ABSTRACT

After making some distinctions between transfer lines and circular machines, certain problems, typical of the type met by a transfer line designer, are discussed. The topics chosen include: measurement of emittance and mismatch, steering, setting tolerances for magnet alignment and excitation and lastly beam blow-up due to scattering in thin windows and screens.

1. DISTINCTIONS BETWEEN TRANSFER LINES AND PERIODIC CIRCULAR MACHINES

Transmission of the position-velocity vector of a particle through a section of a transfer line or circular machine can be simply represented by a $2 \times 2$ matrix (Fig. 1).

\[
\begin{pmatrix}
  y_2 \\
  y'_2 
\end{pmatrix} =
\begin{pmatrix}
  C & S \\
  C' & S'
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
  y'_1
\end{pmatrix} = M_{1+2}
\begin{pmatrix}
  y_1 \\
  y'_1
\end{pmatrix}.
\]

Fig. 1 Transmission through a section of line (linear optics)

\[1\]

The transfer matrix can be found by multiplying together the transfer matrices for the individual elements in the appropriate order. The individual matrices have the form,

\[
M_y = \begin{pmatrix}
\cos (\text{or} \cosh) & \frac{s}{\phi} \sin (\text{or} \sinh) \\
-\frac{s}{\phi} \sin (\text{or} \sinh) & \cos (\text{or} \cosh)
\end{pmatrix}
\]

\[2\]
where

\[ \phi = s\sqrt{|K|} \quad \text{and} \quad |K| = \left| \frac{1}{\rho_p} \frac{d\rho}{dx} + \frac{1}{\rho} \right| \]

in accordance with the earlier lectures by K. Steffen in these proceedings.

However, we often use a parameterized form for the matrix for a section of line, which was also given by K. Steffen.

\[ M_{1+2} = \begin{pmatrix}
\frac{\beta_2}{\beta_1} \left[ \cos \Delta \phi + a_1 \sin \Delta \phi \right] & \frac{\beta_1 \beta_2}{\sqrt{\beta_1 \beta_2}} \sin \Delta \phi \\
(1 + a_1 \alpha_2) \sin \Delta \phi + (a_2 - a_1) \cos \Delta \phi & \sqrt{\beta_1 \beta_2} \left[ \cos \Delta \phi - a_2 \sin \Delta \phi \right]
\end{pmatrix} \]

In the first case of Eq. (1), the matrix is unambiguously determined, but in the second case of Eq. (3), there are in fact an infinite number of sets of parameters \((\beta_1, \beta_2, a_1, a_2 \text{ and } \Delta \phi)\), which satisfy the numerical values of the matrix elements.

This is the root of an important difference between circular machines and transfer lines, which sometimes leads to confusions.

1.1 Circular machines

A circular structure has an imposed periodicity, which imposes the same periodicity on the parameters \(a\) and \(\beta\) and in fact determines them uniquely. If one samples the co-ordinates of an ion after each successive turn in a circular machine, the points will fill out an ellipse in phase space \((y, y')\). Only one set of \(a\) and \(\beta\) values fit that ellipse. It is the periodicity of the structure which makes it possible for that specific ellipse to be returned unchanged after turn and for this reason it is called the matched ellipse. Now suppose one injects a beam of particles, whose spatial distribution defines a different ellipse (see Fig. 2) characterized by some other parameters, say \(a^*\) and \(\beta^*\). The circular machine will not faithfully return this ellipse after each turn. Instead the ellipse will tumble round and round filling out a much larger ellipse of the matched ellipse form.

In a truly linear system, the original ellipse will tumble over and over indefinitely inside the matched ellipse conserving its elliptical form and area, but in a practical system small non-linearities will cause an amplitude frequency dependence, which will distort the ellipse as also shown in Fig. 2. Liouville's theorem requires the phase-space density to
be conserved and in a strict mathematical sense this is true, since as the
figure becomes more wound-up the spiral arms become narrower and the area
is indeed constant. However, it does not take long before the beam is
apparently uniformly distributed over the matched ellipse and for all prac-
tical purposes the beam emittance has been increased. This is called dilu-
tion of phase space by filamentation, which is present to a greater or
lesser extent at the injection into all circular machines.

Since filamentation will quickly transpose any beam ellipse into the
matched ellipse in a circular machine, there is no point in using any \( \alpha \) and
\( \beta \) values other than the matched ones.

