GAUGE-COVARIANT AND DUAL MODEL VERTESES

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ABSTRACT

A brief review of gauge-covariant string theory is given. A simple algorithm for deriving dual-model vertices is presented and used to derive their operator identities. The latter allow one to find the conformal mapping between dual-model and gauge-covariant vertices.

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1. A BRIEF REVIEW OF GAUGE COVARIANT STRING THEORY

Let me begin by first discussing some of the relevant history of string theory. Most of the important developments in modern physics have resulted from a deeper understanding of a symmetry principle. Given this symmetry principle, say general coordinate transformations for the case of general relativity, we can construct invariant actions which are then used to weight the Feynman path integral and derive the S-matrix.

In contrast, the first step in string theory was to guess the four point\[1\] and then the n point\[2\] S-matrix at the tree level. Of course, had their guess included the entire S-matrix, including loop and non-perturbative effects, there would be nothing left to do. The dual model\[3\] was the first step in the understanding of loop effects. Although this model consists of vertices and propagators, the combinatoric rules which specified how these pieces should be assembled to compute the S-matrix are not those of field theory. The old dual model, however, did not incorporate Faddeev-Popov ghosts so when calculating loops the effect of the ghosts had to be inserted by hand with projectors.

From the dual model it was realized that string theory processes could be viewed as a weighted sum over the world sheet of the string.
The appropriate weight in this Feynman path integral being provided by
the correctly gauge fixed Nambu action. Although using these path
integrals one could calculate tree and loop effects the weight of each
diagram had to be separately determined by unitarity.

Despite considerable progress in the above approaches, there was
no attempt to find the deep symmetries which underlie string theory and
are responsible for the many miraculous results of string theory. This
symmetry must include in the relevant strings general coordinate, Yang-
Mills and supersymmetry transformations. A related point is that the
above formulations did not include non-perturbative effects.

Gauge covariant string theory is the search for a formulation of
string theory in which the string symmetry is manifest. At present, this
formulation is a string field theory; that is the primary objects are
functionals of the string co-ordinate $x^\sigma(\sigma)$. This approach embodies the
possibility of computing non-perturbative effects as does point particle
field theory.

It was not entirely clear from the beginning whether field theory
would lead to a good formulation of string theory, but as we will see
the results found so far have a very elegant and simple appearance. We
begin by giving a brief account of the free open bosonic string.

As a result of the two dimensional reparameterization invariance of
the open bosonic string, it is a system which classically has the
constraints

$$L_n \equiv \frac{\pi \kappa^1}{2} \int_{-T}^{T} d\sigma \ \delta^{\gamma n \sigma} \ \left( \partial^\sigma \right)^2 = 0 \quad (1.1)$$

where

$$\partial^\sigma \equiv P^\sigma - \frac{1}{2 \pi \kappa^1} \frac{\partial x^\sigma}{\partial \sigma} \quad (1.2)$$

and $P^\sigma$ is the momentum conjugate to $x^\sigma$ and is easily computed from the
Nambu action for the classical string. These constraints generate the
two dimensional conformal group, namely

$$\{L_n, L_m\} = -i (n - m) L_{n+m} \quad (1.3)$$

The correct constraints\(^4\) to impose on the quantum system after the
usual operator replacements are

$$L_n |\psi\rangle = 0 \quad , \quad n \geq 1 \quad \left( L_0 - 1 \right) |\psi\rangle = 0 \quad (1.4)$$
These constraints lead to a ghost free spectrum when the space-time dimension $D$ is 26$^5$. In equation (1.4) one may view $|\Psi\rangle$ as a functional of $x^U(\sigma)$ or think of it as being in any other basis such as the oscillator basis. An alternative way of finding the on-shell states of string theory is to solve the constraints of equation (1.1) by going to the light cone-gauge which has coordinates $x^\pm(\sigma), x^- = \tau$. The functionals $\psi(x^\pm(\sigma), x^- \tau)$ $i = 1, \ldots , 24$, then only contains the physical degrees of freedom.

The open string contains Yang-Mills fields and equation (1.4) contains within it the constraints $\varepsilon_\mu A^\mu = 0$, $\varepsilon^a A^a = 0$.

The problem of free gauge covariant theory is to find a local action which leads to the correct on-shell spectrum discussed above. For the Yang-Mills fields the result is of course $\varepsilon^a \partial_\sigma A^a = 0$, $\varepsilon^a \partial_\sigma A^a = 0$ which possess the gauge invariance $SA = \partial_\sigma A^a$. The string however contains particles of ever increasing spin with increasing mass. Unlike spins 0, $\frac{1}{2}$ and 1, all higher spins do not possess Lorentz invariant actions which can be constructed only from the fields that occur in the on-shell. Above spin 1, one must include in the Lorentz covariant action new fields that disappear from the on-shell conditions.

For example massive spin 2 is described by the on-shell conditions $h_{\mu \nu} = (\varepsilon^2 - m^2) h_{\mu \nu} = 0$, $\varepsilon^2 h_{\mu \nu} = 0$. To find an action which leads to this on-shell system we must introduce a scalar which can be identified as the trace of $h_{\mu \nu}$. We call these additional fields, supplementary fields.

