SEXTUPOLE CORRECTION AND DYNAMIC APERTURE: NUMERICAL AND ANALYTICAL TOOLS

G. GUIGNARD and J. HAGEL
CERN, 1211 Geneva 23, Switzerland

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We discuss analytical and numerical tools for dealing with nonlinearities arising from the chromatic perturbations and sextupoles in circular accelerators. Chromatic effects are described and the usual methods of correction are discussed. The Hamiltonian formalism as a tool for dealing with the effect of nonlinearities in the transverse coordinates is described. Different approaches (on- and off-resonances) for the nonlinear betatron motion are obtained, and the distortion of nonlinear invariant curves is calculated. In addition, we introduce the concept of dynamic aperture. Finally, we present a new technique of successive linearizations that is directly applicable to the differential equations of motion. With this method, we derive an approximate analytical expression for the nonlinear invariant curves in phase space. In addition, we introduce a semianalytic tool for estimating the dynamic aperture without use of a tracking program.

1. INTRODUCTION

The aim of this report is to discuss the nonlinearities associated with chromatic perturbations and the sextupole fields necessary for their correction. It contains a review of these perturbations and of their influence on the so-called dynamic aperture. We describe the numerical and analytical tools available for studying these problems.

Section 2 describes the chromatic effects and the variables of interest. The question of the cancellation of the linear perturbations in the momentum deviation is summarized, and the influence of the phase advance per cell and sextupole arrangement is discussed. In Section 3 we then describe the nonlinearities in the momentum deviation and in the amplitude. The quantities characterizing the nonlinear effects, such as the dynamic aperture, are defined, and the computer programs used for calculating them, including tracking programs, are mentioned. We also show that the sextupole arrangement can have an effect on the nonlinear perturbations. The Hamiltonian formalism is presented in Section 4 as a tool for dealing with the nonlinearities in the transverse coordinates. The two-dimensional perturbations of the betatron motion are considered, and the perturbation treatment is done up to second order. This allows two different approaches, called the resonance and global approaches. The first one, valid near a strong resonance, is shown to be useful in determining the sextupole strengths which minimize adverse effects. The second one, valid between strong resonances, makes it possible to give distortions of the invariants and the amplitude modulations when the perturbation is not too

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Both methods have important limitations when dealing with the dynamic aperture. Therefore, a new technique of successive linearizations, directly applicable to the differential equations of the betatron motion, is given in Section 5. Solving the recurrence relations associated with the linearized equation of motion, we derive an approximate analytic expression for the nonlinear invariant curves in phase space. A simple one-dimensional application is presented, and the generalization to two dimensions is discussed. In addition, we introduce a semianalytic tool for estimating the dynamic aperture without tracking. As an example, this tool is used for estimating the dynamic aperture as a function of the phase advance for a FODO lattice, and the result is shown to be in good agreement with the limit obtained by tracking.

2. CHROMATIC PERTURBATIONS AND LINEAR COMPENSATION

Chromatic perturbations are the variations of the optics parameters associated with momentum deviations $\delta = \Delta p/p_0$, $p_0$ being the nominal value of the momentum. Differential bending gives rise to the well-known dispersion function $D_x = \partial x_c / \partial \delta$, where $x_c$ is the off-momentum closed orbit. It must be noted that the function $\eta = x_c / \delta$ is sometimes used instead of $D_x$. Differential focusing in the quadrupoles is first seen as a change in the oscillation frequency and amplitude. Higher-order fields, e.g., sextupole fields, also induce changes in tune and amplitude, as well as nonlinear forces and additional tune variations with the amplitude.

2.1. Global Chromaticity and Chromatic Variables

The changes in the betatron oscillation frequency are evaluated by the successive tune derivatives, $Q'$, $Q''$, $Q'''$ with respect to $\delta$. The global chromaticity $Q'$ depicts the linear variation of the tune with $\delta$. Using the tune shift formula and neglecting the contribution of the dipoles, we have

$$Q' = \frac{dQ}{d\delta} = -\frac{1}{4\pi} \int_0^C \beta(K_0 - D_s K'_0) \, ds,$$

where $C$ is the machine circumference, and $K_0$ and $K'_0$ are the normalized gradient and gradient derivative with respect to $x$. A complete expression, including the contribution of dipole magnets, is given in Ref. 2.

Reference 1 deals with the problem of the betatron amplitude changes. Starting from this formalism, it appeared very useful in the field of circular accelerators to describe the amplitude effects with the vector $\mathbf{W}$, whose components $B$ and $A$ are called chromatic variables.$^3$

Considering two orbits at $\delta = 0$ and $\delta \neq 0$, for which the functions defined by Courant and Snyder$^1$ take the values $\beta(0)$, $\alpha(0)$, $\mu(0)$ and $\beta(\delta)$, $\alpha(\delta)$, $\mu(\delta)$, the
definitions of the chromatic variables are

\[ B = \frac{\beta(\delta) - \beta(0)}{\sqrt{\beta(\delta)\beta(0)}} = \frac{\Delta\beta}{\beta} \] (2)

and

\[ A = \frac{\alpha(\delta)\beta(0) - \alpha(0)\beta(\delta)}{\sqrt{\beta(\delta)\beta(0)}} = \beta\Delta\left(\frac{\alpha}{\beta}\right). \] (3)

The \( B \) component can be associated with the variation of the extreme amplitude of the phase-space ellipse, while the \( A \) component is related to a change in the rotation angle of the same ellipse (see Fig. 1, where \( \varepsilon \) is the emittance).