Since \( \alpha \) and \( \beta \) depend on the whole structure any change at any point in
the structure will in general (matched insertions excepted) change all the
\( \alpha \) and \( \beta \) values everywhere.

![Diagram](image)

1 Matched ellipse  2 Unmatched beam

3 Filamenting beam  4 Filamented beam

Fig. 2 Matched, unmatched and filamenting ellipses
1.2 Transfer lines

In a transfer line, there is no such restriction. The beam passes once and the shape of the ellipse at the entry to the line determines its shape at the exit. Exactly the same transfer line injected first with one emittance ellipse and then a different ellipse has to be accredited with different $\alpha$ and $\beta$ functions to describe the two cases. Thus $\alpha$ and $\beta$ depend on the input beam as well as the structure. Any change in the structure will only change the $\alpha$ and $\beta$ values downstream of that point. There is an infinite number of sets of $\alpha$ and $\beta$ values, which can be used to describe the motion of a single ion in a transfer line (see Fig. 3) and the choice of a particular set depends on the input ellipse shape described by all the particles in the beam.

![Diagram](image)

Numerical coefficients of the matrix remain the same

Fig. 3 Two ellipses from the infinite set that include the test ion

Thus, if you are given a circular machine you can immediately determine the $\alpha$ and $\beta$ functions, but for a transfer line you need an external criterion to help you decide, such as the matched ellipse of a circular machine at the entry or exit to the line. The underlying transfer matrices, however, depend solely on the structure and never change.

This has some important consequences for emittance measurements, position of monitors, matching, aperture calculations and so on.

2. ORBIT CORRECTION IN TRANSFER LINES

Orbit correction, or steering, is basically straightforward in transfer lines, whereas in circular machines we could fill an entire course on the subject.
The usual philosophy is illustrated in Fig. 4.

Fig. 4  Basic layout of diagnostic and correction elements for transfer line steering

(i) At the entry to the line, it is useful to have a very clear diagnosis of beam position and angle and qualitative information on the shape, since this is usually the ejection from an accelerator and often a boundary of responsibility between groups. A pair of pickups and knowledge of the transfer matrix between them is in principle all that is needed to find the entry angle and position, but in practice, the precision and reliability of this measurement and its credibility as a diagnostic tool are greatly improved by having only a drift space between the pickups. The qualitative knowledge of the beam shape is most easily obtained with a luminescent screen and is of obvious diagnostic use. The quantitative information on beam shape, i.e. emittance and matching considerations are treated in Section 4.

(ii) In the central section of the line, steering magnets are paired with pickups approximately a quarter betatron wavelength downstream, so that the trajectory can be corrected stepwise along the line.

(iii) At the exit to the line, the last two dipole correctors are used as a doublet to steer the beam to the angle and position, dictated by the closed orbit of the following accelerator or by a target. For maximum sensitivity, the dipoles should be approximately a quarter betatron wavelength apart.
The horizontal and vertical planes should be independent for correction elements. For example, tilted dipoles are sometimes used in the lattice of a transfer line, but correction coils for steering should be avoided on such magnets. Skew quadrupoles are occasionally used to interchange emittances between the horizontal and vertical planes. Such insertions also exchange the planes for steering. While being novel, this is quite acceptable, as long as no corrector is placed inside the skew quadrupole insertion, which would cause a coupling of its effect to both planes rather than a simple exchange.

Some care is needed in positioning elements for the best sensitivity. The monitor controlling a steering magnet should be on the adjacent peak of the downstream beam oscillation (see Fig. 5), i.e. for the section of line from the steering dipole to the pickup, the matrix element b in Eq. (1) must be relatively large or in other terms $\Delta \phi = \pi/2$ in Eq. (3).

![Fig. 5 Positioning of correction elements](image)

The monitors and magnets should be sited near maxima in the $\beta$-function, since these are the most sensitive points for controlling and observing. This depends on the choice of input beam ellipse.

Monitors can also be profitably placed in bends at points where off-momentum particles would have their maximum deviations. Using three well-placed pickups a bend can be used for momentum analysis.

The simple linear matrices make the analysis of such systems very easy.

In a long line, a global correction may well be possible, followed by an exact beam steering at the end using two dipoles.
Fig. 6 Trajectories in the TT6-TT1 Antiproton transfer at CERN
Figure 6 shows computer output of the beam trajectory in the TT6-TT1 antiproton transfer line at CERN. The first two pickups measure the incoming angle and position in both planes. These pickups are separated by exactly 10.75 m of free space.