Since the string contains an infinite number of higher spin particles we must introduce an infinite number of supplementary fields which must be identified in terms of string functionals. The most convenient formulations of gauge covariant string theory in fact contain an infinite number of supplementary string functionals. The problem now arises as to how to handle these additional fields. The answer is to introduce two additional anticommuting co-ordinates $c(\sigma)$ and $\bar{c}(\sigma)$ making with $x^U(\sigma)$ a 28 dimensional space. The encoding works much the same way as it does in the superspace formulation of supersymmetric theories, except in this case an algebraic constraint is used to eliminate all the anticommuting and even some of the commuting component string functions. We therefore consider functionals $\chi(x^U(\sigma), c(\sigma), \bar{c}(\sigma))$. The free action is of the form

$$<\chi | \Phi | \chi> = \int \mathcal{D}c(\sigma) \mathcal{D}\bar{c}(\sigma) \mathcal{D}x^U(\sigma) \chi \Phi \chi$$  (1.5)
where

\[
\mathcal{Q} \equiv \mathcal{Q} \left[ x^{\mu}(\sigma), c(\sigma), \bar{c}(\sigma), \frac{S}{\partial c(\sigma)}, \frac{S}{\partial \bar{c}(\sigma)}, \frac{S}{\partial x^{\mu}(\sigma)} \right]
\]

The object \( \mathcal{Q} \) has in fact a deep group theoretic interpretation.

Since this object has been met before in other contexts and is likely to play an important part in future developments I will give a general definition. Consider a Lie group \( G \) whose generators are represented by \( L_n \) and which obey the commutation relations

\[
[L_n, L_m] = \frac{f_{nm}}{f_{nm}} L_p
\]

If we are dealing with a constrained system, the constraints which generate \( G \) will satisfy in the classical theory a Poisson Bracket relation as they do for the string (see equation (1.3)).

We now introduce two anticommuting oscillators \( \beta_n, \bar{\beta}_n \) for each generator, \( L_n \) of \( G \). These oscillators obey the relations \( \{ \beta_n, \bar{\beta}_m \} = \delta_{m+n,0} \) while all other anticommutators vanish. We define

\[
\hat{Q} \equiv \sum_n \beta_n L_n - \frac{1}{2} \sum_{n,m} \bar{\beta}_n \bar{\beta}_m P_{nm} P_{-n} P_{-m}
\]

This object arises in the B.R.S.T. quantization of gauge theories. It is straightforward to demonstrate that \( \hat{Q}^2 = 0 \). For the string we take \( G \) to be the \( D = 2 \) conformal group and due to normal ordering ambiguities in the quantum theory we actually use the object

\[
\mathcal{Q} \equiv \hat{Q} - P_0
\]

In reference [6] it was shown that \( \mathcal{Q}^2 = 0 \) in \( D = 26 \).

In fact, the \( \mathcal{Q} \) in equation (1.8) is the \( \mathcal{Q} \) given above in the expression for the free string action and the \( \beta_n, \bar{\beta}_m \) are constructed from the coordinates as follows

\[
\bar{\beta}(\sigma) = \frac{S}{\delta c(\sigma)} - \frac{1}{2\pi} \bar{c}(\sigma) = \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \bar{\beta}_n e^{in\sigma}
\]

\[
\beta(\sigma) = -\frac{S}{\delta c(\sigma)} + \frac{1}{2\pi} c(\sigma) = \frac{1}{\pi} \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \beta_n e^{in\sigma}
\]

Clearly the above free action which was found in references [7] and [8] is invariant under

\[
S \left[ \alpha \right] = \mathcal{Q} \left[ \alpha \right]
\]

provided the dimension of space-time is 26. By using the oscillator formalism one can work out the action of equation (1.5) in terms of component fields. In particular, one can show that it contains the free
Yang-Mills action, if in a somewhat unusual first order form.

In fact, it is not difficult to demonstrate that the above action does lead to the correct on-shell spectrum of string theory. In this proof one finds that the light-cone count in 24 dimensions emerges as 24 = 26-1-1. This is not just a numerical coincidence, but the result of the bosonic partition function in 26 dimensions being multiplied by the fermionic partition functions of c and \( \bar{c} \). Thus c and \( \bar{c} \) act in the on-shell count, so as to reduce the 26 dimensions of \( x^a(\sigma) \) by two. This suggests a type of Parisi-Sourlas type mechanism. We will return to this point latter.

Yet a further method of arriving at the addition of c(\( \sigma \)) and \( \bar{c}(\sigma) \) to \( x^a(\sigma) \) is to consider the first quantized string. In order to gauge fix the string we must introduce two sets of ghosts corresponding to the two dimensional general coordinate group. The associated BRST conserved charge is none other than the \( Q \) described above. What is not so clear in this approach is that the first quantized BRST theory should provide all the necessary tools for the gauge covariant second quantized field theory.

The general procedure for the description of first quantized constrained relativistic systems in terms of \( Q \) was given in an important, but up till now, somewhat ignored paper, of reference [9]. From the example of the string we see that the power of this formalism generalizes to the description of the second quantized field theory. It would seem likely that this procedure will lead to a deeper understanding of second quantized field theory itself.

The other free gauge covariant string theories with the exception of the closed superstring, can be constructed along the same lines as for the open bosonic string given above. For more details of the latter and a review of the former the reader may consult reference [10] where further references may also be found.

We now consider the interacting open bosonic string as formulated in references [11] and [12] and independently in reference [13]. An alternative description can be found in reference [8]. Given two strings, the most obvious way to join them to form a third is to join them at their end points. Clearly, the length of the final string is the sum of the lengths of the original two strings.