The additional definitions for the average phase advance and the gradient difference are needed:

\[ \mu = \frac{1}{2}[\mu(\delta) + \mu(0)] \] (4)

and

\[ \Delta K = K(\delta) - K(0). \] (5)

From the known relations for the betatron functions,\(^1\) the exact equations for \( A \) and \( B \) are (including all orders in \( \delta \))

\[ \frac{dB}{ds} = -2A \frac{d\mu}{ds} \] (6)

and

\[ \frac{dA}{ds} = +2B \frac{d\mu}{ds} + \sqrt{\beta(0)\beta(\delta)} \Delta K. \] (7)

Let us now consider two cases:

(i) \( \Delta K = 0 \), meaning that the corresponding section is achromatic. The following

\[ \tan{\omega} = -\frac{\alpha}{\beta} \]

\[ \sqrt{\beta \varepsilon} \]

FIGURE 1 Phase space ellipse.
relations can then be deduced from Eqs. (6) and (7):

\[
\frac{d}{ds} (A^2 + B^2) = 0
\]

and

\[
\frac{d^2 B}{d\mu^2} + 4B = 0.
\]

These equations indicate that the modules of \( \mathbf{W} \) remains constant, while the vector itself rotates with twice the phase advance \( \mu \).

(ii) \( \Delta K \neq 0 \), meaning that there exist local gradient variations. In the presence of quadrupoles and sextupoles, \( \Delta K \) becomes

\[
\Delta K = (-K_0 \delta + K_0 \delta^2 - \cdots + K_0 \eta \delta - K_0 \eta \delta^2 + \cdots)
\]

and the local errors induce \( \Delta A \) kicks proportional to \( \sqrt{\beta_0 \beta} \Delta K \Delta s \). This expression and Eq. (10) show clearly that the main contributors to the \( \mathbf{W} \) perturbations, linear and nonlinear in \( \delta \), are the quadrupoles of the low-\( \beta \) insertions, for which \( K_0 \) and \( \beta \) are large. The linear term (in \( \delta \)) due to sextupoles is finite if \( \eta \neq 0 \) and is used to control the linear \( \beta \) modulation initiated by the quadrupoles. Since sextupoles also induce nonlinear effects and the control of \( \beta \) cannot always be achieved by a local compensation, the chromaticity correction may become difficult.

2.2. Cancellation of the Linear Perturbations

The linear part in \( \delta \) of \( B \) and \( A \) [Eqs. (2) and (3)] associated with the perturbations of a magnetic channel can be calculated exactly.\(^4\) Considering a channel limited by the points 1 and 2 (Fig. 2), the unperturbed and perturbed states can be described by the matrices \( M \) and \( M^* \), respectively, while \( \Delta \mu \) represents the phase advance between 1 and 2.

The components \( B \) and \( A \) at point 2 can then be written:

\[
B_2 = \Gamma_1 \cos (2\Delta \mu) + \Sigma_1 \sin (2\Delta \mu)
\]

and

\[
A_2 = -\Sigma_1 \cos (2\Delta \mu) + \Gamma_1 \sin (2\Delta \mu).
\]

![FIGURE 2 Perturbed magnetic channel.](image-url)
where
\[ \Gamma_1 = 2\left(c_{11} - \frac{\alpha_1}{\beta_1} c_{12}\right) \]

and
\[ \Sigma_1 = \beta_1 c_{21} + 2\alpha_1 c_{11} + \frac{1 - \alpha_1^2}{\beta_1} c_{12}. \]

The quantities \( c_{ij} \) are the coefficients of \( M^{-1} \cdot (M^* - M) \), i.e. they contain the chromatic perturbation of \( M \) to first order in \( \delta \), easily calculable for quadrupoles and sextupoles.\(^4\) The parameters \( \alpha_1 \) and \( \beta_1 \) are the values of \( \alpha \) and \( \beta \) at the entrance of the channel, and \( \Gamma_1 \) and \( \Sigma_1 \) are functions of the strength and length of the elements described by \( M \), as well as of \( \eta \) and \( \eta' \).

Equations (11) and (12), which again show that \( \mathbf{W} \) rotates with twice the phase advance, can be used to calculate analytically the effect of sextupole families\(^5\) at one point of the lattice. Let us consider a magnetic structure (e.g., LEP) made of \( n \) groups of \( k \) cells (Fig. 3) and therefore containing \( nk \) sextupoles of each type (focusing and defocusing). Every sextupole has a linear contribution given by Eqs. (11) and (12), and the total effect at point \( P \) is given by the sum of all these contributions (using for \( \Delta\mu \) the individual phase advances of the sextupoles represented in Fig. 3).

Since the calculation involves sums of sines and cosines, we can make use of
\[ \sum_{j=1}^{n} \left( \frac{\cos}{\sin} \right)^{2jk\mu_0} = \left( \frac{\cos}{\sin} \right)^{(n+1)k\mu_0} \frac{\sin (nk\mu_0)}{\sin (k\mu_0)}. \quad (13) \]

The ratio of sines appearing in Eq. (13) tells us for which conditions the sextupole effects of the \( n \) groups of cells may add linearly:
\[ \frac{\sin (nk\mu_0)}{\sin (k\mu_0)} = \pm n \quad \text{if} \quad k\mu_0 = \left\{ \frac{(2m+1)\pi}{2m\pi} \right\}. \quad (14) \]

The sextupoles are more effective when this condition is satisfied and the total \( \mathbf{W} \) vector can be written formally as follows:
\[ \begin{cases} B_P = n \sum_j a_{1j} X_j & 1 \leq j \leq 2(k - 1). \\ A_P = n \sum_j a_{2j} X_j \end{cases} \quad (15) \]

The coefficients \( a_{1j} \) and \( a_{2j} \) are given explicitly for \( \mu_0 = 60^\circ \) and \( \mu_0 = 90^\circ \) in Ref. 5. The variables called \( X_j \) are linear differences of sextupole strengths (see Ref. 5 and Table 1).