The next two pickups are in a long bend and act as a momentum analyser, in conjunction with an angle and position measurement made using the first two pickups.

The remaining pickups have associated steering magnets.

At the end of the line two dipoles match the beam to the ISR’s closed orbit.

For the example shown in Fig. 6, it was found that a single corrector could virtually correct the whole trajectory with the result shown in Fig. 6.

This type of correction is only practical with non-destructive pick-ups, which reliably record the complete trajectory in one shot, and an on-line computer for logging, display, analysis and application.

The TT6 line achieved 0.1 mm accuracy with as little as $10^9$ particles. All readings were logged and stored for trends, for later analysis. The steering magnets were also equipped with Hall probes (temperature stabilized to $\pm 0.1 \degree C$ for outside ambient temperatures 15 $\degree C$ to 34 $\degree C$). These probes made relative field changes extremely accurate, eliminating any hysteresis errors. This rather careful approach was justified by the scarcity of antiprotons and since setting-up could not be done with the reverse injection of protons.

3. MATCHING TRANSFER LINES

Ideally long transfer lines comprise a regular cell structure over the majority of their length with matching sections at either end to co-ordinate them to their injector and user machines. The regular part of the structure is then regarded as periodic and the simple FODO cell theory, given in earlier lectures by K. Steffen, applies. Usually thin-lens formulae are quite sufficient. The matching sections are complicated and a complete course could be given on this. Basically one needs to match $\beta$, $\alpha$, $D$, $D'$ in both planes. In theory eight variables, that is eight quadrupole strengths and sometimes positions, need to be adapted. Some analytic solutions exist, but usually one uses a mixture of theory, intuition and computer optimization programmes. Some of these problems are dealt with in the lecture on insertions by K. Steffen.
4. EMMITTANCE AND MISMATCH MEASUREMENT

With semi-destructive monitors, such as secondary emission grids or digitized luminescent screens, a density profile can be obtained for a beam. This profile is a projection of the population of the phase space ellipse of the beam onto the transverse co-ordinate axis. In general the profile is a near-Gaussian, but this is not really important for the following (see lecture by K. Potter in these proceedings "Luminosity Measurements and Calculations" where the representation of distribution is discussed). From the profile, the standard deviation of the distribution, \( \sigma \) can be found and this can be used to define a beam width, \( W \). As was explained in the lectures by E.N. Wilson in these proceedings, \( W \) is then used to define the emittance \( \varepsilon \), but unfortunately several definitions are current.

\[
\varepsilon = \begin{cases} 
\frac{\langle \pi \rangle (2 \sigma)^2}{\beta} & \text{Mostly used in proton machines, with or without } \pi \\
\frac{\sigma^2}{\beta} & \text{Mostly used in electron machines, usually without } \pi
\end{cases}
\]

(4)

Somewhat arbitrarily, \( \varepsilon = \pi(2\sigma^2)/\beta \) will be used in this paper.

If \( \beta \) is known, then a single profile measurement determines \( E \) by Eq. (4), but as can be understood from Section 1.2, it is not easy to be sure which \( \beta \) to use, or rather, whether the beam that has been measured is matched to the \( \beta \)-values used for the line. Indeed, the measurement of any mismatch is as important as the emittance itself. This problem can be resolved by using three monitors (see Fig. 7), i.e. the three width measurements determine the three unknowns \( a, \beta \) and \( \varepsilon \) of the incoming beam.

![Fig. 7 Layout for emittance measurement](image-url)
By definition, Eq. (4),

\[ e = \pi \frac{W^2}{\beta_0} = \pi \frac{1}{\beta_1} = \pi \frac{2}{\beta_2} \]  \hspace{1cm} (5)

where \( \beta_0, \beta_1 \) and \( \beta_2 \) are the \( \beta \)-values corresponding to the beam and are therefore uncertain.

We may not know \( \beta \) and \( \alpha \), but since we know the transfer matrices, we know how \( \beta \) and \( \alpha \) transfer (see lectures by K. Steffen in these proceedings).

\[
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}
=
\begin{pmatrix}
C^2 & -2CS & S^2 \\
-CC' & CS+SC' & -SS' \\
C'S' & -2C'S' & S'^2
\end{pmatrix}
\begin{pmatrix}
\beta \\
\alpha \\
\gamma
\end{pmatrix}
\]  \hspace{1cm} (6)

where \( \gamma = (1 + \alpha^2)/\beta \).