In fact, it is this simple picture that emerges from the one gauge
in which the second quantized field theory of strings is known, namely the light cone gauge\textsuperscript{(14)}. Of course this gauge is very special, but one may expect some of its features to be generic. We adopt the above picture illustrated in the figure below. We assign a length $\alpha_r$ to the $r$\textsuperscript{th} string and demand, in the interaction, that the lengths be preserved. In the light cone gauge, the lengths are proportional to $p^*$ and so are automatically preserved.

\[ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_r
\end{array} \begin{array}{c}
1 \\
2 \\
3
\end{array} \]

The above picture has a well defined meaning: at the interaction time we identify string 1 with the lower part of 3 and string 2 with the upper part of string 3. That the vertex is given by

\[ \mathcal{U} \sim \mathcal{N} \mathcal{S}(\frac{2}{\alpha^2} \Theta(\eta_3) - \Theta(z^r - \eta_3) \frac{2}{\alpha^2} \Theta(\eta_3 - z^r) \frac{2}{\alpha^2} \Theta(\eta_3)) \]  

(1.11)

where $\eta_r$ parameterizes the $r$\textsuperscript{th} string, $\eta_{int}$ is the interaction point measured on string three and $z^r = (x^v, c^v/c^f/\alpha^r)$. The factor $\mathcal{N}$, as we will see, is necessary and is in fact proportional to the ghost coordinate.

The action is of the form

\[ A = \frac{1}{2} \langle \chi | q | \chi \rangle + \frac{3}{2} \mathcal{U} (\chi_1, \chi_2, \chi_3) + \text{a possible quartic term}. \]  

(1.12)

The above is written in oscillator form and so $V$ is of the form

\[ \mathcal{U} = \langle 0| \langle 0 | \langle 0 | \mathcal{S}(\alpha^r, \alpha^\nu_r, \beta^r, \beta^\nu_r) \rangle \]  

(1.13)

It must contain among other terms the usual Yang-Mills term: $g(\alpha^{\mu_{j-1}} \alpha^{\mu_j} - \delta^\mu_{j-1} \delta^\mu_j)$.

To find this term is just a matter of oscillator algebra once one takes $|\chi\rangle$ to have the generic form $\alpha^+ \chi_0$. In functional language the interaction term takes the form

\[ \int \mathcal{D} z^r \mathcal{S}(\frac{1}{\alpha^2} \mathcal{K}_{\eta_3} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c} \mathcal{K}_{\eta_3^c}) \]  

(1.14)

where $V$ is of the generic form of equation (1.11).

The transformation law will contain the inhomogeneous given above as well as a term which is linear in the field $|\chi\rangle$ and parameter $|\chi\rangle$. It is given by
\[ S^\alpha_2 \leq \lambda I = \frac{1}{9} \lambda \left( \Phi^2 + U \left( \frac{\alpha_2}{\alpha_3} \Phi^1 \lambda_1 \lambda_2 - \frac{\alpha_1}{\alpha_3} \Phi^2 \lambda_1 \lambda_2 \right) \right) \]

This law will contain the usual Yang-Mills result; namely \( S_\alpha = g A^\alpha \wedge \{ A, A \} \).

We can vary the action under the transformation law of equation (1.15) and keeping only terms of order \( g^0 \) we find that the condition for invariance is given by

\[ \nabla \sum_{\nu=1}^{2} \Phi^\nu = 0 \]  

(1.16)

One may also test the commutator of transformations, keeping the order \( g^{-1} \) terms we find that

\[ [S^\alpha_1, S^\alpha_2] \leq \lambda I = \frac{1}{9} \left( \frac{\alpha_2}{\alpha_3} \Phi^1 \lambda_1 \lambda_2 - \frac{\alpha_1}{\alpha_3} \Phi^2 \lambda_1 \lambda_2 \right) \]

(\( \lambda \leftrightarrow \lambda_1 \))

(1.17)

We must recognise this result as a transformation of the type we already have. It can only be of the form

\[ [S^\alpha_1, S^\alpha_2] \leq \lambda I = \frac{1}{9} \lambda \leq \lambda \left( \Phi^2 \right) \]

where

\[ \leq \lambda \left( \frac{\alpha_2}{\alpha_3} \Phi^1 \lambda_1 \lambda_2 - \frac{\alpha_1}{\alpha_3} \Phi^2 \lambda_1 \lambda_2 \right) \]

(1.19)

It is easily verified that this will be the case provided equation (1.16) holds.

One could use equation (1.16) to determine \( V \), but in reference (13), \( V \) was found by utilizing the overlap condition. The result is given by

\[ V = \sum_{\nu} \exp \left( \frac{\alpha_2}{\alpha_3} \right) \left( \lambda_1 \Phi^\nu \lambda_2 + \lambda_2 \Phi^\nu \lambda_1 \right) \]

\[ V^2 = \sum_{\nu} \exp \left( \frac{\alpha_2}{\alpha_3} \right) \left( \lambda_1 \Phi^\nu \lambda_2 + \lambda_2 \Phi^\nu \lambda_1 \right) \]

where \( N^{rs}_{nm}, N^rs_{nm} \) and \( R^{rs}_{nm} \) are complicated functions of the string lengths which can be found in references [11] and [12]. This form for \( V \) was then shown to satisfy equation (1.16).