Let us now assume that, in a low-\( \beta \) insertion distant from point \( P \) (Fig. 3) by a phase \( \mu^* \), strong quadrupoles generate a perturbation \( \mathbf{W}^* = (0, A^*) = (0, -\beta_{\text{max}} \ell K) \). To a first approximation, this vector rotates by \( 2\mu^* \) between the insertion and \( P \), the channel being almost achromatic, and the corresponding
FIGURE 3  Lattice structure with sextupole families.
TABLE I
Independent variables for the cancellation of the linear perturbations.

<table>
<thead>
<tr>
<th>2m + 1 or 2m</th>
<th>k</th>
<th>( \mu_0 )</th>
<th>2(k - 1)</th>
<th>Number of independent variables (2k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>90°</td>
<td>2</td>
<td>4 variables: ( Y_1, Y_2 ) ( X_1 = K_{F1} - K_{F2}, X_2 = K_{D1} - K_{D2}. ) In principle, it is not sufficient, but ( \mu_{x,z}^* ) can be used as well.</td>
</tr>
<tr>
<td>3</td>
<td>60°</td>
<td>4</td>
<td>6</td>
<td>6 variables: ( Y_1, Y_2 ) ( X_1 = K_{F1} - \frac{1}{2}K_{F2} - \frac{1}{2}K_{F3}, X_3 = ) same as ( X_1 ) except ( F \rightarrow D, ) ( X_2 = K_{F2} - K_{F3}, X_4 = ) same as ( X_2 ) except ( F \rightarrow D. ) Optimum for the linear problem</td>
</tr>
<tr>
<td>4</td>
<td>45°</td>
<td>6</td>
<td>8</td>
<td>8 variables: more than necessary, but focusing is low.</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>108°</td>
<td>8</td>
<td>10 variables: large separation between identical kicks (see later)</td>
</tr>
<tr>
<td>2</td>
<td>72°</td>
<td>8</td>
<td>10</td>
<td>10 variables not “retained” because</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>80°</td>
<td>12</td>
<td>14 variables ( k \mu_0 ) is an even multiple</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>80°</td>
<td>16</td>
<td>18 variables of ( \pi ) (see later)</td>
</tr>
</tbody>
</table>

vector \( \mathbf{W} \) at \( P \) will be

\[
\begin{align*}
B_j &= -A^* \sin (2\mu^*) \\
A_j &= +A^* \cos (2\mu^*). 
\end{align*}
\]  

(16)

The cancellation of the linear perturbations is obtained by solving these six equations. Let us briefly discuss the number of independent variables as a function of the phase advance per cell \( \mu_0 \), the number \( k \) of cells per group (Fig. 3), and the multiple of \( \pi \) considered in the Eq. (14), these three parameters being interdependent. Table I summarizes some of the results. Considering the LEP phase advances of 60° and 90°, Table I shows that the first choice allows an exact
cancellation of the linear chromatic perturbation in general, while the second one allows it only for given phase advances $\mu_x^*$ and $\mu_z^*$ between the insertion quadrupoles and the first sextupole.

3. NONLINEAR PERTURBATIONS AND NUMERICAL TOOLS

As seen in Section 2, the low-$\beta$ insertions are the major source of chromatic errors. Since they are usually located in the middle of an extended region without dispersion $D_x$, the compensation is made with sextupoles installed in the lattice ($D_x \neq 0$), as shown in Fig. 3. In the drift between the insertion and the lattice, the vector $\mathbf{W}^*$ rotates to a first approximation (Section 2.2), but nonlinearities in $\delta$ develop themselves. In the lattice, each sextupole contribution adds to the others and rotates with the phase advance, but nonlinearities in $\delta$ also appear. Finally, the presence of sextupoles implies nonlinear forces in the transverse coordinates $x$ and $z$.

3.1. Nonlinearities and Relevant Quantities

The nonlinearities in $\delta$ mentioned above are present in two forms: (i) amplitude nonlinearities, which are usually weak, and (ii) phase nonlinearities, which are large and follow the phase spread. Of course, the nonlinearities which appear through the lattice and in the drift near the insertion are not the same. To illustrate this, let us represent the vectors $\mathbf{W}$ tracked from the lattice and from the insertion to point $P$ (Fig. 3) as a function of $\delta$. Figure 4 gives the results for the LEP structure. The sextupole strengths were calculated as discussed in Section 2.2 so that the two $\mathbf{W}$ vectors are equal and opposite for $\delta = 0$, and it was expected that this holds approximately for small $\delta \neq 0$. Figure 4 shows, however, that the two vectors do not compensate well for absolute $\delta$ values between 0.5% and 1.5%. It is also obvious from Fig. 4 that the phase nonlinearities dominate.

