

Weak coupling of two propagating waves

<table>
<thead>
<tr>
<th></th>
<th>Phase velocities</th>
<th>Energies</th>
<th>Dispersion relation</th>
<th>Properties of roots</th>
<th>Behaviour of system</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>+, +</td>
<td>+, +</td>
<td>( (k - \frac{\omega}{V_t})(k - \frac{\omega}{V_s}) = k_0^2 )</td>
<td>k real for all real ( \omega ) and ( \omega ) real for all real ( k )</td>
<td>Propagating waves. Stable.</td>
</tr>
<tr>
<td>B</td>
<td>+, -</td>
<td>- , -</td>
<td>( (k - \frac{\omega}{V_t})(k + \frac{\omega}{V_s}) = -k_0^2 )</td>
<td>There are complex roots of ( k ) for real ( \omega ) and ( \omega ) real for all real ( k )</td>
<td>Evanescent waves. Stable.</td>
</tr>
<tr>
<td>C</td>
<td>+, +</td>
<td>+ , -</td>
<td>( (k - \frac{\omega}{V_t})(k - \frac{\omega}{V_s}) = -k_0^2 )</td>
<td>There are complex roots of ( k ) for real ( \omega ) and all double roots of ( k ) occur for real ( \omega ). There are complex roots of ( \omega ) for real ( k ).</td>
<td>Convective instability.</td>
</tr>
<tr>
<td>D</td>
<td>+, -</td>
<td>, + , -</td>
<td>( (k - \frac{\omega}{V_t})(k + \frac{\omega}{V_s}) = k_0^2 )</td>
<td>k real for all real ( \omega ) ( \Rightarrow ) There are complex roots of ( \omega ) for real ( k ): ( \omega = \omega_0 = -2i k \left( \frac{1}{V_t} + \frac{1}{V_s} \right) )</td>
<td>Absolute instability</td>
</tr>
</tbody>
</table>
with number density

\[ n_e = \int f(V) \, dV \, . \]

Note that in a relativistic treatment \( \omega_p^2 \) has to be replaced by \( \omega_p^2/\gamma \). Assuming the function \( f(V) \) to be well-behaved \([f(V) \to 0 \text{ for } V \to \infty]\), this equation may be simplified by integrating by parts

\[ 1 = - \frac{\omega_p^2}{k^2 n_e} \int \frac{2E/3V}{\omega/k - V} \, dV \, . \quad (5.21) \]

For a Maxwellian plasma, with the one-dimensional distribution function of Subsection 1.2.1, the integral of Eq. (5.21) is found to be

\[ \omega^2 = \omega_p^2 + \frac{3kT}{m} k^2 + i \left( \frac{\omega}{k} \right)^{3/2} \frac{1}{\sqrt{2}} e^{-i m \omega^2 / (2k^2 \gamma^2) \lambda_D^2} \, , \quad (5.22) \]

where Boltzmann's constant \( k' \) is to be distinguished from the wave number \( k \). For a cold plasma \((T = 0)\) we find the longitudinal undamped plasma oscillation frequency \( \omega = \omega_p \) again. In the case of a warm plasma, where the waves travel much faster than the electrons \((\omega/k > V)\), we may write approximately

\[ \omega^2 = \omega_p^2 + \frac{3kT}{m} k^2 = \omega_p^2 \left[ 1 + 3k^2 \lambda_D^2 \right] \, , \]

where \( \lambda_D \) is the Debye shielding distance defined in Subsection 1.2.1.

![Graph](image)

**Fig. 5.10** Oscillation frequency for a plasma of temperature \( T \)

The diagram of Fig. 5.10 shows that the frequency increases slightly as \( k \) increases. When the phase velocity approaches \( \sqrt{kT/\gamma} \), the imaginary term of Eq. (5.22) cannot be neglected. As a consequence, \( \omega \) is complex and the waves are damped in spite of the absence of collisions. This surprising damping mechanism, in literature normally called "Landau damping", occurs when the wave phase velocity is in the vicinity of the thermal velocity of the electrons. It has its origin in the strong interaction between a longitudinal plasma wave and those particles which are in synchronism with these waves. If the resonant particles are in
the tail of the distribution function, as indicated by Fig. 5.11, then there are more slow particles that are accelerated than fast particles that are decelerated. Hence energy is removed from the wave and damping occurs. This statement is examined in more detail below.

![Graph showing Landau damping](image)

**Fig. 5.11** Landau damping of waves if the wave phase velocity is near the thermal velocity of the particles

Conversely, if $\partial f/\partial V$ is positive at a value of $V$ which is equal to $\omega/k$ for the wave, energy is added to the wave resulting in an amplification. If resonance takes place at phase velocities where the distribution function $f(V)$ has a flat top, there is neither damping nor amplification. What happens then for a velocity distribution like the one shown in Fig. 5.12? There is a theorem, the so-called Penrose criterion, which gives a necessary and sufficient condition for instabilities for such distribution functions. It states that if $f(V)$ has a minimum at some $V = V_{\text{min}}$ and if

$$\kappa = \int \frac{f(V) - f(V_{\text{min}})}{(V - V_{\text{min}})^2} \, dV > 0,$$

then there is an instability. Though this criterion is quoted without proof, we may note that the required minimum of the distribution function implies at least two maxima so there are at least two identifiable streams which may cause a two- or multi-stream instability, as has been discussed before.

