MASSLESS SPECTRUM AND CRITICAL DIMENSION OF THE SUPERMEMBRANE

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ABSTRACT

We show the existence of massless particles in the supermembrane. These occur in the sector of a completely collapsed membrane and receive zero vacuum energy thanks to supersymmetry. Only in the 11 dimensional supermembrane the massless states include a graviton. Thus, in the critical dimension d=11 the massless particles correspond to the supergravity multiplet. We also find that higher extended objects beyond the supermembrane have no critical dimension in which the supergravity multiplet emerges.

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In recent years we have learned that the superstring theory can provide a framework for unifying all forces, including quantum gravity. The massless states of the superstring determine the observable fields at low energy. It is natural to ask whether higher extended objects share similar properties to the superstring. In this regard one of the most important questions is the existence of massless states in higher extended objects.

The simplest extended object beyond the string is the membrane. The question of massless states in the bosonic membrane was studied by Kikkawa and Yamada [1]. By studying membrane solutions that correspond to the leading Regge trajectory and analysing quantum fluctuations, they conclude that zero-mass particles do not arise in any dimension.

Recently a Green-Schwarz type action for the supermembrane was suggested [2]. The local world-volume fermionic symmetry of the classical action requires the dimension of space-time to be only d=11,7,5,4. The question arises whether massless states could arise in this theory. This was studied by Duff et.al. [3] who considered a classical solution for a toroidal membrane propagating in a space-time with topology $R^d \times S^5 \times S^3$. It was shown that, unlike the bosonic membrane of ref.[1], the vacuum energy of quantum fluctuations was zero. This happened thanks to a cancellation between bosonic and fermionic contributions, as a result of a global space-time supersymmetry of the background membrane solution. Another attempt to find massless states was made by Mezincescu et.al.[4] who generalized the calculations of ref.[1] to the supermembrane. In a semiclassical analysis, in a particular background solution which is not supersymmetric, they find that the vacuum energy is non-vanishing and that massless states do not occur.

Our argument for massless states will be based crucially on supersymmetry. First of all, if supersymmetry remains unbroken non-perturbatively, we expect the vacuum state to have zero mass. In order to determine this non-perturbative question one may attempt to compute the Witten index for the supermembrane, which is currently under investigation. Assuming a dynamically unbroken (global) supersymmetry we immediately conclude that there are massless particles, whose quantum numbers will be determined by examining the quantization of zero modes. The zero modes in the supermembrane are analogous to the zero modes in the Green-Schwarz superstring and they alone determine the quantum numbers of the massless states. These zero modes are the only degrees of freedom of a membrane that has completely collapsed to a point.

In the absence of a non-perturbative proof, we can set up a perturbative calculation scheme with a supersymmetric background, which guarantees zero mass for the perturbative vacuum state. At this stage, one knows of such supersymmetric solutions, such as the toroidal background of ref.[2], and its generalizations [5]. The characterization
of a supersymmetric solution will be given below. In this case too, the zero modes will determine the quantum numbers of the massless states. By contrast, if the perturbative expansion is set up in a non-supersymmetric background, there is no guarantee that the perturbative vacuum will be at zero energy in lowest order. Indeed, this energy is expected to shift with interactions in an unpredictable way. On the other hand, in a supersymmetric expansion, the supersymmetry is expected to maintain the zero energy of the vacuum order by order.

The action for the supermembrane propagating in a flat d=11,7,5,4 dimensional space-time is [2]

\[ I = \frac{1}{2} \int d^9 \xi \left[ \sqrt{-g} \left( \frac{g^{ij}}{2} \Pi^\mu \Pi_{ij} - \sqrt{-g} \right) + i e^{i \phi} \bar{\psi}_I \gamma^\mu \Pi_{ij} \psi \right], \tag{1} \]

where \(X^I(\xi), \phi^I(\xi)\) are the coordinates of d=11,7,5,4 superspace, \(\xi = (r, \sigma, \rho)\) are the coordinates on the world-volume and \(g_{ij}(\xi)\) is its metric. Furthermore,

\[ \Pi^I = \partial_{\mu} X^I - i \psi \Pi_{ij} \partial_{\mu} \psi. \tag{2} \]

In d=11 and 4 \(\psi\) is a 2\(^2\) = 32 and 2\(^2\) = 4 component Majorana spinor, respectively. Recall that Majorana spinors of SO(d,1) can be defined in only d=(2,3,4)mod8 dimensions. In d=7 and 5 \(\psi\) is a 4x4 = 16 and 2x2 = 8 component symplectic Majorana spinor, respectively. Such spinors can be defined for SO(d-1,1) in d=(5,7)mod8 dimensions and satisfy the pseudo-reality condition \(\psi^* = \Pi^{i*} C_{ij} \psi\), where \(\Pi^{i*}\) is the invariant antisymmetric tensor of a symplectic group. In our case of d=7,5 the index \(A\) takes the values 1,2 indicating that the action possesses a global \(USp(2)\) invariance. We will use this group later in the classification of the zero mass spectrum. The index \(A\) is understood to be summed over all in the fermionic bilinears.