Nonlinear transverse forces in $x$ and $z$ induce nonlinear kicks in the betatron motions, i.e., resonances, $x$–$z$ coupling, instabilities, and blowup. It is important to define quantities characterizing these nonlinear effects. For the $\delta$ nonlinearities, they are, for instance, the variations of the tune and betatron function with $\delta$. As an example, Fig. 5 shows the variations obtained for LEP. The figure indicates that the linear correction is achieved and that the instability in the integer tune is reached at $\delta = 1.8\%$, but it does not give obvious indications about the quality of the correction. For the nonlinear forces in $x$ and $z$, calculable quantities are the excitation and stabilizing coefficients of the third-order resonances and the fourth-order ones (due to the propagation of the sextupole effects via the nonlinear trajectory oscillations).

One possible way to have a global description of all effects and to estimate the quality of correction is to look at the dynamic aperture. The dynamic aperture $d_y$ ($y$ being either $x$ or $z$) and the dynamic acceptance $A_y = d_y^2/\beta_y$ describe the
betatron amplitudes at which particles can circulate in a machine indefinitely, as functions of the momentum deviation $\delta$. The dynamic aperture corresponds to the initial amplitudes below which the $\beta$ motion is stable, meaning that it remains bounded. For electrons which radiate, the condition of boundedness must prevail only during a time comparable to the damping times.

In principle, the stable region is a volume in $(E_x, E_\gamma, \delta)$ space centered around

FIGURE 4 W-tracking in LEP version 13.
FIGURE 5 Results of chromaticity correction (LEP 13): variation of betas with momentum at the experimental insertion (left) and at the nonexperimental insertion (center); tune variations with momentum (right).
the origin. $E_x$ and $E_z$ are the emittances related to the amplitudes of betatron oscillations, and $\delta$ is the relative amplitude of synchrotron oscillations. One looks for the largest surface of simple shape falling inside the stable region, thus eliminating stability islands away from the origin. Sometimes, to simplify the presentation, only cuts of the stable volume are shown graphically (for instance the cut $E_z = \frac{1}{2}E_x$, corresponding to full coupling in electron machines).

Up to now, the dynamic aperture could be obtained only by tracking particle trajectories over a long enough time, i.e., one damping time for electrons and a time long with respect to the synchrotron period for protons. However, analytical tools have been sought at the same time, in hopes that they would both illuminate the dynamic aperture problem and reduce the tracking time required for hadrons.

3.2. Dependence on the Sextupole Arrangement

Two nonlinear kicks of the same strength, due to sextupoles $S_1$ and $S_2$, may cancel each other if the phase separation is equal to an odd multiple of $\pi$ and if there are no other nonlinear perturbations in between (Fig. 6).

Under these conditions, the transverse aberrations are cancelled to all orders. However, $\Delta \mu = (2m + 1)\pi$ can only be satisfied for one value of $\delta$ (i.e., $\delta = 0$), and it may be difficult in practice to avoid the presence of other sextupoles over such a distance. The consequences of these remarks are the following:

(i) It is, in principle, advantageous to make the linear correction with $k\mu_0 = \pi$, i.e., $(2m + 1) = 1$ [in Eq. (14)].

(ii) Interleaved sextupole schemes, as in Fig. 3, do not satisfy the second condition above; nevertheless, their average strength is lower.

(iii) Non-interleaved sextupole schemes (pairs of sextupoles distant by $\pi$ without nonlinear elements in between) have the advantage of almost satisfying the two conditions.$^{10,11}$ However, the average strength of the sextupoles is higher, since the number of elements is smaller.

![FIGURE 6 Betatron oscillation and sextupole kicks.](image)
Qualitative characteristics of the dynamic aperture can be drawn up for these two schemes (Fig. 7 gives examples for LEP configurations with different working points):

Scheme (iii): Since $\Delta \mu = \pi$ at $\delta = 0$, $d_y$ is very large at the origin (but finite, because of the finite sextupole length). Since the strengths are high and $\Delta \mu \neq \pi$ at $\delta \neq 0$, $d_y$ decreases rapidly with $\delta$.

FIGURE 7 Examples of dynamic aperture obtained for LEP.
Scheme (ii): Because of the presence of sextupoles within a pair of elements separated by $\pi$, $d_y$ is smaller at the origin. Since the strengths are lower, it is possible to limit the decrease of $d_y$ with $\delta$.

Let us note that interleaved sextupoles with “high” strengths correspond to an extreme case combining disadvantages. This pleads against a local compensation of the insertion perturbations with interleaved sextupoles, e.g., at lattice extremities.

It is interesting to mention additional means\textsuperscript{12,13} for modifying high-order tune derivatives with respect to $\delta$. First, the phase advance $\mu^*$ (Section 2.2) can also be used when $\mu_0 \neq 90^\circ$ to change the $\beta$ modulations at the sextupoles for $\delta \neq 0$. Second, one sextupole family (whose members are all equivalent for the linear correction) can be split in subgroups having clearly different $\beta$ modulations for $\delta \neq 0$.

3.3. Chromaticity Correction and Computer Calculations

Let us describe as an example how we have proceeded in calculating the chromaticity correction of LEP and which computer programs have been used.

Starting with a given phase advance per cell in the lattice, $\mu_0$, the linear correction is calculated, satisfying the condition $k\mu_0 = (2m + 1)\pi$ [Eq. (14)] and using the variable $\mu^*$ (Fig. 3), if necessary. The computer program HARMON\textsuperscript{14} is then used to find the sextupole strengths that minimize the linear perturbations, as well as the $\delta$ nonlinearities ($Q''$, $Q'''$, $\beta''$, . . .) and the resonance excitations mentioned in Section 3.1. To show that the linear correction is a good starting point, Table II compares the strengths obtained by HARMON with the solution of the linear problem for LEP with $\mu_0 = 90^\circ$ and interleaved sextupoles.\textsuperscript{15}

The optics program MAD\textsuperscript{16} then allows us to calculate for the obtained solution the variations of the betatron parameters (tunes and $\beta$'s) with $\delta$ (see Fig. 5), in order to check the linear correction and to visualize the residual effects of the $\delta$ nonlinearities.

