UNITARITY BOUNDS AT HIGH ENERGIES

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ABSTRACT

We review the variable-\(t\) bounds on the elastic absorptive amplitude at high energies. Applying these bounds, particularly the upper unitarity bound, to ISR and SppS data tests the self-consistency of these data sets, notably with regards to normalization, rapid changes of slopes and determination of global quantities such as \(\sigma_{el}\), \(\sigma_{tot}\) and the forward elastic slope parameter.

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1. Introduction

The history of unitarity bounds dates back quite a few years. The very first one, and still the most famous, is the Froissart–Martin\(^1\) (FM) bound on the rise of \(\sigma_{\text{tot}}\): it contains no reference to any other experimentally observable quantity. Later, MacDowell and Martin\(^2\) (MDM) derived a lower bound on the forward absorptive elastic slope \(B^A(s,t=0) = d/dt(\ln(d\sigma^A/dt))|_{t=0}\), given experimental values for \(\sigma_{\text{el}}^A\) and \(\sigma_{\text{tot}}\):

\[
B^A(s,t=0) \geq B_{\text{MM}} = \sigma_{\text{tot}}^2/(18\pi\sigma_{\text{el}}^A)
\]

(1)

where the superscript \(A\) stands for absorptive, i.e. due to the imaginary part only. Using the same experimental inputs, Roy and Singh\(^3\) (RS) then derived bounds on the absorptive elastic amplitude, but these were applicable only in the very forward direction, Auberson and Roy\(^4\) (AR), noticing that the MacDowell–Martin bound was nearly saturated by the measured \(B^A(s,t=0)\), i.e. that the ratio

\[
R = B^A(s,t=0)/B_{\text{MM}} \geq 1
\]

(2)

was very close to one, decided to improve the Roy–Singh bound by adding as input \(B^A(s,t=0)\) itself; this resulted in extending the range of applicability of the bounds greatly. In Figure 1, one can compare\(^4\) these bounds for various values of \(R\).

Recently\(^5\)-\(^9\), I have extended still more this range of application by specifying yet another experimental quantity, namely the value of the absorptive elastic amplitude at an arbitrarily chosen physical \(t\)–value (hence the name variable \(-t\) bound). In general, to be able to compare directly with the measured elastic differential cross–section, one has to remove the real part contribution from the latter; over the range of \(-t\) reported here, the real part contributes less than the errors in \(d\sigma/dt\), unless otherwise noted. For a more careful treatment of real part effects, I refer the reader to the original references.\(^6\)-\(^7\) The effect of spin can be neglected, as always, since bounds derived with the inclusion of spin effects differ so little from the ones without.\(^8\)
II. Background to the problem

We must first state our conventions: the elastic amplitude is normalized such that \( \frac{d\sigma}{dt} = \pi |f(s,t)|^2 \) and the optical theorem reads \( \sigma_{\text{tot}} = 4\pi \text{Im} f(s,t) \). The amplitude is then the Fourier–Bessel transform of the elastic profile \( h(b,s) \)

\[
f(s,t) = \int_0^\infty h(b,s) J_0(b\sqrt{-t}) \, db
\]

with \( h(b,s) \) satisfying the unitarity equation

\[
2 \text{Im} h(b,s) = |h(b,s)|^2 + G(b,s)
\]

The positivity of the overlap function \( G(b,s) \) and the neglect of the real part yields immediately that \( 0 \leq \text{Im} h(b,s) \leq 2 \). From now on, I will drop the explicit reference to absorptive part and mention \( h \) instead of \( \text{Im} h(b,s) \). In fact, at high energies, the smallness of the ratio \( \sigma_{\text{el}}/\sigma_{\text{tot}} \) (of order 20%) allows one to construct eikonal and overlap function models for which, by construction, \( h \) lies in the interval \([0,1]\). The profiles for many of the famous bounds are simple: for the Froissart–Martin bound, it is a step function, and for the MacDowell–Martin bound, an inverted parabola. In all cases, the bound profiles assume the existence of a maximum impact parameter which is justified a posteriori by finding the extremal solution.