![Graph showing unstable, marginal, and stable distributions](image)

**Fig. 5.12** Unstable, marginal, and stable distributions
Many papers have been written on Landau damping, and it is discussed in most texts on plasma physics. The original approach of Landau, with slight corrections, is well typified in the paper by Jackson, and in Chapter 8 of Ref. 4. Earlier we stated that, since more particles are moving faster than the wave than slower than the wave, this represents a net energy loss from the wave. This is perhaps not immediately evident, since we know that a particle moving in a wave which travels with fixed velocity, on the average, does not gain or lose energy.

The difficulty arises because we are discussing a resonant situation, and consequently the initial conditions need to be specified in an appropriate manner. Because of the reversibility of the basic equations, a solution with antidamping is also physically admissible. (This situation is true also for Čerenkov radiation, where the "advanced potential" solution is also valid; both phenomena are concerned with resonant interaction of particles and waves).

This feature is well brought out in a treatment by Dawson, in which he considers two groups of electrons: the "main" electrons, and a resonant group with velocity \( \omega/k \pm \Delta v \) (see Fig. 5.11). He considers first a system containing only the main electrons, this does not show damping; the electron velocities and positions are correlated in such a way that a wave with velocity \( \omega/k \), corresponding to the central velocity of the missing resonant electrons, is present. The missing electrons are then inserted at \( t = 0 \), with no spatial structure; they interact with the wave already present in such a way that they extract energy. To calculate this requires a detailed argument; Dawson does the requisite calculation and finds an energy gain which agrees with the amount expected from Landau damping.

That a net energy gain by the particles occurs can be seen physically by considering electrons initially at all phases of the wave, and inquiring what happens next. If the electrons are faster than the wave, bunching occurs in such a way that more electrons are slowed down than are speeded up, and vice versa.

If one considers an infinite system, or alternatively a system with circular geometry (as in an accelerator), it is relatively easy to formulate the problem essentially in terms of time rather than \( x \) and \( t \). It then appears analogous to a lossless continuum of resonators driven by a frequency within the continuum. This approach provides clear physical insight into what is happening, and has been developed at length by Hereward. We describe some of the essential features in the next subsection.

5.4.1 Response of a continuum of lossless resonators

Consider a continuum of lossless resonators, which are driven by a sinusoidal force \( f(\Omega) \) of maximum amplitude \( X \) and frequency \( \Omega \). For a single resonator of natural frequency \( \omega \) we may write

\[
X + \omega^2 x = f(\Omega) = X \cos \Omega t .
\]  

(5.23)

If we take as initial condition \( x = \dot{x} = 0 \) for \( t \leq 0 \), we readily find the solution

\[
x(\omega t) = \frac{2X}{\omega^2 - \Omega^2} \sin \left( \frac{\omega - \Omega}{2} t \right) \sin \left( \frac{\omega + \Omega}{2} t \right), \quad t > 0 .
\]
On resonance, $\omega = \Omega$, the amplitude of a resonator increases linearly with time. Close to resonance, where $|\omega - \Omega| \ll 1/t$, a resonator behaves approximately like the on-resonance one. Initially its amplitude increases with time and it absorbs energy from the applied force. This is illustrated in Fig. 5.13a, where the amplitude is drawn of resonators in the three regions of interest. Figure 5.13b shows a plot of the corresponding absorbed energy.

Let us consider the total power absorbed by the system of oscillators. It is proportional to the square of the amplitude integrated over all natural frequencies. If the rapidly oscillating part $\sin^2[(\omega + \Omega)/2]$ is replaced by its mean value $\frac{1}{2}$, we arrive at

$$W = \frac{X^2 t}{4\pi^2} \int_0^\infty \frac{\sin^2 \left(\frac{(\omega - \Omega)t/2}{2}\right)}{\left[(\omega - \Omega)t/2\right]^2} \, d\left(\frac{\omega - \Omega}{2} \right),$$

where we assumed that the frequencies which contribute the most to the integral are close to the frequency $\Omega$ of the exciting force. The integration may be performed by substituting $z = (\omega - \Omega)t/2$; hence

$$W = \frac{\pi X^2}{8\Omega^2} \, t. \quad (5.24)$$

We can conclude that as time goes on there is a decreasing band (of width $\sim 1/t$) of oscillators of which the amplitude increases linearly with time and hence their individual energy is proportional to the square of time. This explains the linear relationship (5.24) between the total absorbed energy $W$ and time. Therefore the continuum of oscillators behaves as an absorber of energy, as a pure resistance, though there is no real dissipation of energy.