The action is invariant under global super Poincaré and local (\(\xi\) dependent) fermionic transformations, and reparametrizations as shown in ref.[2]. The equations of motion are

\[ \partial_{\mu} \Pi^I = 0, \tag{3} \]

\[ \partial_{\mu} \left( \sqrt{-g} \left( \frac{g^{ij}}{2} \Pi^I \Pi_{ij} \right) + i e^{i \phi} \bar{\psi}_I \gamma^\mu \Pi_{ij} \psi \right) = 0. \tag{4} \]

\[(1 - \Gamma) \sqrt{-g} \left( g^{ij} \Pi^I \Pi_{ij} \partial_{\mu} \psi \right) = 0, \tag{5}\]

where

\[ \Gamma = \frac{1}{6 \sqrt{-g}} e^{i \phi} \bar{\psi}_I \gamma^\mu \Pi_{ij} \Pi_{ij} \bar{\psi}_I. \tag{6} \]

On shell \(\Gamma^2 = 1\), and therefore \(\frac{1}{2} (1 + \Gamma)\) become projection operators.

The local symmetries can be used to fix a lightcone gauge which is specified by [6]

\[ \bar{X}^4 = p^4, \]

\[ g_{ij} = -\det(h); \quad h_{ab} = \delta_{ab}, \quad a, b = 1, 2, \]

\[ g_{ij} = 0, \]

\[ \Gamma^I \psi = 0, \tag{7} \]

where \(X^I = \frac{1}{\sqrt{2}} (X^a \pm X^a')\) and \(X^a = \frac{1}{\sqrt{2}} (X^a \pm X^a')\). Then in an appropriate gamma matrix representation \(\psi\) takes the form

\[ \psi = \phi(0, S), \tag{8} \]

where half of the components of \(\psi\) have been eliminated. Thus the number of real fermionic components in \(S\) is 16,8,4,2 in d=11,7,5,4 respectively.

From the gauge conditions one finds that \(X^4\) satisfies the following equations

\[ \partial_{\mu} X^4 = -\frac{1}{2p^4} [\bar{X}^4 \gamma^\mu + \partial \bar{h}] + \partial S \partial S, \quad I = 1, 2, \ldots (d - 2), \]

\[ \partial_{\mu} X^I = \frac{1}{p^4} \bar{X}^I \partial_{\mu} X_I + i \partial S \partial S. \tag{9} \]

The integrability condition for the second equation results in a constraint on the remaining transverse \(X_I\)

\[ e^{\phi} [\partial_{\mu} \bar{X}^I \partial_{\mu} X_I + i p^4 \partial_{\mu} S \partial_{\mu} S] = 0. \tag{10} \]

Furthermore, for topologically non-trivial membranes the line integral of this equation leads to a global constraint as in ref.[3]. In the quantum version of the theory we will choose to satisfy these local as well as global constraints on states. Thus the equations of motion satisfied by the remaining transverse degrees of freedom take the SO(d-2) covariant form

\[ \bar{X}^I = \epsilon^{\mu \nu \rho \sigma} \partial_{\mu} X_I \partial_{\nu} X_J \partial_{\rho} X_K - i p^4 \epsilon^{\mu \nu \rho \sigma} \partial_{\mu} S \partial_{\nu} S, \tag{11} \]
\[ \dot{S} = -e^t \partial_t X^I \Gamma^I \partial_t S \]  

(11)

Now, including the constraint (10) one can count d-3 physical bosonic degrees of freedom. Furthermore, as noted above the number of real fermion fields corresponds to 2(d-3). However, since they satisfy a linear equation of motion in the time coordinate, this corresponds to (d-3) degrees of freedom. Thus, as expected in a supersymmetric theory, the number of bosonic and fermionic degrees of freedom are equal.