Given the vertex in the oscillator basis we can, using the appropriate 'transition function', find it in any basis. In particular, in the coordinate basis we find that[12]
\[ \mathcal{U}_1 \mathcal{U}_2 = (c^2 + \sum_{n=1}^{\infty} e_n c_n^2) \mathcal{S} (\mathcal{Z}(\eta_3) - \mathcal{O}(\eta_3 - \eta_3^* \mathcal{Z}(\eta_3)^*) \mathcal{Z}(\eta_3)) \]

(1.20)

where \( e_n \) is a complicated function of the string lengths.

This demonstrates that the beautiful picture of the splitting and joining of strings at their end-points is contained within the above vertex. Presumably, the additional factor is related to the ghost co-ordinate \( c \) at the interaction point when made well defined by a suitable limiting process.

The on-shell scattering of three strings at the tree level has been known\(^{[15]} \) for many years and we must check that the above vertex does recover this result on-shell. In fact, it can be shown\(^{[12]} \) that \( V \) satisfies the equation

\[ \mathcal{U} = V^{CSV} e_{x} \prod_{n=1}^{\infty} \frac{\mathcal{Z}(\eta)}{\mathcal{Z}_{x}^{x}} \right) \]

(1.21)

where \( V^{CSV} \) is the known result. Clearly, for on-shell states (i.e. \( L_{n} |\varphi> = 0; n \geq 1 \))

\[ \mathcal{U} |\varphi> = V^{CSV} |\varphi> \]

(1.22)

A demonstration of this result can be found in the next section.

The calculation of the four point function goes much the same way as in the light-cone gauge. The extra anticommuting oscillators in the covariant theory just correct for the two additional bosonic integrations that are not present in the light cone gauge.

We also observe that all the string length, \( x^F \) dependence in \( V \) is contained in the conformal mapping term on the right hand side of equation (21) and so disappears for on-shell quantities. Given the complicated dependence of \( V \) on the string lengths \( x^F \) this result requires a considerable conspiracy. It leads one to strongly suspect that all scattering amplitudes are \( x^F \) independent. In fact, this lack of \( x^F \) dependence can be explained by an underlying local symmetry in an extended formalism. The reader is referred to reference (17) where a detailed exposition is given. A different approach to string length can be found in reference \( 181 \).
2. THREE-STRING VERTICES AND THEIR OPERATOR IDENTITIES

We begin with the three-string vertex of Sciuto\textsuperscript{15}) and Caneschi, Schwimmer and Veneziano\textsuperscript{16}) used in the dual model\textsuperscript{3}) and given by

\[
\mathcal{V}_{\text{CSV}} = \sum_{\frac{1}{2}}^{\frac{1}{3}} \frac{1}{2} \leq 1 \leq 1 \leq \frac{1}{3} \exp \frac{\alpha}{2} \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_m \alpha_n (-1)^m \Gamma(n)}{\Gamma(l + m) \Gamma(l + n - m)}
\]

(2.1)

where \( \alpha_n > 1 \) for \( n > 1 \). A more concise method of writing this vertex is given by

\[
\mathcal{V}_{\text{CSV}} = \sum_{\frac{1}{2}}^{\frac{1}{3}} \frac{1}{2} \leq 1 \leq 1 \leq 1 \exp \frac{1}{2 \pi n} \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_r}{2} \int_0^\infty \frac{\alpha d\alpha}{\alpha} \mathcal{P}_+ (\alpha) \mathcal{Q}_+ (\frac{1}{1-\alpha})
\]

(2.2)

where

\[
\mathcal{P}_+ (\alpha) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} \alpha_n \alpha_{-n}, \quad \mathcal{Q}_+ (\alpha) = \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \alpha_{-n}
\]

(2.3)

Using \((L_0-1)^{-1}\) as a propagator and summing with unit weight over all topologically distinct diagrams, one can correctly compute all tree graphs for the scattering of strings. Note that the above combinatoric instruction, which was found by demanding unitarity, is different from that obtained for a Feynman path integral. Consequently, using \(\mathcal{V}_{\text{CSV}}\) in a field theory with only a three-point function cannot lead to the correct tree graphs, as was indeed explicitly verified\textsuperscript{19}). For loop graphs, in the old dual model one had to remove by hand the propagation of negative norm states, which are required in any Lorentz-invariant treatment of spin-one particles and above. This required the use of the Brink-Olive projector. At one loop this procedure was used successfully, but its higher-loop implementation was not carried out. Even overlooking this point in order to ensure the correct covering over moduli space, the integration must be restricted to a fundamental domain by hand.
Despite these drawbacks, the dual-model vertex is simpler than those found in string field theory; it was used for the quick evaluation of the tree and, after adjustment by hand of certain factors, also of loops. Indeed, general $n$ string vertices $^{20)}$ extending $V_{CSV}$ were found and shown to factorize into two such vertices. What was not discussed, with one exception $^{21)}$, were the operator identities that the $V_{CSV}$ and higher-point vertices satisfied. In particular, it was not shown that these vertices were physical vertices in the sense that if all but one leg was put on-shell, then the one remaining leg was automatically on-shell. To demonstrate this property, requires a knowledge of how the $L_n$'s pass through the vertex.