Following the idea of Section 3.1, the program PATRICIA\textsuperscript{17} is used to track particle trajectories for different initial conditions in horizontal amplitude $x$ ($x'$ being always zero) and for different amplitudes $\delta$ of the synchrotron oscillations. It begins the tracking at a low-$\beta$ insertion and follows the trajectories over a few hundred turns, assuming that $E_x = \frac{1}{2}E_x$. For every $\delta$ value, the program indicates when the motion remains bounded during the time considered and gives the acceptance $A_x$ or the quantity $(A_x)^{1/2}$ (Section 3.1). The curve $(A_x)^{1/2}$ as a

| TABLE II |
| Sextupoles strengths, in m\textsuperscript{3}, for LEP with 90° per cell. |

<table>
<thead>
<tr>
<th></th>
<th>$K'_{SP1}$</th>
<th>$K'_{SP2}$</th>
<th>$K'_{SD1}$</th>
<th>$K'_{SD2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear cancellation</td>
<td>0.216</td>
<td>0.141</td>
<td>0.288</td>
<td>0.128</td>
</tr>
<tr>
<td>HARMON</td>
<td>0.209</td>
<td>0.125</td>
<td>0.260</td>
<td>0.143</td>
</tr>
</tbody>
</table>
function of $\delta$ represents the dynamic aperture defined in Section 3.1 (Fig. 7). In general, the curve obtained does not show narrow dips where resonances occur, since the synchrotron motion attenuates the effects. However, inspection of the trajectories in phase space may show cases where stability islands occur in connection with resonances and changes of the tunes with the amplitudes (Fig. 8). When the horizontal amplitude is close to the stability limit, the separatrix associated with these resonances can become clearly visible (Fig. 9).

4. ANALYTICAL APPROACH WITH HAMILTONIAN FORMALISM

As mentioned in Section 3.1, analytical tools can also be very useful in dealing with the problems of the nonlinear forces associated with sextupoles introduced for chromaticity correction. Two formalisms are considered in this report. The first one, which is based on the classical Hamiltonian treatment, is described here; the second one, starting directly from the differential equation, is the subject of Section 5. We discuss possible applications and limitations of both formalisms, keeping in mind the question of the dynamic aperture.

4.1 Nonlinear Perturbations in Two Dimensions

The basic principles of the Hamiltonian approach are described in Ref. 18 (for example) for the one-dimensional betatron motion. In this section, we are interested in the two-dimensional betatron motion, and the general treatment described in Refs. 9, 19, and 20 is applied.

Let us start with the linear "unperturbed" motion, which is described by the following equations and the Hamiltonian function $H$ (the longitudinal coordinate $s$ being the independent variable):

\begin{align*}
  x'' + K_1x &= 0, \\
  z'' + K_2z &= 0, \\
  H_0 &= \frac{1}{2}(K_1x^2 + K_2z^2 + p_x^2 + p_z^2).
\end{align*}

It has been shown\(^1\) that the general solution of these equations has the form

\begin{align*}
  x &= A_x \sqrt{\beta_x} \exp(i\mu_x), \\
  z &= A_z \sqrt{\beta_z} \exp(i\mu_z). \tag{21}
\end{align*}

In the presence of nonlinear fields, the perturbed motion can be depicted by:20

\begin{align*}
  x'' + K_1x &= -\frac{\partial H_1}{\partial x}, \\
  z'' + K_2z &= -\frac{\partial H_1}{\partial z}, \tag{22}
\end{align*}

\begin{align*}
  H_1 &= \sum_{N} H^{(N)} = \sum_{k_1+k_2=N} \frac{\pm 1}{k_1! k_2!} K^{(N-1)}_{x,x} x^{k_1} z^{k_2},
\end{align*}
where $H_1$ is the Hamiltonian of the perturbation and $K^{(N-1)}_{x,x}$ are the $N-1$ normalized derivatives of the field components $B_x$ or $B_y$. If we now use for the complete motion (defined by $H = H_0 + H_1$) the action-angle variables $I_x$, $I_z$, $\Phi_x$, and $\Phi_z$, and the independent variable $\theta$ (geometric angle at the machine center), the two parts of $H$ can be written as follows:

$$H_0 = (Q_x I_x + Q_z I_z)$$

and

$$H_1 = \sum_{jklm} h_{jklm}(j+k)/2 I_x^{(j+k)/2} (l+m)/2 \exp \{i[(j-k)(\Phi_x + Q_x \theta) + (l-m)(\Phi_z + Q_z \theta)]\},$$

with

$$h_{jklm} = h_C = \frac{1}{2^{(N+1)/2}} \frac{1}{j! k! l! m!} \int_{0}^{2\pi} \left( \frac{\beta_z}{R} \right)^{(j+k)/2} \left( \frac{\beta_x}{R} \right)^{(l+m)/2} K^{(N-1)}_{x,x} d\theta \times \exp \{i[(j-k)(\mu_x - Q_x \theta) + (l-m)(\mu_z - Q_z \theta)]\},$$

where $Q_x$ and $Q_z$ are the tunes and $C$ is used for the group of indices $jklm$. In the special case where $H_1 = 0$, $H_0$ is a constant of the motion which defines the invariant curves. In the case of interest, where $H_1 \neq 0$, the method used consists of finding a canonical transformation which generates a new Hamiltonian $G$ either almost independent of $\theta$ for low frequencies or exactly independent of $\theta$ to first order in the perturbation. This canonical transformation can be defined by a generating function of the form

$$S_{tot}(\Phi_x, J_x, \Phi_z, J_z, \theta) = \Phi_x J_x + \Phi_z J_z + S(\Phi_x, J_x, \Phi_z, J_z, \theta).$$