The lightcone gauge of eq.(1) is maintained under a surviving global supersymmetry that corresponds to a linear combination of the original global and local supersymmetries, as in the case of the Green-Schwarz superstring [7]. For the supermembrane these transformations are derived in ref.[8] and they take the form

\[ \delta X^I = 2i\alpha^I S + e^t \partial_t X^I \Gamma^I \partial_t S \]

\[ \delta S = -\frac{1}{2p^t} (X^I \Gamma_I \gamma^t) \alpha + e^t \partial_t S \Gamma^I \partial_t S + \beta \]  

(12)

where \( \alpha \) and \( \beta \) are 16 dimensional SO(0,16) spinors, which can be reinterpreted as eight two-component spinors on the world volume in 3 dimensions. Thus the original space-time and local supersymmetries metamorphose into world-volume supersymmetries in the lightcone gauge, just as in the Green-Schwarz superstring.

Now we are in a position to define the meaning of a supersymmetric background solution. In a background that the fermions vanish, we obtain \( \delta X = 0 \) automatically, while only for certain solutions there would be some non-zero components of \( (\alpha, \beta) \) such that \( \delta S = 0 \). In a perturbative set up alluded earlier in the paper, we will consider only such backgrounds. This will then guarantee that, at the linearized level, the quadratic part of the Hamiltonian will be supersymmetric. Thus the perturbative vacuum will have zero energy for all such backgrounds.

We have argued in the beginning of the paper that massless states would emerge in a perturbative and hopefully also in a non-perturbative supersymmetric vacuum state. In such a state it is natural to consider the degrees of freedom of a collapsed membrane to determine the quantum numbers of the particles. Classically a collapsed membrane is described by the zero modes which are purely \( r \) dependent functions \( X_r^\alpha \) and \( \psi_r^\alpha \). For these the covariant or lightcone gauge equations collapse to

\[ \rho^r = 0, \quad \rho^0 = 0, \quad p_r \Gamma^r \psi = 0, \]  

(13)

where \( \rho^r \) is the center of mass momentum which becomes equal to \( \Pi_r^\alpha = X_r^\alpha - i\psi_r^\alpha \Gamma^r \psi = \rho^r \). As expected these are the equations of the superparticle. This solution is indicating a massless spectrum. In the lightcone gauge chosen above, the nonzero components of \( \psi_r \) reduce to the zero modes \( S_0 \) whose numbers were indicated following eq.(8). The degeneracy of the solution is determined by quantizing these zero modes.

The expectation value of the induced metric corresponding to this quantum state is identically zero. However this is not alarming as far as the field equations (in this state) are concerned as is evident by working directly in the lightcone gauge. To see how this works in the covariant field equations, consider any induced metric that has the form (which can always be achieved as a gauge choice [6])

\[ g_{ij} = \begin{pmatrix} -det(h) & 0 \\ 0 & h_{tt} \end{pmatrix}, \quad (\Pi_{ij})^2 = -det(h). \]  

(14)

Then

\[ \sqrt{-det(g)} g^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & h_{tt} \end{pmatrix}. \]  

(15)

In the limit of \( h_{tt} = 0 \), we obtain

\[ g_{ij} = 0, \quad (\Pi_{ij})^2 = 0, \quad \sqrt{-det(g)} g^{ij} = diag(-1, 0, 0), \]  

(16)

which shows that the covariant equations remain well defined and are satisfied by the solution displayed in eq.(13).

Considering now the quantum properties of the zero modes, we naturally expect that the center of mass coordinate and momenta \( q \) and \( p \) are canonically conjugate to each other. Furthermore, we note that the lightcone gauge action that yields eq.(11) for \( S^I, \sigma, \rho \) has the form \( \tilde{S}^I + ... \). From this we deduce that the canonical conjugate to a real \( S \) is again \( S \). Therefore \( S \) satisfies locally a Clifford algebra. Hence, the 2(d-3) real fermionic zero modes must also satisfy the Clifford algebra

\[ \{ S^I_0, S^J_0 \} = 2 \delta^{IJ}_0 \]  

(17)

This conclusion is confirmed by noting that the supercharge, that generates the rigid supersymmetries that leave the solution invariant (as discussed above), is linearly related to the zero mode \( S_0 \) as in the case of the point superparticle. The anticommutation rules of this supercharge are identical to those of \( S_{0} \), as it should be. Again, similar arguments apply in superstring theory [7].

To study the degeneracy and quantum numbers of the zero mass states we now concentrate on the fermionic zero modes. There are 2(2(d-3)) such (pseudo) real modes forming a Clifford algebra. This algebra is then realized on \( 2^{d-3} \) states with \( 2^{d-4} \) bosons and
2 - 4 fermions. These states are classified under the relevant transverse $SO(d-2)$ group as well as the internal global $USp(2)$ group (in $d=7,5$) that was mentioned following eq (2). The classification of the states follows from the classification of the $S_n$'s which is conveniently given in the following table, where B and F label bosons and fermions respectively.