III. Finding the extremal solutions

The problem, in general, can be stated as follows in terms of the dimensionless scaling variable \( x = (b/b_{\text{max}})^2 \). Given experimental measurements for \( \sigma_{\text{tot}}, \sigma_{\text{el}}, B(t=0) \) and \( f(t_1) \), we have the following constraints on \( h \)

\[
\sigma_{\text{tot}} = 2\pi \beta_m \int_0^x h(x) \, dx
\]

\[
\sigma_{\text{el}} = \pi \beta_m \int_0^x h^2(x) \, dx
\]

\[
B(t=0)\sigma_{\text{tot}} = \pi \beta_m^2 \int_0^x h(x) x \, dx
\]

\[
f(t_1) = \beta_m^2/2 \int_0^x h(x) J_0(\sqrt{-\beta_m^2 t_1 x}) \, dx
\]

with the shorthand notation \( \beta_m = b/b_{\text{max}} \). The existence of \( \beta_m \) implies further that \( h(x \geq 1) = 0 \)

\[\text{and positivity of} \ h \ \text{has to be imposed for} \ x \leq 1\]
\[ h(0 \leq x \leq 1) \geq 0 \]  

The problem of finding the extrema of \( f(t) \), subject to the four constraints (5) – (8) and eq. (9) is done by the method of Lagrange multipliers. These can be chosen so the solution for the rescaled profile \( \mathcal{H} = 2\pi \beta_m h / \sigma_{\text{tot}} \) reads in terms of the four multipliers \( \alpha_i \) and a parameter \( d \) which is obtained from \( \beta_m \) by \( d = 3 \pi \sigma_{\text{el}} \beta_m / \sigma_{\text{tot}}^2 \):

\[
\mathcal{H} = \alpha_1 J_0(\sqrt{-\beta_m'(x)}) - \alpha_2 - \alpha_3 x - \alpha_4 J_0(\sqrt{-\beta_m'(x)})
\]  

All five dimensionless parameters are to be determined by substituting into the five equations (5) – (9), with the added inequality constraint (10) to be imposed (in (9) we take \( x = 1 \) as the last time \( h \) becomes 0).

The extremal amplitudes are then just obtained by Fourier–Bessel transforming

\[
f(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathcal{H}(x) J_0(\sqrt{-\beta_m'(x)}) \, dx
\]  

It is important to note that the profile \( \mathcal{H}(x) \) is actually a function of \( t \), both explicitly through the argument of the first term of (11) and implicitly through the \( t \) – dependence of the \( \alpha_i \) and \( d \) once these are solved for, at any given \( t \). As such \( \mathcal{H}(x) \) does not correspond to any one (data – reproducing) amplitude, but is rather the envelope of all possible amplitudes consistent with unitarity, positivity as in (10) and the experimental measurements that are the inputs to the left – hand sides of (5) – (8). The method of obtaining the five parameters numerically and their \( t \) – dependence are described in the literature.\textsuperscript{5,6}

The choice of the \( \alpha_i \) is such that requiring or giving information about any one of the experimental quantities mentioned above makes one of the \( \alpha_i \) non – zero. Hence knowledge of the amplitude makes \( \alpha_1 \neq 0 \); similarly, to \( \sigma_{\text{tot}} \) corresponds \( \alpha_2 \), to \( B(t=0) \) \( \alpha_3 \), and to \( f(t_{1}) \) \( \alpha_4 \). All previously known bounds mentioned in the introduction are then simply obtained: the FM bound has \( \alpha_2 \neq 0 \) only and constant, the MDM bound has both \( \alpha_2 \) and \( \alpha_3 \neq 0 \) and constant, the RS bound has only \( \alpha_3 \) and \( \alpha_4 \) equal to zero and the AR bound only \( \alpha_4 \) equal to zero. Clearly only the variable – \( t_1 \) bound has all Lagrange multipliers non – vanishing in general. The reason the variable – \( t_1 \) bound is much superior
to the previously obtained bounds resides in the fact that the extra constraint (8), corresponding to a non-zero $\alpha_{4}$, has an oscillating factor of our choice, which can probe much finer structures in the profile $h$, than can the global constraints (5)–(7) which involve only powers of $h$ and $x$. Consequently, the bound profiles $h(x)$, although $t$–dependent, represent much better the actual (unique) profile obtained by direct Fourier–Bessel transformation of the data. Since the data will be found to saturate the upper bound over a wide $t$–range (see the next section), the bound profile $h(x)$ is a very slowly varying function of $-t$ over that interval, to such an extent as to be able to reproduce the energy trends found by the experimental analysis of the data in impact parameter space done by Amaldi and Schubert.