Thus, from Eq. (6)

\[
\beta_2 = C_2^2 \beta_0 - 2C_2S_2a_0 + \frac{S^2}{\beta_0} (1 + \alpha^2) \]  \hspace{1cm} (7)

\[
\beta_1 = C_1^2 \beta_0 - 2C_1S_1a_0 + \frac{1}{\beta_0} (1 + \alpha^2) \]  \hspace{1cm} (8)

and from Eq. (5),

\[
\beta_0 = \pi \frac{W^2}{e} \]  \hspace{1cm} (9)

\[
\beta_1 = \left( \frac{W_1}{W_0} \right)^2 \beta_0 \]  \hspace{1cm} (10)

\[
\beta_2 = \left( \frac{W_2}{W_0} \right)^2 \beta_0 \]  \hspace{1cm} (11)

From Eqs. (7) and (8), we can find \( \alpha_0 \) and using Eqs. (10) and (11), we can express \( \alpha_0 \) as,

\[
\alpha_0 = \frac{1}{2} \beta_0 \Gamma \]  \hspace{1cm} (12)

where
\[ \Gamma = \frac{(W_2/W_0)^2/S_2 - (W_1/W_0)^2/S_1 - (C_2/S_2)^2 + (C_1/S_1)^2}{(C_1/S_1) - (C_2/S_2)} . \]  

(13)

Since \( \Gamma \) is fully determined, direct substitution back into Eq. (7) or Eq. (8), using Eq. (10) or Eq. (11) to re-express \( \beta_1 \) or \( \beta_2 \), yields \( \beta_0 \), which via Eq. (9) gives the emittance,

\[ \varepsilon = (\pi W_0^2)^{1/2} \left[ \left( (W_1/W_0)^2/S_1 - (C_1/S_1)^2 + (C_1/S_1) \Gamma - r^2/4 \right) \right]^{1/2} \]  

(14A)

where

\[ \beta_0 = \frac{1}{\sqrt{[(W_1/W_0)^2/S_1 - (C_1/S_1)^2 + (C_1/S_1) \Gamma - r^2/4]^{1/2}}} \]  

(14B)

The mismatch parameters \( \Delta \beta \) and \( \Delta \alpha \), the differences between what is expected and what exists, can now be found directly from Eqs. (14B) and (12).

5. SMALL MISALIGNMENTS AND FIELD RIPPLE ERRORS IN DIPOLES AND QUADRUPOLES

One problem, which always faces a transfer line designer, is to fix the tolerances for his magnet alignment and excitation currents. Although the following is rather idealistic and does not include such real-world problems as magnets with correlated ripple because they are on the same transformer, it does give a basis for fixing tolerances.\(^1\)

5.1 Dipole and misalignment errors in transfer lines

The motion of a particle in a transfer line can be written as

\[ y = A \sqrt{\beta} \sin (\phi + B) \]  

(15)

As we now know, this motion is an ellipse in phase space with

\[ y' = \frac{A}{\sqrt{\beta}} \cos (\phi + B) - \frac{A \alpha}{\sqrt{\beta}} \sin (\phi + B) . \]  

(16)

Rearranging we have

\[ Y = y/\sqrt{\beta} = A \sin (\phi + B) \]  

\[ Y' = y\alpha/\sqrt{\beta} + y'\sqrt{\beta} = A \cos (\phi + B) , \]  

(17)

where \((Y, Y')\) are known as normalized phase-space coordinates since with these variables particles follow circular paths. Note: \( y' \) denotes \( dy/ds \) while \( Y' \) denotes \( dY/d\phi \) and \( \alpha = -1/2 \) \( d\beta/ds \).
The transformation to \((y, y')\) is conveniently written in matrix form.

\[
\begin{pmatrix}
y'
\end{pmatrix} = \begin{pmatrix}
1/\sqrt{\beta} & 0 \\
\alpha/\sqrt{\beta} & \sqrt{\beta}
\end{pmatrix} \begin{pmatrix}
y
\end{pmatrix}.
\]

(18)

Consider now a beam for which the equi-density curves are circles in normalized phase space. If this beam receives an unwanted deflection, \(D\), it will appear at the time of the deflection as shown in Fig. 8(a). However, this asymmetric beam distribution will not persist. As the beam continues along the transfer line, the particles will re-distribute themselves randomly in phase, while maintaining their distance from the origin, so as to restore rotational symmetry. This effect is known as filamentation (see also Section 1.1). Thus after a sufficient time has elapsed the particles, which without the deflection \(D\) would have been at point \(P\) in Fig. 8(b), will be uniformly distributed at a radius \(D\) about the point \(P\).