The following discussion is an account of work carried out in collaboration with André Neveu and published in Refs. 22) and 23). Let us begin with the $\alpha_n^\mu$ oscillators contained in

$$ P^\mu (z) = \sum_{n=-\infty}^{\infty} \alpha_n^\mu z^{-n} \tag{2.4} $$

We recall that an operator $R(z)$ has conformal weight $d$ if

$$ g R(z) g^{-1} = R(z') \left[ \frac{z}{z'} \frac{d}{dz} \right]^d \tag{2.5} $$

where $g = \exp \left( \sum_{n=-\infty}^{\infty} a_n L_n \right)$ and the transformation $z$ to $z'$ is that induced by the representation $L_n z = z^{n+1} d/dz$. One may easily find that $p^\mu$ has $d = 1$ while $L(z) = \sum_{n=-\infty}^{\infty} L_n z^n$ has $d = 2$. One can show the following identity

$$ 0 = V_{CSV} \left\{ A(z) P^\mu \left( \frac{1}{1-z} \right) + B(z) \frac{P^\mu z}{z} \left( \frac{z-1}{z} \right) + C(z) P^\mu \right\} \tag{2.6} $$

provided

$$ \frac{A(z)}{[-z]} + B + \frac{C(z)}{[z-1]} = 0 \tag{2.7} $$

This result is found by pushing the creation operator onto the vacua in $V_{CSV}$ and cancelling the terms that come from the exponential. Note that if
C and B are non-zero, then on any state the creation operators in \( p^\mu \) and \( p^{\mu \nu} \) are non-zero, and so for convergence we require \(|z| < 1\) and \(|(z-1)/z| < 1\), which implies \(1/(1-z) > 1\) and so we must put \( A(z) = 0\). Consequently, in using the above relation, we must, for convergence, require that one of A, B or C be zero.

Consider now the transformation \( g: z + z^1 = 1/(1-z) \). This is an element of \( SL(2,\mathbb{R}) \) and has the property that it cubes to the identity transformation. It maps the points 0, 1 and \( \infty \) in the following way

\[
0 \rightarrow 1 \rightarrow \infty \rightarrow 0
\]  

(2.8)

and one may suspect that what this transformation is really doing is permuting the three strings. This will be confirmed later.

Let us now apply this transformation to the identity of Eq. (2.6), using the fact that \( p^\mu(z) \) has conformal weight one. We find that

\[
O = \mathcal{V}^{CSV} \mathcal{g}^{-1} \mathcal{g} \sum \mathcal{A}(z) \mathcal{P}^\mu \left( \frac{1}{1-z} \right) + \mathcal{B}(z) \mathcal{P}^{\mu \nu} \left( \frac{z-1}{z} \right) + \mathcal{C}(z) \mathcal{P}^{\mu \nu}(z)
\]

\[
= \mathcal{V}^{CSV} \mathcal{g}^{-1} \left\{ \sum \bar{\mathcal{B}} \mathcal{P}^{\mu \nu} \left( \frac{z-1}{z} \right) + \bar{\mathcal{C}} \mathcal{P}^{\nu}(z) + \bar{\mathcal{A}} \mathcal{P}^{\mu \gamma} \left( \frac{1}{1-z} \right) \right\}
\]

(2.9)

where

\[
\bar{\mathcal{B}} = \frac{\mathcal{A}(z)}{(-z)}, \quad \bar{\mathcal{C}} = \mathcal{B}(z)(z-1), \quad \bar{\mathcal{A}} = \mathcal{C}(z) \left[ \frac{-z}{z-1} \right]
\]

(2.10)

At first sight one might think that this new equation is just a different identity for a different vertex. However, we find that

\[
\bar{\mathcal{B}} + \bar{\mathcal{A}} \left[ \frac{1}{(-z)} \right] + \bar{\mathcal{C}} \left[ \frac{1}{(z-1)} \right] = \frac{\mathcal{A}}{(-z)} + \frac{\mathcal{C}}{(z-1)} + \mathcal{B} = 0
\]

(2.11)

and so the condition is just the same as before. In fact, as we will see in more detail later, the identity of Eq. (2.6) uniquely determines \( \mathcal{V}^{CSV} \) to be given by Eq. (2.1). Since \( \mathcal{V}^{CSV}_{g^{-1}} \) obeys the same identity as \( \mathcal{V}^{CSV} \), we
may conclude that
\[ V^{\text{CSV}} g^{-1} = V^{\text{CSV}} \]  
(2.12)

This result may also be found\(^{22}\) by a direct evaluation of \( g^{-1} \) on \( V^{\text{CSV}} \).

Let us now consider a general operator identity involving an operator \( R(z) \)
of conformal weight \( d \). It will obey an identity which, by analogy with that for \( p^{\text{HR}}(z) \), we take to be
\[ 0 = V^{\text{CSV}} \left\{ A(z) R^1 \left( \frac{1}{1-z} \right) + B(z) R^2 \left( \frac{z-1}{z} \right) + C(z) R^3 (z) \right\} \]
(2.13)

This relation will only hold subject to one condition which can be written in the form
\[ \frac{A(z)}{f(z)} + \frac{B(z)}{g(z)} + \frac{C(z)}{h(z)} = 0 \]  
(2.14)

As before, one of \( A \), \( B \) and \( C \) must be taken to vanish. The one condition results from pushing the creation operators to the right through the exponential which swops them for destruction oscillators of another species. These must then cancel again terms already in the identity.