It is interesting to note that the energy of the system is hidden in that it increases without showing up as an increase of the averaged amplitude $\langle x \rangle$ of the oscillators, because it is progressively concentrated into a decreasing number of oscillators. More detailed analysis was carried out by Hereward, who shows that the response of the system (5.23) of oscillators of a well-behaved distribution function $G(\omega)$ is given by
\[ \frac{\langle x \rangle}{f(\Omega)} = \frac{1}{2\pi} \left[ \text{P.V.} \int \frac{G(\omega)}{\omega - \Omega} \, d\omega - \text{i} \pi G(\Omega) \right], \quad (5.25) \]

where P.V. denotes Cauchy's principal value. Furthermore, it is pointed out that the resistive term of the response (5.25) could be obtained equivalently by putting a damping term \( \epsilon \dot{x} \) in Eq. (5.23) and evaluating the response \( \langle x \rangle/f(\Omega) \) in the limit \( \epsilon \to 0 \).
6. WAVES ON BEAMS OF FINITE DIAMETER

In the previous section we studied electromagnetic waves in free space. We saw that, by superposition of plane waves, boundary conditions can be satisfied for all types of geometries. Thus for a rectangular waveguide one needs only four plane waves, whereas for a circular waveguide a manifold of plane waves is required in order to comply with the boundary conditions. It was pointed out that the difference between the wave propagation in waveguides of different structure is physically trivial, although the mathematical description of the fields may be very distinct.

We also looked at electromagnetic waves propagating in a plasma at rest. It was verified that the dielectric constant is a tensor rather than a scalar. This anisotropy led to dispersion equations which were different for the longitudinal and transverse wave propagation. Then by introducing streaming motions in plasma beams we discovered that resonant interaction between plasma waves and electrons can take place, thus leading to growth of the waves or to Landau damping.

So far we have considered only plasma beams of infinite extent. For a finite geometry, in principle, we can proceed as for plane electromagnetic waves bounded by media or waveguides, namely by superposition of a set of plane waves to meet the requirements of the boundaries. The general aim is as in Subsection 5.2, first to derive the conductivity tensor from the dynamics of the charged particles, and then to introduce it into Maxwell's equations to find the refractive indices and polarizations of the characteristic waves but for the finite geometry. Then by integration over the velocity distribution of a large or infinite number of beams the dispersion relation may be obtained. However, since we have already discussed plasma kinetic theory in Section 2, we shall use the Vlasov equation in connection with Maxwell's equations to derive the dispersion relation for a plasma which is bounded to form a cylinder.

6.1 Dispersion relation for longitudinal waves on a beam of finite transverse dimensions

The finite-beam analysis is simplified by assuming that an infinite static axial magnetic field exists so that no transverse currents are possible. This situation is a good approximation for the case where a large longitudinal magnetic field is required in order to focus the particle beam. Equivalently we could assume that there are transverse focusing fields which constrain the beam to one direction but do not influence its longitudinal motion.

As we shall consider relativistic streaming motions, Eq. (2.3b) is the appropriate Vlasov equation to describe the dynamic behaviour in the longitudinal direction, say parallel to the z-axis:

\[
\frac{\partial f}{\partial t} = - v \frac{\partial f}{\partial z} - q (E_z + (\nabla \times B)_{\parallel}) \frac{\partial f}{\partial p_z},
\]

(6.1)

where \( p = mv \gamma \) is the linear momentum in the z-direction and \( v \) the corresponding velocity. Due to our assumption above, the static magnetic field is then parallel to the z-axis and thus the second term in the bracket of the Vlasov equation becomes zero. The difference from the non-relativistic Vlasov equation (2.3a) is immediately seen by noting that

\[
\frac{\partial f}{\partial p} = \frac{1}{mv^2} \frac{\partial f}{\partial v}.
\]

(6.2)
The unperturbed beam is uniform in azimuth and constant in time, so it may be described by a distribution function

\[ f_d(v) = n_0 \delta \left( \frac{P}{M_Y} - v \right) \]

with the normalization

\[ \int f_d(v) \, dv = n_0 \]

which, multiplied by the unit charge \( q \), equals the averaged unperturbed charge density:

\[ \rho = q n_0 = q \int f_d(v) \, dv . \]  \hspace{1cm} (6.3)

We consider now an infinitesimal perturbation that allows us to write the distribution function as

\[ f(v,z,t) = f_d(v) + \tilde{f}(v,z,t) . \]  \hspace{1cm} (6.4)

Inserting Eq. (6.4) into Eq. (6.1), and linearizing, results in

\[ \tilde{f}(v,z,t) = - \frac{q}{m} \frac{\partial f_d}{\partial p} \int \frac{\tilde{E}}{\tilde{v}} \, dt . \]

Now let

\[ \tilde{E} = \frac{E}{E} \exp \left[ i(\omega t - \omega z) \right] , \]  \hspace{1cm} (6.5)

with \( z = pt/(m\gamma) \), and the integration yields

\[ \tilde{f}(p,z,t) = i \frac{q}{m \gamma} \frac{\tilde{E}}{\omega - kv} \frac{\partial f_d}{\partial v} , \]  \hspace{1cm} (6.6)

where we made use of relation (6.2).