(a) Beam directly after deflection, \(D\)  
(b) Particle distribution after phase randomization (Filamentation)

**Fig. 8 Effect of an unwanted deflection**

For one of these particles the projection onto the \(y\)-axis will be

\[
y_2 = y_1 + D \cos \theta,
\]

where the subscripts 1 and 2 denote the unperturbed and perturbed positions respectively.

Taking the square of this amplitude

\[
y^2_2 = y^2_1 + 2y_1D \cos \theta + D^2 \cos^2 \theta
\]
and then averaging over the particles around the point P after filamentation has randomized the kick gives

\[ \langle Y^2 \rangle_p = \langle Y^2 \rangle_{1'}^p + 2 \langle Y_1 D \cos \theta \rangle_p + \langle D^2 \cos^2 \theta \rangle_p. \]

Since \( Y_1 \) and \( D \) are uncorrelated (i.e. \( D \) does not depend on \( Y_1 \)), the second term can be written as

\[ 2 \langle Y_1 D \cos \theta \rangle_p = 2 \langle Y_1 \rangle_p \langle D \cos \theta \rangle_p. \]

The second factor is zero, since \( D \) is a constant [Fig. 8(a)], which gives,

\[ \langle Y^2 \rangle_p = \langle Y^2 \rangle_{1'}^p + \frac{1}{2} \langle D^2 \rangle_p = \langle Y^2 \rangle_{1'}^p + \frac{1}{2} D^2. \]

However, this result is true for any \( P \) at any radius \( A \) and hence it is true for the whole beam and

\[ \langle Y^2 \rangle_p = \langle Y^2 \rangle_{1'}^p + \frac{1}{2} D^2. \quad (19) \]

The emittance blow-up will be

\[ \varepsilon_2 = \varepsilon_1 + 2\pi D^2, \quad (20) \]

where, by definition, \( \varepsilon = 4\pi \langle Y^2 \rangle \). The subscripts 1 and 2 refer to the unperturbed and perturbed emittances respectively, and remember that \( Y = y/\sqrt{\beta} \).

Expanding the deflection, \( D \),

\[ D^2 = (\Delta Y)^2 + (\Delta Y')^2 = (\Delta Y)^2 + (1 + \frac{a^2}{\beta}) + (\Delta Y')^2 \beta \quad (21) \]

and substituting into (20) gives the emittance blow-up, in terms of the basic errors. Thus

\[ \varepsilon_2 = \varepsilon_1 + 2\pi [(\Delta Y)^2 \left(1 + \frac{a^2}{\beta}\right) + (\Delta Y')^2 \beta], \quad (22) \]

where

\( \Delta y \) is a beam alignment error,

\( \Delta y' = \frac{\alpha B}{B \beta} \) an angle error from a field error \( \Delta B \) of length \( \ell \).
5.2 Gradient errors in transfer lines

Consider once again a beam for which the equi-density curves are circles in normalized phase space. If this beam sees a gradient error, \( k \), the equi-density curves directly after the perturbation will be ellipses as shown in Fig. 9(a). Since the object of this analysis is to evaluate the effects of small errors, it is sufficient to regard this gradient error as a thin lens with the transfer matrix

\[
\begin{pmatrix}
  y_2' \\
  y_1'
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  k & 1
\end{pmatrix} \begin{pmatrix}
  y_2 \\
  y_1
\end{pmatrix}
\]

(23)

where

\[
k = \frac{1}{B_0} \Delta G, \quad \text{an amplitude dependent kick arising from the gradient error } \Delta G \text{ of length } \lambda.
\]

Denoting the matrix in Eq. (18) as \( T \), it is easy to show that

\[
\begin{pmatrix}
  y_2' \\
  y_1'
\end{pmatrix} = T \begin{pmatrix}
  1 & 0 \\
  k & 1
\end{pmatrix} T^{-1} \begin{pmatrix}
  y_2 \\
  y_1
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  k \beta & 1
\end{pmatrix} \begin{pmatrix}
  y_2 \\
  y_1
\end{pmatrix}.
\]