We now apply \( g \) of Eq. (2.13) to the above identity and use \( V^{\text{CSV}} = V^{\text{CSV}} g^{-1} \) to find the result
\[ 0 = V^{\text{CSV}} \left\{ \sum \frac{A(z)}{(-z)} R^1 \left( \frac{1}{1-z} \right) + B(z) (z-1) d R^2 (z) \right. \]

\[ + \left. C(z) \left[ \frac{z}{z-1} \right]^d R^3 \left( \frac{1}{1-z} \right) \right\} \]
(2.15)

Using the fact that \( V^{\text{CSV}} \) is cyclic and that it can only obey one identity, we must insist that the one restriction is the same, namely we require
\[ \left[ \frac{z}{z-1} \right]^d \frac{C(z)}{f(z)} + \frac{A(z)}{(-z)^d} + \frac{B(z)}{g(z)} (z-1) d = 0 \]
(2.16)
This implies that
\[ f(z) = (-z)^d, \quad g(z) = (z-1)^d, \]   \hspace{1cm} (2.17)

thus determining the identities for a general operator of conformal weight \( d \).

The above chain of argument can be reversed. Let us assume that the dual-model vertex is invariant under the action of \( g^{-1} \), that is, it obeys Eq. (2.12). Then the one condition is deduced to be of the form of Eq. (2.7) by applying the above argument for the case \( R = p_{\text{HT}} \) and \( d = 1 \). This identity, however, uniquely determines the vertex to be \( V_{\text{CSV}} \). Consequently, given that the vertex satisfies Eq. (2.12) and the general shape of the identity, we can deduce what the vertex is. It is remarkable that the \( V_{\text{CSV}} \) vertex can be deduced from such a simple assumption. Presumably, this characterization of vertices can be extended to the higher-point vertices which describe the scattering of strings. The \( N \) point vertex will depend on \( N-3 \) parameters and should be invariant under the transformation which permutes the strings and whose \( N^{\text{th}} \) power is unity. In this way, one can find a simple characterization of tree-level string scattering amplitudes which might be extendable to loop effects.

It is often more useful to obtain the operator identities in integrated form. Consider the quantity
\[ V_{\text{CSV}} = \oint dz \frac{dz}{z} C(z) R^3(z) \]   \hspace{1cm} (2.18)

The contour is taken to be a small circle \( |z| < 1 \) round the origin, since here \( R^3(z) \) is a convergent quantity. We now deform the contour to go round the points \( z = 1 \) and \( z = \infty \). As we do this, we must, in the appropriate regions, for convergence swap \( C \) for \( A \) and \( B \) using Eq. (2.13) and the relation
\[ \frac{A}{(-z)^d} + B + \frac{C}{(z-1)^d} = 0 \]

which we found above. Since the only singularities occur at \( z = 0, 1 \) or \( \infty \), we find the result
\[ \nu^{CSV} \sum \frac{\phi(z) R^3(z)}{z} + \sum \frac{1}{z-1} \frac{\phi(z) R^2(z+1)}{z} \]
\[ + \sum \frac{\phi(z)(-z)}{(z-1)} \frac{1}{d} R^1 \left( \frac{1}{1-z} \right) \sum = 0 \tag{2.19} \]

By changing variables to \( \xi_1 = z, \xi_2 = 1/1-z \) and \( \xi_3 = 1-1/z \), we may rewrite this result as
\[ \nu^{CSV} \sum \frac{\phi(z) R^3(z)}{\xi_1} + \sum \frac{\phi(z) R^2(z)}{\xi_2} \frac{1}{(z-1)^{d-1}} \]
\[ + \sum \frac{\phi(z) \left[ \frac{-z}{z-1} \right]}{(z-1)^{d-1}} \frac{R^2(z)}{\xi_3} \sum = 0 \tag{2.20} \]

In particular, taking \( R = p^\mu(z), d = 1 \) and \( \phi(z) = z^{-n} \), we find that
\[ 0 = \nu^{CSV} \sum \frac{\alpha^m}{z^{-n}} + \sum \frac{\left( \frac{n}{p} \right)}{p=0} \left( \frac{(-1)^p}{p} \right) \alpha^m \]
\[ - \sum \frac{\left( \frac{-n}{p} \right)}{p=0} \frac{\alpha^{m+1}}{p+n} \sum \] \[ = 0 \tag{2.21} \]

It is straightforward to deduce from this equation that \( \nu^{CSV} \) is indeed given by Eq. (2.1).

Taking \( R = L(z) \) and \( d = 1 \), we find the identity
\[ 0 = \nu^{CSV} \sum L^{(1)}_n - L^{(1)}_0 + \sum \frac{(n+1)!}{p!(n+1-p)!} \frac{(-1)^p}{p} L^{(3)}_{p-1} + L^{(3)}_0 \]
\[ -(-1)^n \sum \frac{(n+1)!}{p!(n+1-p)!} \frac{1}{p} L^{(2)}_{n+p} + L^{(2)}_0 - L^{(2)}_0 \sum = 0 \tag{2.22} \]

We can now verify that \( \nu^{CSV} \) is a physical vertex in the sense discussed above. Namely, given that \( \mid \chi \rangle_1 \) and \( \mid \chi \rangle_2 \) are physical states [i.e.,
\[ L_{n}^{(2)} |\chi_{2} = L_{n}^{(3)} |\chi_{3} = 0 \text{ for } n > 1 \text{ and } (L_{o}^{(2)} - 1) |\chi_{2} = (L_{o}^{(3)} - 1) |\chi_{3} = 0, \]

then

\[ \langle \psi \rangle = \sqrt{\oint \frac{dz}{2\pi i} \left( z^{L_{o}^{(1)} - 1} \right) |\chi_{2} \rangle |\chi_{3} \rangle^{(2.23)} \]

is a physical state.