Maxwell's equation for the perturbed electric field inside the beam is

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{E} = \frac{1}{\varepsilon_0} \nabla \tilde{\rho} + \mu_0 \frac{\partial}{\partial t} (\tilde{\beta} v) . \]  \hspace{1cm} (6.7)

This inhomogeneous wave equation for \( \tilde{E} \) accounts for the presence of a perturbed axial charge density \( \tilde{\rho} \) and a perturbed current density \( \tilde{\beta} v \), where

\[ \tilde{\beta} = q \int \tilde{f}(v) \, dv . \]  \hspace{1cm} (6.8)

Splitting the differential operator \( \nabla^2 \) into the transverse and the longitudinal part, \( \nabla^2 = (\partial^2/\partial z^2) + (\nabla^2/\partial z^2) \), from Eq. (6.7) it follows, with Eqs. (6.5) and (6.6), that
\[
\left[ v_1^2 - k^2 + \frac{\omega^2}{c^2} \right] \vec{E} = \frac{q}{\varepsilon_0} \int \frac{\partial \vec{F}}{\partial z} \, dv + \omega_0 \int \frac{\partial \vec{F}}{\partial t} \, v \, dv
\]
\[
= \left[ - \frac{q^2}{\varepsilon_0 m \gamma^3} \int \frac{(\omega v/c^2) - k}{\omega - kv} \, \frac{\partial F}{\partial v} \, dv \right] \vec{E} .
\]

Integration by parts leads to an operator of the form
\[
v_1^2 + \left( \frac{\omega^2}{c^2} - k^2 \right) \left[ 1 - \frac{q^2}{\varepsilon_0 m \gamma^2} \, I(\omega, k) \right] = 0 .
\]

After another integration by parts of the integral
\[
I(\omega, k) = \int \frac{F_0}{(\omega - kv)^2} \, dv .
\]
the dispersion relation for a transverse finite plasma stream is obtained:
\[
v_1^2 + \left( \frac{\omega^2}{c^2} - k^2 \right) \left[ 1 - \frac{q^2}{\varepsilon_0 m \gamma^2} \int \frac{\partial F / \partial v}{(\omega/k) - v} \, dv \right] = 0 . \tag{6.9}
\]

For non-relativistic velocities of the particles we may put \( \gamma = 1 \).

Substitution of unperturbed and perturbed charge density, Eqs. (6.3) and (6.8), respectively, into the hydrodynamic equations (5.13) determines \( \tilde{\rho} \) and \( \tilde{\nu} \) as a function of known quantities. It is left as an exercise for the reader to show that introducing these perturbed quantities into Maxwell’s equation (6.7) yields the dispersion relation (6.9). So this example reveals the equivalence of the hydrodynamic equations (5.13) and the collisionless Boltzmann or Vlasov equation (2.3).

For an infinite plasma, \( v_1^2 = 0 \), we obtain from Eq. (6.9) already known dispersion relations in their relativistic notation:
\[
1 = \frac{\omega_p^2}{\gamma^2 k^2 n_0} \int \frac{\partial F / \partial v}{(\omega/k) - v} \, dv . \tag{6.10}
\]

For a beam of constant streaming motion, \( F_0 = \delta(v) \), we obtain
\[
1 = \frac{\omega_p^2}{\gamma^2 (\omega - kv)^2} . \tag{6.11}
\]

This relation again gives the dispersion relation plotted in Fig. 5.6, with \( \nu k^2 = \omega + \omega_p / \gamma \). For a stationary cold plasma there is simply
\[
1 = \frac{\omega_p^2}{\omega^2} .
\]
and for a stationary hot plasma we may apply Eq. (6.10), but with $\gamma = 1$. Note that the plasma frequency $\omega_p$ measured in the proper frame of the particle beam is

$$\omega_p^2 = \frac{q^2 n_e}{\gamma c m}.$$  

We also get from Eq. (6.9) the "dispersion relation" for a light wave, i.e.

$$\omega = ck.$$  

This corresponds to transverse waves, which do not interact with the charges since they are prevented by the magnetic field from moving perpendicular to the z-direction.

6.2 Plasma beam bounded by a resistive cylinder

Now we try to solve Eq. (6.9) for a cylindrical beam which is centred in a conducting cylinder as shown in Fig. 6.1. The cylinder wall supports image charges which tend to reduce the strength of the axial space-charge field.

Fig. 6.1 Plasma beam in a conducting cylinder

If we assume the wavelength to be large compared to the radii $a$ and $b$ the dispersion equation (6.9) may be written

$$V_1^2 \gg k^2 c^2,$$  

$$V_1^2 + T_0 \left( \frac{\omega_p^2}{\gamma^2 k^2 n_0} \int \frac{\partial f_e/\partial v}{\omega/k-v} \, dv \right) = 0,$$  

where

$$T_0^2 = k^2 (1 - \beta_w^2),$$  

with

$$\beta_w = \frac{\omega}{ck}$$  

the normalized phase velocity of the wave. To solve Eq. (6.13) we need to solve the transverse wave equation

$$[V_1^2 + T^2] E_z - H_\phi = 0$$  

(6.14)
with the appropriate boundary conditions. For no angular ($\phi$) variation and the lowest mode this has been done by Birdsall and Whinnery for arbitrary impedance $Z_a = -E_z/H_y$ at the edge of the beam\textsuperscript{18}). Their result is

$$Z_a = \frac{i T_0^2 \ J_0(T_a)}{\omega_0 T \ J_1(T_a)} .$$

With the assumptions $\lambda > b/\gamma_w$, the Bessel functions $J_0$ and $J_1$ may be approximated by the first two terms, thus

$$Z_a \approx \frac{2i T_0^2}{\omega_0 a T^2} \left[ 1 - \frac{(T_a)^2}{4} \right] \approx \frac{2i T_0^2}{\omega_0 a T^2} \left[ 1 - \frac{(T_a)^2}{8} \right] .$$  \hspace{1cm} (6.15)

In the case $b > a$, the plasma stream does not fill the space cut to the wall, so that the impedance $Z_a$ seen by the stream is that of the wall impedance $Z_w$ transformed through the space between the wall and the stream essentially by cut-off transmission line equations.