It is now convenient to find a new coordinate system \((YY, YY')\), which is at an angle \( \theta \) to the \((Y, Y')\) system, and in which the perturbed ellipse is a right ellipse [see Fig. 9(b)].
\[
\begin{pmatrix}
YY_2 \\
YY_2'
\end{pmatrix}
= \begin{pmatrix}
c & -s \\
s & c
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
k \beta & 1
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y_1'
\end{pmatrix}
\tag{24}
\]

where \(s\) and \(c\) denote \(\sin \theta\) and \(\cos \theta\) respectively.

Introducing the initial distribution \(Y_1 = A \sin (\phi + B)\), \(Y_1 = A \cos (\phi + B)\), in the above expression for the new distribution, gives

\[
YY_2 = A \sqrt{1 + s^2 k^2 \beta^2 - 2sc \beta \sin (\phi + B + \theta)}
\]
\[
YY_2' = A \sqrt{1 + c^2 k^2 \beta^2 + 2sc \beta \sin (\phi + B + \theta')} \tag{25}
\]

where

\[
\theta = \tan^{-1} \left( \frac{-s}{c - sk \beta} \right) \quad \text{and} \quad \theta' = \tan^{-1} \left( \frac{-c}{s + ck \beta} \right).
\]

The \((YY_2, YY_2')\) ellipse will be a right ellipse when \((\theta - \theta') = \pi/2\), which gives the condition

\[
\tan (2\theta) = \frac{2}{k \beta}. \tag{26}
\]

Equations (25) can be simplified using (26) and the relationship \((\theta - \theta') = \pi/2\) and rewritten as

\[
\begin{pmatrix}
YY_2 \\
YY_2'
\end{pmatrix}
= \begin{pmatrix}
\tan \theta & 0 \\
0 & 1/\tan \theta
\end{pmatrix}
\begin{pmatrix}
YY \\
YY'
\end{pmatrix}
\tag{27}
\]

where

\[
YY_1 = A \sin (\phi + B') \quad \text{i.e.} \quad Y_1 \text{ and } Y_1' \text{ with a phase shift}
\]

\[
YY_1' = A \cos (\phi + B')
\]

\[
B' = (B + \theta) = \left[ B + \tan^{-1} \left( 1/\tan \theta \right) \right].
\]

Thus it has been possible to diagonalize Eq. (24) by introducing a phase shift \(\theta\) into the initial distribution. Equation (27) is therefore not a true point-to-point transformation, as is Eq. (24) but since the initial distribution is rotationally symmetric the introduction of this phase shift has no effect.
The distance from the origin of a perturbed particle is given by Eq. (27) as

\[
\langle Y^2 + Y'^2 \rangle = \frac{1}{2} (\tan^2 \theta + \frac{1}{\tan^2 \theta}) \langle A^2 \rangle .
\]

Averaging over \(2\pi\) in \(\phi\) gives

\[
\langle Y^2 + Y'^2 \rangle = \frac{1}{2} (\tan^2 \theta + \frac{1}{\tan^2 \theta}) \langle A^2 \rangle ,
\]

but

\[
\langle A^2 \rangle = \langle Y^2 + Y'^2 \rangle = \langle Y^2 + Y'^2 \rangle
\]

and from (26)

\[
(\tan^2 \theta + \frac{1}{\tan^2 \theta}) = k^2 \beta^2 + 2 .
\]

Thus,

\[
\langle Y^2 + Y'^2 \rangle = \frac{1}{2} (k^2 \beta^2 + 2) \langle Y^2 + Y'^2 \rangle .
\]

As in the previous case for dipole errors, the asymmetric beam distribution will not persist. The beam will regain its rotational symmetry by filamentation or phase randomization. Each particle, however, will maintain its distance from the origin constant. Once filamentation has occurred, the distribution will not distinguish between the YY and YY' axes and Eq. (28) can be rewritten as

\[
\langle Y^2 \rangle = \frac{1}{2} (k^2 \beta^2 + 2) \langle Y^2 \rangle
\]

and hence the emittance blow-up will be

\[
\epsilon_2 = \frac{1}{2} (k^2 \beta^2 + 2) \epsilon_1 .
\]