Applying \( L_{-n}^{1} \) to Eq. (2.23) and using Eq. (2.22), we find that

\[ 0 = \langle \psi | L_{-n}^{1} |\psi \rangle = \langle \psi | \oint \frac{dz}{2\pi i} \left( z^{L_{o}^{(1)} - 1 + n} \right) \left( z^{L_{o}^{(1)} + n - 1} \right) \] (2.24)

As a result of the projector, \( |\psi \rangle \) also satisfied

(2.25)

Consequently, if we build a four-point function and after factorizing examine the residues of the poles, we will find only physical states.

3. RELATION BETWEEN DUAL-MODEL AND GAUGE-COVARIANT VERTICES

Using the results of the previous section, we may now demonstrate\(^{12}\) that the \( V_{CSV} \) vertex and the three-vertex used in the gauge-covariant formulation of Refs. 11, 12 and 13) are related by a conformal transformation. To establish this result, we use the Mandelstam map\(^{24}\) from the world strip to the upper half plane (see Fig. 1), which is given by

\[ \mathcal{S} = \mathcal{C} = \alpha_{c} \ln (\mathcal{S} - 1) + \alpha_{b} \ln \mathcal{S}. \] (3.1)
The inverse map denoted by $g$ from the half plane to the strip is given by\textsuperscript{24)

$$
\begin{align*}
2 = \begin{cases} 
- \frac{1}{3^1} e^{-Y_3(3^1)} & \text{for string } 3 \\
\frac{\alpha_1}{\alpha_2} Y_1(3^1) & \text{for string } 1 \\
\frac{\alpha_1}{\alpha_3} Y_3(3^2) & \text{for string } 2
\end{cases}
\end{align*}
$$

(3.2)

where

$$
\begin{align*}
\varsigma_\nu &= e^{\tau_0 + s_\nu} \\
y &= \varsigma_\nu \ln (1 + \gamma_\nu e^{Y_\nu}) \\
\gamma_\nu &= -\frac{\alpha_{\nu+1}}{\alpha_\nu} \\
(\alpha_{3+1} &\equiv \alpha_1)
\end{align*}
$$

(3.3)

One can easily verify by substituting in Eq. (3.1) that this is the correct inverse map. One may also show that the map which takes $\xi_1 = z + 3^3$ takes $\xi_2 \to 3^2$ and $\xi_3 \to 3^1$. This confirms our previous suspicion that the transformation $z \to 1/z$ (i.e., $\xi_r \to \xi_{r+1}$, with $\xi_4 \equiv \xi_1$) maps the three strings into each other. While the gauge-covariant vertex "lives" on the strip, the dual-model vertex "lives" on the upper half plane. Hence, to
find the covariant vertex, we expect to apply the inverse conformal mapping \( \tilde{g} \) of Eq. (3.2) to \( V^{CSV} \).

Applying \( \tilde{g}^{-1} \) to the right-hand side of Eq. (2.19), we map the identity from the upper half plane to the strip. The first term becomes, in the case of \( R = p^\mu \),

\[
V^{CSV} \tilde{g}^{-1} \oint_{z=0} \frac{dz}{z} \phi(z) \tilde{g} \mathcal{P}^{3\mu}(z) \tilde{g}^{-1} = \oint_{z=0} \frac{dz}{z} \phi(z) \mathcal{P}^{3\mu}(z)
\]

\[
= \oint_{z=0} \frac{dz}{z} \phi(z) \mathcal{P}^{3\mu}(z)
\]

where

\[
\bar{V} = V^{CSV} \tilde{g}^{-1}
\]

Evaluating all the terms for a general conformal operator \( R \), we find that Eq. (2.19) now becomes

\[
\bar{V} \sum_{\nu=1}^{3} \oint_{3^\nu} \frac{dz}{z} \phi(z) (z - z^{nu}) d^{-1} R^{nu}(3^\nu) (3.6)
\]

where \( z^{nu} \) is the interaction point of the three strings marked by "x" in Fig. 1. We observe that identities are required to be more and more subtracted at the interaction point as the conformal dimension \( d \) of \( R \) increases.

In particular, taking \( R = p^\mu \), we find that

\[
\bar{V} \sum_{\nu=1}^{3} \oint_{3^\nu} \frac{dz}{z} \phi(z) \mathcal{P}^{3\nu}(3^\nu) = 0
\]

This identity uniquely determines \( \bar{V} \). Comparing with Refs. 11) and 12), where the identity for \( p^{nu}(3^\nu) \) acting on the covariant vertex, denoted by
were given, we realize that the identities for \( \tilde{V} \) and \( V^c \) are the same, and so

\[
V^{cSV} \tilde{g}^{-1} = \tilde{V} = V^{cov}
\]

(3.8)

One may also verify that the other identities of Refs. 11) and 12), such as for \( R = L(z) \) are in agreement with Eq. (3.6). From the way the \( \alpha \) identity is mapped between \( V^{cSV} \) and \( \tilde{V} \), we may conclude that the conformal map \( \tilde{g} \) is constructed out of \( L_n^i \)'s with \( n > 1 \) only, and so

\[
V^{cov} = U^{cSV} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n L_n^{(n)} \right]
\]

(3.9)

Putting three physical states on each side of this equation, we find

\[
U^{cov} |\chi_1\rangle |\chi_2\rangle |\chi_3\rangle = U^{cSV} |\chi_1\rangle |\chi_2\rangle |\chi_3\rangle
\]

(3.10)

and consequently the covariant vertex gives the known result12) for the scattering of the three on-shell strings.