<table>
<thead>
<tr>
<th>$d$</th>
<th>group</th>
<th>classification of $S_n$</th>
<th>classification of states</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$SO(9)$</td>
<td>16</td>
<td>$[44 + 84]_a + 128_r$</td>
</tr>
<tr>
<td>7</td>
<td>$SO(5) \times USp(2)$</td>
<td>(4,2)</td>
<td>$[(5,1) + (1,3)]_a + (4,2)_r$</td>
</tr>
<tr>
<td>5</td>
<td>$SO(3) \times USp(2)$</td>
<td>(2,2)</td>
<td>$(1,2)_a + (2,1)_r$</td>
</tr>
<tr>
<td>4</td>
<td>$SO(2)$</td>
<td>helicity $1/2$</td>
<td>$[helicity 0]_a + [helicity 1/2]_r$</td>
</tr>
</tbody>
</table>

Thus, in $d=11$ we have the gravity supermultiplet, in $d=7$ the Yang-Mills supermultiplet, in $d=5$ the matter supermultiplet, but the $d=4$ states do not form a $CPT$ self-conjugate supersymmetry multiplet. Therefore, only the $d=11$ collapsed supermembrane contains gravity at low energies. If we apply the above analysis to the closed Green-Schwarz superstring, we arrive also at a similar conclusion. Namely, the zero modes of the superstring yield the supergravity multiplet in $d=10$, the Yang-Mills supermultiplet in $d=6$, the Wess-Zumino matter multiplet in $d=4$ and a $(1 + 1)$ multiplet in $d=3$. We find that the number of zero mass states in the closed string theory are the same as those of the membrane theory in one more dimension. The massless string states and their classification groups can directly be obtained from the above table by compactifying one dimension. We then see that the above supermultiplet set of states correspond to the massless sector of the type-IIA superstrings. Thus, type-IIA superstrings are closely related to supermembranes in one higher dimension. Other aspects of such a relationship can be seen in ref. [8].

Since $d=11$ is singled out by the presence of gravity, we briefly outline the determination of the $SO(9)$ content of the physical states, using the fact that $S_n$ is classified as the $16$ dimensional spinor. It is simplest to embed $SO(9)$ into $SO(16)$. The reason for $SO(16)$ is the $16$ dimensional Clifford algebra of eq (17). Now, this Clifford algebra can be represented on $128_a + 128_r$ dimensional spinor representations of $SO(16)$. Their decomposition under $SO(9)$ directly yields $128_a \rightarrow [4_4 + 84]_a$ while the $128_r \rightarrow 128_r$. $44$ is the second rank symmetric traceless tensor (graviton), $84$ is the third rank antisymmetric tensor and $128$ is the vector-spinor (gravitino) of $SO(9)$.

In the above discussion we have ignored the possibility of zero mass solitonic configurations of the supermembrane which might carry non-trivial representation, $R$, of the $SO(d-2)$ group, and which may therefore change the quantum numbers of the vacuum in solitonic sectors. If this happens, then the quantum numbers of massless states might be determined by combining the quantum numbers of $R$ with those of the zero modes discussed above. Then the graviton multiplet might arise in $d=7,5,4$ as well. For example, in $d=7$, if we take the direct product of the Yang-Mills multiplet in the above table with a solitonic $SO(5) \times USp(2)$ representation $(5,1)$, one obtains [9] the $d=7, N=0$ supergravity multiplet which consists of $[(14,1) + (10,1) + (3,1) + (5,3)]_a + [(4,2) + (16,2)]_r$. It is difficult to imagine how such solitonic quantum numbers could arise unless additional variables (such as extra dimensions) are included. Thus, we will assume that, as in the case of the superstring, that there are no massless solitonic states unless additional variables are added in the supermembrane action.

Our discussion of the massless sectors for the supermembrane and superstrings in various dimensions can also be generalised to the super $p$-branes, which are generalisations of the superstring (1-brane) and supermembrane (2-brane) to higher extended objects. According to ref. [10] the super $p$-branes exist classically for the following values of $p$ and dimensions $d$:

\begin{align}
1) & (d, p) = (10, 1), (11, 2), \\
2) & (d, p) = (6, 1), (7, 2), (8, 3), (9, 4), (10, 5), \\
3) & (d, p) = (4, 1), (5, 2), (6, 3), \\
4) & (d, p) = (3, 1), (4, 2), 
\end{align}

After taking into account gauge fixing, the number of fermionic or bosonic degrees of freedom in each of these chains of super $p$-branes is $8, 4, 2, 1$, respectively. It is interesting that these numbers correspond to the division algebras of octonions, quaternions, complex numbers and real numbers, respectively. If we apply our analysis on the fermionic zero modes, as above, we find that chain 1) contains the supergravity multiplet, chain 2) the Yang-Mills supermultiplet, chain 3) the Wess-Zumino supermultiplet, and chain 4) the CPT non-invariant $1 + 1$ supermultiplet, in the appropriate dimensions. Thus, only (the octonionic) chain 1) contains the graviton among the massless states that are determined by our analysis of the zero modes. This property distinguishes the $d=10$ superstring and the $d=11$ supermembrane from all other super $p$-branes.