Thus we have expressions for what happens after an alignment error, a dipole error and a quadrupole error, assuming that the beam is observed after phase randomization has taken place. A series of errors can be treated by taking them in beam order and assuming complete phase randomization between each error.
If a circular machine is at the end of the transfer line, filamentation will take place there and the above gives a way of calculating the emittance blow-up due to a single error in the preceding transfer line, but transfer lines are usually too short to even show the effects of filamentation and certainly there is never complete randomization between elements in a line. In the real world adjacent magnets are often on the same transformer, which also gives correlated errors. Having pointed out these deficiencies, the above method does give a basis upon which to fix tolerances. The assumption that full randomization takes place between elements will give a pessimistic result for the usual case of tens of elements spread over several betatron wavelengths, which errs on the correct side for fixing tolerances. Small numbers of elements, very close in phase with possibly correlated errors can be underestimated by this analysis.

6. EMITTANCE BLOW-UP DUE TO THIN WINDOWS IN TRANSFER LINES

Transfer lines are often built with a thin metal window separating their relatively poor vacuum from that of the accelerator or storage ring that they serve. The beam must pass through this window with as little degradation as possible. Luminescent screens are also frequently put into beams with the same hope that they will have a negligibly small effect on the beam emittance. It is therefore interesting to know how to calculate the blow-up for such cases.

The root mean square projected angle due to multiple Coulomb scattering in a window is given by\(^2, 3\)

\[
\sqrt{\langle \theta^2 \rangle} = \frac{0.0141}{p_C^0} Z_{inc} \sqrt{\frac{L}{L_{rad}}} (1 + \frac{1}{5} \log_{10} \frac{L}{L_{rad}}) \text{ [radian]}, \quad (31)
\]

where

- \(Z_{inc}\) is particle charge in units of electron charge
- \(p_C = v/c\)
- \(L\) is thickness of scatterer
- \(L_{rad}\) is radiation length in material of scatterer.

Consider a particle with a projected angular deviation of \(\theta_1\) at the window due to the initial beam emittance. This particle receives a net projected kick in the window of \(\theta_s\) and emerges with an angle \(\theta_2\) given by

\[
\theta_2 = (\theta_1 + \theta_s).
\]
By squaring and averaging over the whole beam this becomes

$$\langle \theta^2 \rangle_2 = \langle \theta^2 \rangle_1 + \langle \theta^2 \rangle_5 + 2\langle \theta_1 \theta_5 \rangle$$

but, since the initial $\theta_1$ is in no way correlated to $\theta_5$,

$$2\langle \theta_1 \theta_5 \rangle = 2\langle \theta_1 \rangle \langle \theta_5 \rangle = 0$$

and the above simplifies to

$$\langle \theta^2 \rangle_2 = \langle \theta^2 \rangle_1 + \langle \theta^2 \rangle_5.$$  \hspace{1cm} (32)

This describes the situation immediately after the scattering (see Fig. 10) when the beam is no longer matched.

Fig. 10  Effect of a thin scatterer in normalized phase space

Using the same arguments as in Section 5.1 we see that this initial distribution filaments and the average angular divergence becomes

$$\langle \theta^2 \rangle_2 = \langle \theta^2 \rangle_1 + \frac{1}{2} \langle \theta^2 \rangle_5.$$  \hspace{1cm} (33)

For conversion to beam sizes or emittance the following relationships can be used,

$$\langle \theta^2 \rangle = \sigma^2 \frac{(1 + \alpha^2)}{\beta^2} = \frac{\varepsilon}{4\pi} \frac{(1 + \alpha^2)}{\beta}.$$ \hspace{1cm} (34)
where $\sigma$ is the r.m.s. amplitude in the plane of projection.
$\varepsilon$ is the beam emittance in the plane of projection.

Combining (33) and (34) gives the beam blow-up,

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\langle \theta^2 \rangle}{\langle \theta^2 \rangle_1} = \frac{\sigma^2}{\sigma_1^2} = 1 + \frac{\langle \theta_5^2 \rangle}{\langle \theta_5^2 \rangle_1} \frac{2\pi B}{E_i(1 + \sigma^2)} ,$$  

(35)

in which $\langle \theta_5^2 \rangle$ is given by Eq. (31).

* * *

REFERENCES


3) E. Fischer, private communication.