A SCALING LAW DERIVED FROM A BROADBAND IMPEDANCE APPLICATION TO SPEAR

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The bunch length in high-brightness synchrotron radiation sources is an important performance parameter. It is critically dependant on the \( \mu \)-wave instability. Usually the SPEAR scaling law is used to compute the expected bunch length. In this paper we show that the SPEAR scaling law is compatible with a broadband impedance. This makes it possible to calculate the appropriate scaling law for a machine like the one proposed in Berkeley assuming that the impedance is known from measurements and/or calculations.

1. INTRODUCTION

A new 1–2 GeV/c synchrotron light source is proposed for LBL. In Ref. 1 an overview of the performances is given. For the emittance growth and the beam lifetime two distinct curves are considered for these items depending on whether the SPEAR scaling is assumed. The difference can be as large as a factor of two. Therefore it is legitimate to ask whether the exponent of the SPEAR scaling law\(^2\) is applicable to this new machine.

The scaling law relates an intensity parameter \((\xi)\) with the RMS bunch length \((\sigma)\). In Section 2 we derive this relation. A value for \(Z/n\) appears in this expression. The experimental data for SPEAR suggest an exponential dependance of this impedance on the bunch length. In Section 3 this \(Z/n\) factor is computed from a broadband impedance. A reasonable choice of the resonator parameters indeed produces the correct experimental exponential dependance on bunch length.

2. THE IMPEDANCE SCALING LAW

The instability that is responsible for the blow-up of the bunch length is a high-frequency longitudinal instability also known as the \( \mu \)-wave instability. The criterion for the threshold is, following Boussard,\(^3\):

\[
(Z/n)\hat{I} = 2\pi(dp/p)^3(E/e) |\eta| \beta^2 d^{-1},
\]

where \(\hat{I}\) is the peak current in the bunch, \(dp/p\) the half-momentum spread, \(\eta\) the phase slip factor, \(E\) the beam energy, \(\beta\) the relativistic velocity factor (which we
set equal to 1), and $d^{-1} = 0.175$, a factor derived from an analysis involving a stability diagram.

The following substitutions are used to modify Eq. (1):

- For bunches that are short with respect to the RF bucket it is possible to write
  \[ \frac{dp}{p} = 2\tau_e \Omega Q_\gamma / |\eta|, \]
  where $Q_\gamma$ is the synchrotron tune, $\Omega$ the angular revolution frequency, and $\tau_e$ the RMS bunch length(s).
- The peak current can be written in terms of the average current and the RMS bunch length:
  \[ \bar{I} = I_0 \sqrt{(2\pi) R / \sigma_\epsilon}, \]
  where $R$ is the machine radius and $\sigma_\epsilon$ is the RMS bunch length (m).
- For very high $\gamma$ (the momentum factor) it is possible to replace $\eta$ by the expression:
  \[ \eta = - D / R, \]
  where $D$ is the dispersion.

This then leads to
\[ \xi = \left[ I_0 / (Q_\gamma^2 E / e) \right] = 1.75 (R / D) (\sigma_\epsilon / R)^3 \frac{1}{z/n}, \]
where $\xi$ is the so-called scaling parameter, which depends on known or measurable quantities. For SPEAR a power law was found experimentally between $\xi$ and $\sigma_\epsilon$, with $\xi = \sigma_\epsilon^a$ ($a = 1.32$).

3. THE EFFECTIVE IMPEDANCE

In what follows the machine coupling impedance is assumed to be that of a low-$Q$ high-frequency resonator, i.e. the usual broadband model.

The criterion for the $\mu$-wave instability is usually applied for rather long bunches. In that case it is generally accepted that the $(Z/n)$ factor in Eq. (1) is simply the shunt resistance of the resonator. For the short electron bunches this fact is not so evident. At this point we wish to introduce the notion of effective impedance.

Assume that the beam consists of very long bunches, i.e., is in the limit a dc beam. The spectrum will be extremely peaked around the dc component. This beam is excited at a frequency much higher than the bunch frequency. This excitation may be applied externally or may be the result of the beam-environment interaction (wake field). The quasi-dc beam will react at the excitation frequency. The response will be a high-frequency signal modulated by the beam intensity; in other words, the bunch spectrum is shifted by the excitation frequency. The same is true for short bunches. It can be shown that the maximum resistive response of a broadband resonator occurs at its resonant frequency. Hence, we will take this frequency as the frequency of the excitation.
The power of the perturbation can be found by computing the convolution integral between the shifted power spectrum of the beam intensity and the resistive part of the impedance. The effective impedance is a fictive impedance independent of frequency in which the bunch spectrum develops the same power as the perturbation does in the broadband resonator.