By definition, the relations between the old variables $(\Phi_x, I_x, \Phi_z, I_z)$ and the new ones $(\Psi_x, J_x, \Psi_z, J_z)$, and between the old and new Hamiltonians are

$$I_x = J_x + \partial S_{\phi_x}, \quad \Psi_x = \Phi_x + \partial S_{J_x},$$
$$I_z = J_z + \partial S_{\phi_z}, \quad \Psi_z = \Phi_z + \partial S_{J_z},$$
$$G = H + \partial S_{\theta},$$

where $\partial S_{\phi_x}$, $\partial S_{J_x}$ and $\partial S_{\theta}$ are used for the partial derivatives with respect to $\Phi$, $J$, and $\theta$, respectively. Formally, the function $S$ can be written as $H_1$, [Eq. (24)] where $h_C$ is replaced by $s_C$, and $I_{x,z}$ by $J_{x,z}$. Similarly, the new Hamiltonian $G$ can have the same form as $H$, where $h_C$ is replaced by $g_C$, $I_{x,z}$ by $J_{x,z}$, and $\Phi_{x,z}$ by $\Psi_{x,z}$. The three functions $S$, $H$, and $G$ hence have the same form, but with different variables. Expressing all of them as functions of the $S$ variables implies the use of Taylor’s expansions for the other variables via the definitions of Eqs. (26). Doing this and using the last expression among Eqs. (26), we obtain for the new Hamiltonian $G$

$$G(\Psi_x, J_x, \Psi_z, J_z) = G(\Phi_x, J_x, \Phi_z, J_z) + \partial G_{\psi_x} \partial S_{J_x} + \partial G_{\psi_z} \partial S_{J_z} + \cdots$$
$$= H_0(J_x, J_z) + Q_x \partial S_{\phi_x} + Q_z \partial S_{\phi_z} + \cdots + \partial S_{\theta}$$
$$+ H_1(\Phi_x, J_x, \Phi_z, J_z) + \partial H_{1,x} \partial S_{\phi_x} + \partial H_{1,z} \partial S_{\phi_z} + \cdots,$$

(27)
remembering that \( Q = \partial H_0/\partial \theta \). Since the perturbation appears in the functions \( G, S, \) and \( H_0 \), the products of two partial derivatives are of second order in the perturbation and the only first-order terms are those underlined in Eq. (27). For circular accelerators, the functions are all periodic in \( \theta, \Phi_x, \) and \( \Phi_z \), with a period of \( 2\pi \). Therefore, it is possible to analyse them in a Fourier series with respect to these variables, e.g.,

\[
H_1 = \sum_{c,p=-\infty}^{\infty} h_{cp} J_{x}^{(j+k)/2} J_{z}^{(l+m)/2} \times \exp \{i[(j-k)(\Phi_x + Q_x \theta) + (l-m)(\Phi_z + Q_z \theta) - p \theta]\}. \tag{28}
\]

Similar expressions can be written for \( S \) and for \( G - H_0 \), with coefficients \( s_{cp} \) and \( g_{cp} \), respectively. Introducing these expressions in Eq. (27) and keeping only first-order terms in the perturbation give the following relation:

\[
g_{cp} = iQ_x(j-k)s_{cp} + iQ_z(l-m)s_{cp} - ip s_{cp} + h_{cp}, \tag{29}
\]

where \( C \) again replaces the 4 indices \( j, k, l, \) and \( m \). The generating function \( S [\text{Eqs. (26)}] \) can be chosen arbitrarily, provided Eq. (29) is satisfied. Let us therefore solve Eq. (29) with respect to the coefficients \( s_{cp} \):

\[
s_{cp} = i\frac{h_{cp} - g_{cp}}{(j-k)Q_x + (l-m)Q_z - p}. \tag{30}
\]

This is the key equation. The canonical transformation \( I \to J \) and \( \Phi \to \Psi \) must remain finite, and the convergence of \( S \) implies that \( s_{cp} \) be bounded. Two cases can now be distinguished:

(i) Resonance approach. The denominator of Eq. (30) vanishes, and low-frequency terms of the type \( n_x Q_x + n_z Q_z = p \), corresponding to nonlinear resonances, are retained. Since \( s_{cp} \) must be bounded, the numerator of Eq. (30) must be equal to zero, which means \( h_{cp} = g_{cp} \). Hence, \( s_{cp} \) is a constant chosen to be zero. In this case, the first-order terms \( g_{cp} \) of the new Hamiltonian are nonzero, and the well-known first-order perturbation theory\(^8,9,19,20\) applies; thus, the concepts of single resonance lines, excitation and stabilizing coefficients, bandwidths, resonance curves, and separatrices can be introduced.