The result in the approximation (6.12), obtained from the more general formula given by Birdsall and Whinnery, is

$$Z_a = \frac{a}{b} \left( \frac{i T_0}{\omega_0} \right) \frac{Z_w + \frac{i T_0^2}{\omega_0} \ b \ln \frac{b}{a}}{1 + \frac{i T_0}{\omega_0} \ \frac{b}{a} \left( \frac{b}{a} + \frac{a}{b} \right) T_0 a} .$$  \hspace{1cm} (6.16)

Assume $Z_w$ small enough so that the $Z_w$ term in the denominator can be neglected. Then Eqs. (6.15) and (6.16) yield

$$\frac{-1}{T^2} = \frac{i \omega_0 a^2}{2 T_0^2 b} \left[ Z_w + \frac{i T_0^2 b}{\omega_0} \left( \frac{1}{4} + \ln \frac{b}{a} \right) \right] .$$

This, together with Eq. (6.14) substituted back into the dispersion relation (6.13), results in

$$1 = \frac{b k}{\epsilon_0 \beta_b c} \left[ 1 - \epsilon_k^2 \right] \left( \frac{1}{4} + \ln \frac{b}{a} \right) - i Z_w \ \frac{a^2 \epsilon_0 \beta_b c^2 \omega_0^2}{2 b y^2 n_b k} \int \frac{\partial f_\delta(v)}{\omega/k - v} \ dv .$$  \hspace{1cm} (6.17)

We examine this equation first with two simplifications. We take $f_\delta(v)$ as a $\delta$-function, and $Z_w = 0$, so that the wall is purely conducting. Hence Eq. (6.17) becomes

$$1 = \frac{A k^2 \omega_0^2}{\gamma (\omega - k v)^2} ,$$  \hspace{1cm} (6.18)

where

$$A = \frac{b^2}{2} \left[ 1 - \beta_b^2 \right] \left( \frac{1}{4} + \ln \frac{b^2}{a^2} \right) .$$
This is represented by two straight lines through the origin. We must remember, however, that we have imposed assumptions (6.12); if these are not imposed the curves become asymptotic to the lines \( \omega - kv = \omega_p/\gamma \) already shown in Fig. 5.6. This is demonstrated in Fig. 6.2. Note that in the limit of \( \omega_p \to 0 \), that is no beam current, the curves degenerate to the single line \( \omega = kv \).

![Graph showing dispersion function for a beam with \( \delta \)-function velocity distribution in a tube with lossless wall](image)

### 6.3 Negative mass and longitudinal resistive instabilities of coasting beams in particle accelerators

Before we establish a dispersion equation for circular accelerators, let us demonstrate why Eq. (6.17) may predict an instability by choosing \( f_b(v) = n_0 \delta(v - v_0) \), which represents a beam with all particles having the same velocity. Then the dispersion relation (6.17) is of the form

\[
1 = (A - iB) \frac{k^2 \omega_p^2}{(\omega - kv)^2},
\]

from which it follows that

\[
\omega = kv \pm \frac{k \omega_p}{\sqrt{2}} \left\{ \sqrt{A^2 + B^2} + A - i \sqrt{A^2 + B^2} - A \right\}.
\]

The plasma frequency \( \omega_p \) is

\[
\frac{\omega_p}{\gamma} = \sqrt{\frac{n_0 A}{e^2 m^*}},
\]

where \( m^* \) is the effective mass already defined in Subsection 3.1.4:
\[ m^* = -\frac{ym}{\eta}, \quad \eta = \frac{1}{\gamma^2} - \frac{1}{\gamma^2}. \]  

(6.21)

Above transition energy, \( \eta > 0 \), the plasma frequency is purely imaginary: \( \omega_p = i|\omega_p| \). Remember now that we always assume a perturbation of form \( \exp(i\omega t) \), hence \( \text{Im} \,(\omega) < 0 \) means growth and \( \text{Im} \,(\omega) > 0 \) results in damping of the perturbation. With these preliminaries, different cases can be distinguished according to whether \( B = 0 \) (wall impedance \( Z_w \) reactive) or not. Table 6.1 shows the situations in which growth of the plasma waves may occur. For \( B \neq 0 \) the sign of \( A \) does not matter, but it is obvious that above transition energy, in the régime of the "negative mass", an instability may arise. This is the so-called "negative mass instability".