The following derivation may clarify this argument.

We consider a resonator with quality factor $Q$, resonant frequency $\omega_r$, and inductance $L$. If we call $x$ the frequency normalized to the resonant frequency $(\omega/\omega_r)$ then

$$ (Z/n) = Q\omega L[x - j(x^2 - 1)]/[x^2 + Q^2(x^2 - 1)^2]. $$

The power spectrum of the bunch is

$$ h^2(\omega) = \exp[-(\omega \tau_e)^2]. $$

During the instability, the bunch spectrum will be excited in the neighborhood of the resonant frequency $\omega_r$. The effective impedance $Ze/n$ is computed with the following convolution integral:

$$ Ze/n = \int (Z/n)h^2(\omega - \omega_r)\,d\omega / \int h^2(\omega)\,d\omega. $$

The $Ze/n$ can be normalized with the inductive impedance at the revolution frequency ($QL = Z_0/n$). We will keep the resistive term and obtain the relative effective real impedance, which is a function of bunch length $\sigma_e$ and $\omega_r$. An example is given in Fig. 1 for a $Q = 1$ resonator.

At this point it is worth mentioning that the same reasoning can be applied to the transverse case of the $\omega$-wave instability, usually called the transverse-mode coupling instability. 4

![FIGURE 1 Real effective impedance, $Q = 1$ resonator.](image)
4. THE EXPERIMENTAL SPEAR SCALING LAW

The scaling parameter $\xi$ was measured in SPEAR for a wide range of bunch lengths. We reproduce the experimental data in Fig. 2, where we used linear scales.

To test the validity of the calculation of the effective impedance we will ask the following questions. Is it possible to find the resonator parameters that match the experimental data, and are these parameters reasonable? We proceed to answer these questions in the following way.

A double logarithmic plot is used that just covers the experimental data. Then we choose sets of $\omega_r$ and $Q$ for the resonator and look for sets that produce, as accurately as possible, the experimental slope of the plot. This slope is the power coefficient that earlier was called $a$. If the vertical position of the curve is adjusted it is then possible to determine the inductance $L$ of the resonator or the low-frequency value of $Z_0/n = \Omega L$. A sample is given in Table I. For 3 cases in which a good fit for the slope was obtained $Z_0/n$ was computed.

An example is shown in Fig. 3. The dots are the experimental data, whereas the curve is the scaling parameter $\xi$ computed with the following resonator parameters:

$$Z_0/n = 16.1 \Omega$$
$$\omega_r/2\pi = 1 \text{ GHz}$$
$$Q = 3.$$  

The straight line is the average slope of the curve and of the data points.

5. DISCUSSION

It was shown in the preceding section that the experimental SPEAR results can be explained by a broadband resonator. In fact, several candidate resonators can

![Figure 2: Bunch length in SPEAR.](image)

![Figure 3: Scaling law for $Q = 3$ (frequency = 1.0 GHz).](image)
TABLE I
Fitting resonators to SPEAR experimental data

<table>
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<tr>
<th>$Q$</th>
<th>$\omega_r/2\pi$ (GHz)</th>
<th>$a$</th>
<th>$Z_0/n$ (Ω)</th>
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be retained. To single out one of the resonators, information from other experiments is needed, for example, experiments that yield the low-frequency value of $Z_0/n$. Let us choose the $Q = 3$ resonator. The computation of the scaling parameter $\xi$ can easily be extended to much lower values of $\sigma_r$, assuming that the model is still valid for these very short bunches. The power coefficient $a$ can be calculated for these small bunches as well and is shown in Fig. 4.

SPEAR is not the only machine where bunch lengthening was measured and studied. This was done in the SPS as well as for proton bunches. The results, presented in a linear scale, are shown in Fig. 5. Although we have specialized the present analysis for small bunches and high $\gamma$, it is perfectly possible to extend it

![FIGURE 4](image1.png)  
**FIGURE 4** Power coefficient $a$ for $Q = 3$ resonator.

![FIGURE 5](image2.png)  
**FIGURE 5** Bunch length in SPS.
to the long SPS bunches and the low momentum factor $\gamma$. However, the measurement span in terms of $\sigma_x$ is so small in the SPS that no attempt was made to test a scaling law on these data.

REFERENCES