(ii) Global approach. The tune values \( Q_x \) and \( Q_z \) are such that the working point is far from every strong resonance and can only be close to weak and negligible resonances. In this case, the denominator of Eq. (30) is supposed to be always different from zero, and it is possible to arbitrarily choose \( g_{cp} = 0 \). Hence, all first-order terms of the new Hamiltonian vanish and the \( \theta \) dependence is moved to second order in the perturbation.\(^21\) This approach allows us to study the motion between strong resonances and to evaluate the distortions of the invariant curves, as well as the amplitude modulations, within certain limits of validity.
4.2. Resonance Approach

As explained in the previous section, the coefficients \( g_{Cp} \) are taken equal to \( h_{Cp} \). In the complete expression of the new Hamiltonian [Eq. (27)], the higher-order terms which were neglected in establishing Eq. (30) must be reintroduced. Hence \( G \) has the form

\[
G = H_0 + \sum_{C,p} h_{Cp} f_1^{(j+k)/2} f_2^{(l+m)/2} \exp i(\cdots) + \sum_{C'qC''q''} h_{C'qC''q''} f_1(C', q', C'', q'') f_2(C'', q'', C', q') \exp i(\cdots) + O(h^3),
\]

(31)

where the functions \( f_1 \) and \( f_2 \) of the indices and the phases are not given explicitly.

Looking at first-order perturbations of degree \( N \), it follows by definition that

\[
j + k + l + m = j' + k' + l' + m' = j'' + k'' + l'' + m'' = N.
\]

Then the terms of the double sums are of degree

\[
j' + k' + j'' + k'' + l' + m' + l'' + m'' - 2 = 2(N - 1)
\]

in amplitude. Consequently, sextupole magnets which excite third-order resonances \( (N = 3) \) due to first-order perturbations, also excite fourth-order resonances \( (2N - 2 = 4) \) due to second-order perturbations. The physical explanation is the following: The trajectory oscillations due to the nonlinear kicks propagate around the ring and affect the kicks of other sextupoles. Of course, sextupoles also excite higher-order resonances associated with higher-order terms in the perturbation, neglected in Eq. (31).

The approach discussed here is used in the program HARMON,\(^{14}\) which adjusts the sextupole strengths to minimize adverse effects of a particular quadrupole-sextupole configuration. The basic principle is to minimize a given set of coefficients \( h_{Cp} \). For instance, third-order (in amplitude) excitation coefficients \( h_{1020}, h_{1002}, \) and \( h_{1011} \) are minimized because of coupled motions, while \( h_{3000} \) and \( h_{2100} \) are of less concern. The fourth-order (in amplitude) excitation coefficients are indirectly reduced (products of \( h \)), and they are calculated.

The equations of motion

\[
\frac{d\Psi_z}{d\theta} = \frac{\partial G}{\partial J_z}, \quad \frac{dJ_z}{d\theta} = -\frac{\partial G}{\partial \Psi_z},
\]

\[
\frac{d\Psi_x}{d\theta} = \frac{\partial G}{\partial J_x}, \quad \frac{dJ_x}{d\theta} = -\frac{\partial G}{\partial \Psi_x}
\]

(33)

can then be used to express the variations of the tunes and amplitudes with \( \delta \), as well as the tune variations with amplitude.

Let us look, for instance, at the tune variations in the horizontal plane

\[
Q_x = Q_{x0} + \frac{d\Psi_x}{d\theta} = Q_{x0} + \frac{\partial G}{\partial J_x}.
\]

(34)
From Eq. (31) the tune can be written

$$Q_x = Q_{x_0} + \sum_{r,s} J_x^{-1} J_s f_{r,s \text{rreso}} + f(C', C'', q', q'') h_{c'q', c''q''},$$

(35)

where $f$ is a function of the indices with $(j' + k' + j'' + k'' - 4)/2 = r - 1$ and $(l' + m' + l'' + m'' - 4)/2 = s$. Let us consider two special examples:

(i) The term in Eq. (35) corresponding to $r = 1$ and $s = 0$ is

$$h_{11000} - 4 \sum_q \left( \frac{h_{2000q} h_{0200-q}}{2 Q_x - q} + \frac{h_{1000q} h_{1200-q}}{Q_x - q} \right).$$

(36)

The sum gives some dependence with $\delta$, because of the definition of the $h$'s [Eq. (24)]. The coefficient $h_{1000q}$ contains the dipole field $B$, which corresponds to $B'\eta \delta$ and $B''\eta^2 \delta^2$ in quadrupoles and sextupoles, respectively. The derivative $B'$ is present in $h_{2000q}$, and the sextupoles contribute with $B''\eta \delta$. Hence, the tune dependence on $\delta$ and $\delta^2$ is calculable.

(ii) The term in Eq. (35) corresponding to $r = 2$ and $s = 0$ is

$$h_{22000} - 9 \sum_q \left( \frac{h_{3000q} h_{0300-q}}{3 Q_x - q} + \frac{h_{2100q} h_{1200-q}}{3 Q_x - q} \right).$$

(37)

Since all $h$ coefficients in the sum contain $B''$, there is no $\delta$ dependence. However, because $r = 2$, the tune variation is proportional to $J_x$ in Eq. (35), and the amplitude dependence is calculable.

By minimizing certain coefficients $h_{c'q'}$, it is hence possible to control, to some extent, the tune variations, and this is done in HARMON.14 Besides this, the resonance approach, of course, allows a detailed study of isolated resonances excited by sextupoles. Near one of them, fixed points, resonance curves, and separatrices can be analysed,8,9,19,20 giving information about the unboundedness of the motion. The initial conditions $(x_0, x_0', z_0, z_0')$ at which the separatrices are reached give the dynamic aperture defined in Section 3.1. However, this is only valid in a narrow range of tune values near the resonance line, as illustrated later (Fig. 13) for one-dimensional motion. The knowledge of the dynamic aperture far from strong resonances necessitates different approaches (see below).