**Table 6.1**

**Coefficients of the dispersion relation**

<table>
<thead>
<tr>
<th>( B = 0 )</th>
<th>( A &gt; 0 )</th>
<th>( A &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Below transition ( m^* &gt; 0 )</td>
<td>Both waves stable</td>
<td>Slow wave unstable</td>
</tr>
<tr>
<td>Above transition ( m^* &lt; 0 )</td>
<td>Slow wave unstable</td>
<td>Both waves stable</td>
</tr>
<tr>
<td>Fast wave damped</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( B \neq 0 )</th>
<th>( A \geq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Below transition ( m^* &gt; 0 )</td>
<td>Slow wave unstable</td>
</tr>
<tr>
<td>Fast wave stable</td>
<td></td>
</tr>
<tr>
<td>Above transition ( m^* &lt; 0 )</td>
<td>Slow wave stable</td>
</tr>
<tr>
<td>Fast wave unstable</td>
<td></td>
</tr>
</tbody>
</table>

For a circular accelerator of radius \( R \) let us define

\[ v = R\delta \]
\[ k = n/R, \quad n = \text{mode number} \]
\[ N = 2\pi^2R^2n_\theta \quad \text{total number of particles} \]
\[ k_\theta = -1/(2\pi R^2m^*) \]

(6.22)

The quantity \( k_\theta \) in connection with the definition (6.20) reflects the characteristics of the accelerator guiding field. With the definitions (6.19) to (6.22) and by introducing the canonical momentum \( 2\pi W \) by

\[ \dot{\delta} = \omega_\delta + k_\theta W \quad \text{and} \quad \int f(W) \, dW = 1, \]

we may write Eq. (6.17) as

\[ 1 = iqlZ_w \int \frac{\delta f(W)/dW}{\omega - \omega_\delta} \, dW. \]

(6.24)

The coupling impedance \( Z_w \) is defined as

\[ Z_w = \frac{R}{b} \left[ Z_w + i \frac{bk}{\beta_w c_0} \left( 1 - \beta_w^2 \right) \left( \frac{1}{\delta} + \ln \frac{b}{2} \right) \right] \]
and I is the perturbed current

\[ I = \frac{\beta_c Nq}{2\pi R}. \]

In practice the propagation constant \( \beta_w \) is near \( \beta \); therefore

\[ I = Nqf_0, \quad f_0 = \frac{\beta c}{2\pi R} \quad \text{revolution frequency}. \]

The current I is related to the coupling impedance by

\[ Z_0 = \frac{\int E_z \, ds}{I} = -\frac{2\pi R}{I} \langle E_z \rangle, \quad (6.25) \]

where \( \langle E_z \rangle \) is the average retarding field generated by the perturbed current I. To compare the dispersion relation (6.24) with that of Neil and Sessler \(^{36} \), we introduce the quantities

\[ U = \frac{Nq^2 R}{4\pi \epsilon_0} \left[ (1 - \beta_w^2) \log \frac{b}{a} \right], \quad g_0 = \frac{1}{2} + 2 \ln \frac{b}{a} \]

\[ V = \frac{2Nq^2 Z_w \beta_c c}{4\pi b}. \]

Hence

\[ Z_0 = \frac{1}{qI} (U - iV) \]

and Eq. (6.24) becomes in terms of U and V:

\[ -1 = (U - iV) \int \frac{3f_i(W) / \omega W}{\omega - \mathbb{m}^2} \, dW \quad (6.26) \]

or

\[ nk_0 = (U - iV) \int \frac{3f_i(W)}{\omega W - W - W_i} \, dW, \]

where \( W_i = (\omega - \mathbb{m} \beta_c) / nk_0 \). Equation (6.26) is the dispersion relation for a circular beam of radius \( a \) centred in a circular pipe of radius \( b \) and wall impedance \( Z_w \). It is identical except for \( g_0 \) with the dispersion relation derived by Neil and Sessler \(^{36} \); their geometrical factor \( g_0 = 1 + 2 \ln (b/a) \) takes into account the averaging of the retarding field \( E_z \) on the beam axis, whereas we took \( E_z \) at the edge of the beam.

To evaluate the integral of Eq. (6.26) \( W \) must be regarded as a complex variable. The case of a beam having a velocity distribution like a \( \delta \)-function has already been discussed [see Eq. (6.19)]. For any other distribution function \( f_i(W) \) we may use Landau's prescription
\[
\int \frac{\Delta E(w)/\partial w}{\omega - n\delta} = \text{p.v.} \int \frac{\Delta E(w)/\partial w}{\omega - \delta} + i\pi \frac{\partial E(w)}{\partial w} \bigg|_{w=W_i}
\]

where P.V. denotes Cauchy's principal value. With this prescription the conformal mapping of the real W onto the complex (U,V)-plane yields the stability diagrams of Fig. 6.3, which is drawn for different distribution functions. The coordinates are

\[
U' = \frac{U}{n|k_0|\delta^2}, \quad \delta = \frac{\Delta E}{2f_0}
\]

\[
V' = \frac{V}{n|k_0|\delta^2},
\]

where \(\Delta E\) is the full energy spread, measured at the half height of the distribution function. With some of the definitions above we find a relationship between the quantities \(U', V'\) and the coupling impedance \(Z_{\parallel}\)

\[
U' - iV' = i \frac{2}{\pi} \frac{IZ_0\delta^2}{n(\Delta E/E)^2 [E_0/q]|_{Y}},
\]

(6.27)

where \(E\) is the mean energy and \(E_0\) the energy at rest of a particle.