We have seen from our discussion of the structure of the vacuum state in super $p$-brane theories that in general the vacuum is a super-multiplet. However, the supermembrane in eleven dimensions is singled out as the unique case that has in common
with the ten-dimensional superstring the property that the vacuum supermultiplet is a supergravity multiplet. Now classically, all of the super p-brane theories are formulated in curved superspace, and so all of them can couple classically to the corresponding supergravity background. Thus the results of this paper, which are based on a first-quantized approach, show that the ten-dimensional superstring and the eleven-dimensional supermembrane are singled out as the two cases where the massless spectrum coincides with the background fields to which the classical superspace action can couple. It seems to be quite plausible that any disagreement between the states of the vacuum supermultiplet and the background fields to which the superspace action can couple is an indication of an inconsistency.

In string theory we have developed the intuition that quantum numbers of background fields correspond to quantum numbers of states in the theory. In particular the massless states in the theory have corresponding representatives among the background fields. Thus it seems that disagreement between the set of massless states of the vacuum supermultiplet and the background fields may be taken as evidence for a quantum inconsistency.

It is tempting to conjecture, in view of the fact that the superstring is known to be quantum mechanically consistent only in ten dimensions, that this coincidence of the spectrum and background-field states is crucial for the quantum consistency of any of the super p-brane theories. By the same token, it is highly suggestive that the only other consistent possibility is d=11 and the supermembrane.

To establish that this theory is indeed consistent will require further study of the structure of anomalies. In string theory the conformal group allows to establish that there is only one anomaly which can be cancelled by the choice of the dimension d=10. In supermembrane theory one must study the local fermionic symmetries and the full diffeomorphism group (which contains the conformal group in the string case) and understand precisely how the anomaly structure cannot be cancelled in any dimension except possibly for d=11. This should also correspond to the closure of the lightcone gauge Lorentz algebra in the critical dimension.

We can obtain an indication of the closure of the Lorentz algebra by studying the spectrum of the massive states. A prerequisite is that we need to establish that the massive states form representations of SO(d-1). In string theory in d=10 the manifest lightcone symmetry is SO(8), and one can show that the massive states do form SO(8) multiplets. This is an indication that the Lorentz algebra can close. Of course, for strings the closure is established more directly by brute force commutation. In ref.[11] the analogous problem is investigated for the supermembrane at the first massive level. By means of remarkable group theoretical properties of SO(8), SO(9), SO(10) and SO(16) which look highly miraculous, it is shown that for d=11, starting from a manifest (possibly broken)

lightcone symmetry of SO(8) (8 bosons + 8 fermions), the first level massive states do reassemble into complete multiplets of SO(10), as desired. It is also shown that for d=7,5,4 supermembranes, as well as the d=6,4,3 superstring and all remaining extended objects in eq. (19), the same analysis fails. Therefore, this observation definitely excludes d=7,5,4 supermembranes and p≥3 extended objects as containing incurable anomalies, while it raises the possibility that anomalies may cancel for d=11.

Given the circumstantial evidence that we have presented so far, and the close relationship with the d=10 type-IIA superstring it is tempting to conjecture that the eleven-dimensional supermembrane is actually a consistent theory. Thus, it appears that the eleven-dimensional supermembrane is a plausible consistent theory of extended objects, which contains the ten-dimensional superstring.

What physical consequences could we imagine if the d=11 supermembrane is the underlying unified theory? For this to be possible, we have to further explain how chiral fermions could arise in a vacuum configuration of the supermembrane. To analyse this question, our results suggest that we should look in the direction of type-IIA superstring theory for possible candidate vacuum configurations. In the same manner that type-IIB superstring theories can yield chiral fermions, type-IIA can also provide them in left-right asymmetric compactifications. This is already unlike the case of d=11 Kaluza-Klein supergravity which cannot yield chiral fermions. Similarly, one may also imagine intrinsically membrane mechanisms for producing chiral fermions that are not contained in type II superstring theories. It seems therefore that further study of such open questions, including the search for realistic phenomenological models that may be embedded in vacuum configurations of type-IIA superstrings or supermembranes, is warranted.

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