IV. Comparison with ISR and SppS data

Let us first summarize previously published results on the comparison with ISR data (the reader will find reference to the data in the original article). At 23.5 GeV, the data of Boehm et al. could be renormalized slightly lower to avoid the conflict with the upper unitarity bound shown in Figure 2a. The curves correspond to specifying $t_1$ to be either at the position of the zero in Im$t(t)$ at $t = t_o$ or at more modest values which tremendously improve the bound: the dashed curve is for $-t_1 = .55$ and the solid curve for $-t_1 = .71$ GeV$^2$. At 62.5 GeV, only the first three points of the Kwak et al. data are in conflict with unitarity but the overall normalization appears otherwise good, as can be seen in Figure 2b. At 52.8 GeV, we again have saturation of the data over a large $t$ range, as is shown for small $-t$ in Figure 2c, the full line corresponding to the actual value of $R$ and the dashed line to one with $\sigma_{el}$ and $B(t = 0)$ at their largest experimental values. In Figure 2d, we illustrate the fact that the lower bound is of little practical use (except in the very forward direction) and that we considerably change the look of the AR bound (dashed line) by specifying a moderate $-t_1 = .7$. The AR bound is nearly identical in this $t$ range to the one with $t_1 = t_0$.

At the SppS, an analysis was carried out with preliminary values of the UA4 data which are sufficiently close to the final numbers that the comparison shown in Figure 3 is still valid for the final
data. Again remarkable saturation of the bound by the data can be seen (R ≈ 1.16 in this case). We should point out that experimental quantities are closely correlated by the fact that for an exponential peak and constant ρ(t), we have that

$$
\sigma_{\text{el}}^{A} = \sigma_{\text{el}}^{B} / (1 + \rho^2) = \sigma_{\text{tot}}^{2 / (16 \pi B A)}
$$

which implies that a "natural" value for R is, by (1), 18/16 or 1.125 which is quite close to the experimental value.

If the value of R is sufficiently accurate, the allowed "band" of bounds generated by taking its extremal values can rule out quick breaks of slopes at small $-t^6$. However the errors on the global quantities entering in the definition of R, i.e. $a_{\text{tot}}$, $a_{\text{el}}$ and B($t=0$) are highly correlated by the approximate validity of the assumptions leading to (13). It is therefore important for the experimentalists themselves to quote their best estimate for the value of R. I should point out a minor weakness in the hypotheses leading to (13). In the range of t where one has nearly a unique exponential, Geometrical Scaling (GS) arguments would lead us to expect a zero in ρ(t). Clearly GS is not exact throughout the ISR $-$ SpS energy range$^{10}$ but it should give a better guide to the behaviour of ρ(t) than assuming pure constancy. In particular, more careful analyses$^7$ have shown the persistence of this zero. Since the real part affects fits of do/dt as a squared contribution except in the Coulomb interference region, one would expect to see the largest effect there.

V. Conclusion

The variable $-t_i$ bounds can serve to normalize different experiments at the same energy; it is quite clear, for example, that it would be incorrect to normalize the early UA1 and UA4 1982 data at one value of, say, $-t = .15$ (see Figure 4 which requires downwards rescaling of the early UA1 data a somewhat similar scheme was followed in Ref.11). They can also rule out unusually low or high data points and thus rapidly oscillating data. They can be used proportionally further towards the dip at the SpS than at the ISR, which suggests$^5$ that we may be approaching scaling in $\beta_m^m t$, if d is a sufficiently slowly varying function of t, i.e. scaling in $\sigma_{\text{tot}}^{2 / \sigma_{\text{el}} t}$. They are useless if $t_i$ is chosen to be $t_0$, the loca-
tion of the zero in \( \text{Im}(t) \) estimated somehow by carefully\(^{5^\circ} \) removing the real part that masks it. Finally, let me remind our experimentalist friends that it is crucial to have an estimate of the error on \( R \), given that the quantities entering in its definition are so correlated.

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- Figure 1 -
- Figure 2 -