Fig. 6.3 Stability diagrams for different distribution functions: a) and b) for parabolae of powers 2 and 5, respectively, c) for a Gaussian distribution, and d) for a Lorentz distribution (Ref. 37)
We have instability when \( \omega \) has a negative imaginary part. This corresponds to points in the \((U', V')\)-plane, which lie outside the bell-shaped curve for the distribution function under consideration. Inside the curve \( \text{Im} (\omega) \) is positive and thus there is no instability. From Fig. 6.3 we see that there is a region in the vicinity of the origin of the \((U', V')\)-plane, where for realistic distribution functions no instability can occur. It may be approximated by

\[
|U' - iV'| \leq \frac{2}{\hbar}. \tag{6.28}
\]

Combining Eqs. (6.27) and (6.28) we obtain the Kell-Schnell stability criterion\(^{38}\), which holds for reasonable distribution functions:

\[
\left| \frac{Z_n}{n} \right| \leq \left| \frac{E_n}{q} \right| \frac{\gamma}{\beta^2} \left( \frac{\Delta E}{E} \right)^2.
\]

6.4 Transverse resistive instabilities

We now consider the case of transverse waves in a "thread" beam of small transverse dimensions. Such a wave might represent completely coherent betatron oscillations on a beam with no energy spread, or filamentary cyclotron waves in a microwave tube with a \(B_z\) focusing field. A disturbance of maximum amplitude \(r_0\) in the transverse direction and propagating with velocity \(v\) in the longitudinal \((z)\) direction can be described by

\[
r = r_0 e^{-i(\omega t - k z)}.	ag{6.29}\]

\(r\) here could be \(x\) or \(y\); in general the system does not have circular symmetry, \(Q_x \neq Q_y\). Such a disturbance may appear in the form of coherent oscillations of particles in an accelerator due to the external field which determines the transverse oscillation frequency \(Q_0\); \(Q\) is the number of betatron wavelengths per revolution and \(\omega_0\) equals the angular frequency of the particles. The corresponding force term is

\[
\mathbf{f} = -Q_0^2 r_0^2 e^{iQ t}. \tag{6.30}\]

With the total differential for the transverse displacement with respect to time

\[
\frac{dr}{dt} = \frac{\partial r}{\partial t} + v \frac{\partial r}{\partial z},
\]

Eqs. (6.29) and (6.30) may be combined to yield

\[
(\omega - kv)^2 = Q_0^2 r_0^2. \tag{6.31}\]

This dispersion relation resembles the one plotted in Fig. 5.6 for longitudinal waves in an infinite plasma. It is essentially different from Eq. (6.19) with \(B = 0\). Equation (6.18) represents two straight lines through the origin, whereas (6.31) represents parallel lines. Because of the accelerator restoring force, the particles oscillate even when the particle density is vanishingly small, whereas for longitudinal oscillations, where only space-charge forces act on the particles, the frequency becomes zero for vanishing density. Therefore we can state that betatron oscillations are a special form of transverse oscillations.
If the beam due to a transverse displacement interacts with its surrounding walls, there is another force which contributes to the dispersion equation (6.31). This additional force may provoke an instability in which a dissipative mechanism is supplied by the resistivity of the walls. In the LNS theory\(^{39}\) (Laslett, Neil and Sessler) an analysis of this instability is given for various geometries and a beam with uniform density in the azimuthal direction. Let us just quote the results for the case where the beam and the beam pipe have circular cross-sections as shown in Fig. 6.1.

For a beam rigidly displaced in the r-direction it is possible to derive simple expressions for the perturbed fields \(\vec{E}_z\) and \(\vec{B}_z\) acting on the moving charges. These expressions are valid provided the wavelength of the perturbation is very much larger than the radius of the pipe. From these fields follows the transverse force per unit charge:

\[
\frac{F}{q} = \vec{F}_z - v \vec{B}_z = -\frac{q N R \omega_0}{4\pi \varepsilon_0 a^2 R} \left[ \frac{1 - (a/b)^2}{\gamma^2} - \frac{2 \varepsilon_0 v^2 a^2 (1 + i) \omega}{\omega b^4} \right],
\]

(6.32)

where \(N\) is the total number of particles distributed around the accelerator of radius \(R\) and \(v\) is the particle velocity. Form Eq. (6.32), which is only correct to first order in the wall impedance \(Z_w\), the force term per unit displacement is readily found to be

\[
\frac{F}{m_0 \gamma r_0} = 2 Q \omega_0 [U + (1 + i) V],
\]

where

\[
U = -\frac{N \gamma^2 [1 - (a/b)^2]}{8 \pi \varepsilon_0 a^2 R m_0 \gamma \omega \omega_0},
\]

\[
V = \frac{N \gamma v^2 a^2}{4 \pi^2 b^4 R m_0 \gamma \omega \omega_0}.
\]

Adding this to the left-hand side of Eq. (6.31), we obtain

\[1 = -2 Q \omega_0 [U + (1 + i) V] \frac{1}{(\omega - kv)^2 - Q^2 \omega_0^2}.
\]

This relation is the dispersion equation for a beam in which all the particles have the same maximum amplitude \(\rho\) of axial betatron oscillations, and velocity \(v\), and thus the same canonical momentum \(W\). The LNS theory treats the more general case in which distribution functions \(f_1(W)\) and \(h(\rho)\) for the constants of motion \(W\) and \(\rho\) are assumed, where \(f_1(W)\) is normalized as in Eq. (6.23) and \(h(\rho)\) as

\[
\int h(\rho) d\rho = 1.
\]