We observe that \( V^{cSV} \) is independent of the string lengths, and so therefore is the on-shell scattering amplitude. This result generalizes to all higher-point functions12),13) and is explained in the extended formalism of Ref. 17) by a Parisi- Sourlas mechanism.

The reader will probably have realized that \( V^{cov} \) included not only \( \alpha \) oscillators but also ghost oscillators \( \beta_n \) and \( \tilde{\beta}_n \). We may therefore take \( V^{cov} \) and, by extending the \( L_n^i \)'s to include the ghost oscillators, deduce, by applying \( \tilde{g} \), a vertex

\[
V^{cov} \tilde{g} = V^{cSV} U^{gh} = U^{S}\]

(3.11)

where12)
\[
\omega_{gh} = \sum_{r=1}^{\infty} \exp \left\{ \sum_{n=1}^{\infty} \beta_n^{(r)} \left[ \bar{\omega}_0^{(r)} - (-1)^n \sum_{p=0}^{\infty} \frac{\beta_{n+p}^{(r)}}{\Gamma(n+p)} \frac{(-1)^{p-1}}{p!} \frac{\Gamma(n+2)}{\Gamma(n+2-p)} \Gamma(n+1) \right] \right\}
\]

The vertex \( V^T \) lives on the upper half plane and extends \( V_{CSV} \) to include ghost oscillators, whose presence is required to compensate for the negative metric states in any Lorentz-covariant treatment of higher spin gauge particles. As such, \( V^T \) is a good candidate for the vertex with which to construct a new dual model which does not require projectors in loops. Such a vertex must be a physical vertex, which, in the BRST language, means that

\[
(\text{Vertex}) \left( \sum_{r=1}^{\infty} Q^{(r)} \right) = 0 \quad (3.13)
\]

where \( Q^{(r)} \) is the BRST charge for the \( r^{th} \) string. If Eq. (3.13) holds, then \( Q \) applied to the third state will vanish if the other two states are physical; that is, are annihilated by \( Q \). However, \( V_{CSV} \) did obey just such an equation \(12,13 \). Further, since \( Q \) is composed of operators that have a summed conformal dimension of one, \( Q \) will commute with any conformal transformation; and therefore \( V^T \) does obey Eq. (3.13). If we assume that up to a conformal transformation the condition of Eq. (3.13) determines the vertex uniquely, then \( V^T \) must be the required result. This follows from the observation that the \( a_\mu^n \) piece must coincide with \( V_{CSV} \), and this uniquely fixes the conformal transformation used from any other vertex which satisfies Eq. (3.13).

The vertex of Eq. (3.12) has indeed formed the starting point for some recent work\(^{25} \) to find its higher-point generalizations.

4. DISCUSSION

It is straightforward to apply the algorithm used to deduce \( V_{CSV} \) to find the on-shell vertex for the scattering of three Neveu-Schwarz strings. The steps are as follows. We first find a transformation \( g_s \) that rotates the strings and cubes to one. We then assume that this leaves the vertex
invariant (i.e., \( V = V g^{-1} \)). This condition allows us to deduce the identities which in turn determine the vertex. This derivation can be found in Ref. 23). We note here, however, that for supervertices one is dealing with a superconformal transformation, and so the calculation is most efficiently carried out in superspace. The supervertex also depends on one anticommuting variable, since the general \( N \) string supervertex has \( 3N \) Z variables and \( 3N \) \( \theta \) variables which label the end points of the strings. Of these, three \( Z \) 's and two \( \theta \) 's may be fixed by a super \( SL(2,R) \) transformation. The three-point vertex will then depend on one anticommuting variable. As a check on the vertex \footnote{26} for the scattering between a tachyon on one leg and arbitrary states on the other legs.

To find the corresponding vertex to be used in the gauge-covariant Neveu-Schwarz theory, we can apply a super Mandelstam mapping to the on-shell (i.e., dual model) three-string Neveu-Schwarz vertex. This is most easily achieved by mapping over the oscillator identities, in the same manner as discussed for the bosonic vertex, and using these to deduce the covariant vertex.

The ease with which one can deduce string scattering vertices using their conformal properties and their cyclic nature suggests that one can deduce all perturbative aspects of string theory by symmetry principles alone. That is, rather than start from a symmetry principle to determine an action which is then used to calculate Feynman rules, Green functions and finally an S-matrix, we would deduce the scattering amplitude directly by a symmetry argument along the lines presented in Refs. 22) and 23) and summarized in this paper.

Since the symmetries uniquely determine the string theories up to distinctions like whether they are open or closed, it is perhaps not surprising that they can be used to determine the answer directly. It is interesting to recall in this context that string theory was indeed discovered by a part of the S-matrix being guessed. One of the major advantages of normally starting with an action is that one can, with care, ensure unitarity. However, in the vertex construction of string theory, unitarity is more or less manifest.
One clear advantage over string field theory is that it is much faster to compute the perturbative effects this way. Of course, the only way known at present to compute the, no doubt, essential non-perturbative effects is to use string field theory. However, it is interesting to note that the method discussed here stresses conformal invariance and duality and not space-time-related symmetries.
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