The dispersion equation then is derived either by integrating over all velocity streams as in Section 5 or, following LNS, from the linearized Vlasov equation; it is

\[
1 = Q \omega_0 [U + (1 + i) V] \int \frac{[\rho h(\rho)] \omega_0 f_1 W \rho^2 d\omega dW}{(\omega - kv)^2 - Q^2 \omega_0^2}.
\]

(6.33)
Equation (6.33) is approximate in that it neglects azimuthal density variations and the dependence of Q on the betatron amplitude \( \rho \). By analogy with the longitudinal coupling impedance (6.25) we introduce a transverse coupling impedance by integrating the Lorentz force field (6.32) over the accelerator circumference and dividing by the coherent amplitude \( r_z \) and the current \( I \); hence

\[
Z_t = \frac{1}{\text{sgn} r_z} \int F \, ds.
\]

For realistic particle beams, a stability criterion can be derived from the dispersion relation (6.33):

\[
|Z_t| \leq \frac{n |E_x/q| \beta \nu Q}{RI} \left( |n - Q| \eta + \beta^2 \frac{3Q}{3(\Delta E/E)} \right) \left( \frac{\Delta E}{E} \right),
\]

where \( n \) is the mode number as defined in Subsection 6.3. The LNS theory shows that the modes with \( n > Q \) are unstable. (We could also see this immediately from the fact that such waves are slow negative energy waves with positive phase velocity, so that dissipation causes them to grow. For \( n < Q \) the phase velocity of the slow waves is opposite to that of the group (i.e. particle) velocity, so that dissipation causes attenuation.) Therefore \( \eta \) and \( 3Q/3(\Delta E/E) \) should have the same sign in order to avoid the cancellation of the two terms in the sum of (6.34). It is also interesting to note that the upper limit of the transverse coupling impedance depends linearly on the energy spread of the beam, whereas the upper limit of the longitudinal coupling impedance is proportional to the square of the energy spread. Illustrative examples for the order of magnitude of the impedances \( Z_\parallel \) and \( Z_t \) are given by Keil\(^{39} \) for typical geometrical structures of the Intersecting Storage Rings at CERN.

6.5 Transverse cyclotron waves

In the previous subsection we considered transverse betatron waves on a beam; four waves are possible, one fast and one slow \[\text{[Eq. (6.31)]} \] for vertical oscillations and a similar pair for radial oscillations. An alternative configuration studied in connection with travelling wave tubes is that with a \( B_z \) field, so that particles oscillate with the cyclotron frequency \( \omega_c \). In such a system we again have four waves, but they are usually classified in a different way. Corresponding to Eq. (6.31) we have the dispersion relation

\[
(\omega - kv)^2 = \omega_c^2,
\]

which corresponds to a pair of cyclotron waves. The particles move in spirals composed of a rotation at the cyclotron frequency, plus motion in the \( z \)-direction. The particles of the two waves may be thought of as arranged along the threads of a right- and left-handed screw, respectively, rotating with \( \omega_c \) and moving forward with velocity \( v \). A screw being screwed into a stationary nut represents the particular case of \( \omega/k = 0 \).

The other two waves are the synchronous waves; the particles do not rotate, they simply move along the field lines. The particles themselves form a spiral of any pitch, left- or right-handed, so that these waves have the dispersion relation
\[ \omega = 2\omega_\nu = t_v/\lambda, \]

where \( \lambda \) can have any value. (To produce such waves, take a garden hose from which the water emerges with velocity \( v \), and rotate it in a plane perpendicular to that of flow with frequency \( 2\omega_\nu \).)

It is left to the reader to relate the betatron and cyclotron waves more precisely by making a Larmor transformation into rotating coordinates as discussed in Subsection 2.2.2. The magnetic field is transformed into a radial electric field, with \( Q_x = Q_y \). A discussion of the coupling of these waves to circuits, and the subsequent "instabilities", is given in Louisell's book \(^1\). The "transverse TWT" principle has been demonstrated experimentally, but is not now used in practical devices. A further text which can be recommended for discussing both longitudinal and transverse waves on electron beams in clear physical terms is given by Johnson \(^2\).

6.6 Concluding remarks

An attempt has been made in the last two sections to relate the essential physical properties of continuous beams in accelerators and storage rings to the corresponding problems in plasma physics and microwave tubes. Only a very broad comparison is, of course, possible since in all these fields particular and practically useful calculations rapidly become very specialized. Nevertheless, the physical ideas are associated both with the occurrence of instabilities and with Landau damping of universal application, and thus fresh insights can be obtained by comparing related fields.
REFERENCES


4) P.C. Clemmow and J.P. Dougherty, Electrodynamics of particles and plasmas (Addison-Wesley, Reading, Mass., 1969). This text contains a good discussion of this type of sheet, p. 253, and also of Debye shielding.


10) I.M. Kapchinskij, Particle dynamics in resonant linear accelerators (Atomizdat, Moscow, 1966). (Some parts were translated by B. Schnitzer as CERN MPS/LIN notes.)


21) R.C. Davidson and J.D. Lawson, Particle Accelerators 4, 1 (1972).