4.3. Global Approach

We have seen in Section 4.2 that it was possible to adjust sextupole strengths by minimizing some excitation coefficients and to get information about stability limits in the vicinity of strong resonances. However, we could not say anything about tunes far from resonance values, nor could we determine whether two or more resonances contribute together. In the analysis of the nonlinear motion, we are mainly concerned with the distortions of the invariants and the amplitude beating for tune values far from the strong resonances. In this case, the global approach can be tried.21

Since $g_{Cp} = 0$, the generating function $S$ has the form

$$S = i \sum_{c,p} \frac{h_{c'p}}{Q - p} J_x^{(j+k)/2} J_z^{(l+m)/2} \exp \left\{ i[(j - k)\Phi_x^* + (l - m)\Phi_z^* - p\theta] \right\},$$

(38)
where \( Q = (j - k)Q_x + (l - m)Q_z \) and \( \Phi^*_{x,z} = \Phi_{x,z} + Q_{x,z} \theta \). Using Taylor’s expansion for changing from \( \Phi_{x,z} \) to \( \Psi_{x,z} \) and neglecting second-order terms in the perturbation, the expression for \( S \) can be put in Eqs. (26). We obtain for instance the old horizontal action variable

\[
I_x = J_x - \sum \frac{(j - k)h c}{Q - p} \frac{\pi}{\sin (\pi Q)} \exp \left\{ i[(j - k)\Psi^*_{x} + (l - m)\Psi^*_{z} - p \theta] \right\},
\]

(39)

where \( \Psi^*_{x,z} = \Psi_{x,z} + Q_{x,z} \theta \). The infinite sums over the harmonic number \( p \) of Eq. (39) can be calculated with the help of the formula

\[
\sum \frac{e^{ip \theta}}{Q - p} = \frac{\pi}{\sin (\pi Q)} \exp \left\{ -iQ(\pi - \theta) \right\}.
\]

(40)

The new Hamiltonian \( \tilde{G} \) contains only second order terms in \( h \), as already explained, but is still \( \theta \) dependent. This residual dependence can be suppressed by averaging over \( \theta \), i.e., by keeping only low-frequency terms. Hence, \( \tilde{G} \) becomes independent of the phases \( \Psi_x, \Psi_z \):

\[
\tilde{G} = Q_x I_x + Q_z I_z + \sum_{m,n} B_{mn} J_x^{m/2} J_z^{n/2}.
\]

(41)

Consequently, \( \tilde{G} \) becomes a constant of the motion, along with \( J_x \) and \( J_z \), by virtue of the equations of motion [Eqs. (33)]. The constant \( \tilde{G} \) gives the new invariant: The first two terms of Eq. (41) correspond to the ellipse associated with the linear motion, while the sum gives the ellipse distortions due to nonlinear terms.

The modulation of the amplitudes is given by Eq. (39) (in the H plane), which depends upon the transformed phases \( \Psi_x \) and \( \Psi_z \), including nonlinear effects. For this reason, \( \Psi_x \) and \( \Psi_z \) depend on the number of revolutions \( n_r \), and at the point \( \theta = \theta_c \), \( I_x \) can be rewritten as

\[
I_x = J_x - \sum \frac{\pi}{\sin (\pi Q)} (j - k)h c \frac{J_x^{(j+k)/2} J_z^{(l+m)/2}}{J_x^{(j+k)/2} J_z^{(l+m)/2}} \exp \left\{ -iQ(\pi - \theta) \right\}
\]

(42)

where the sum over \( t \) is the addition of all the sextupole contributions to the integral \( h \) [Eq. (24)]. A similar expression can, of course, be written for \( I_z \). Since there are no correlations between \( \Psi_x \) and \( \Psi_z \), the extreme amplitude modulations are given approximately by

\[
\Delta I_x = \sum \frac{\pi}{\sin (\pi Q)} h c \frac{J_x^{(j+k)/2} J_z^{(l+m)/2}}{J_x^{(j+k)/2} J_z^{(l+m)/2}} \exp \left\{ -iQ(\pi - \theta) \right\}
\]

(43)

In Ref. 21, this treatment is given in detail, compared with some other work\(^{22} \) in which the sextupole propagation appears to be neglected, and applied to a structure containing sextupoles. For given initial conditions, relations of the type:

\[
I_{x_0} = I_{x_0}(J_x, J_z, \Phi_{x_0}, \Phi_{z_0})
\]

(44)

and

\[
I_{z_0} = I_{z_0}(J_x, J_z, \Phi_{x_0}, \Phi_{z_0})
\]

(45)
are valid. They can, in principle, be solved for $J_x$ and $J_z$, which are constants of the motion. Once this is done, the amplitude beating is known from Eq. (43), and the domain of validity can be estimated from the condition $I_{x,z} \geq 0$:

$$\Delta I_{x,z} \leq J_{x,z}. \quad (46)$$

Equations (41) and (43) allow us to calculate invariant distortions and amplitude modulations in the presence of nonlinear magnetic fields. Some results have been compared with tracking results.\textsuperscript{21} It may however, be, difficult to solve the initial condition equations [Eqs. (44) and (45)] analytically, since they are generally nonlinear. Furthermore, third- and higher-order terms of the Hamiltonian $G$ have been neglected, which implies that the perturbation is small enough and that the working point (tune values) is either within stable regions of resonances or between strong resonances. This indicates that the second-order perturbation approach presented here has difficulties in describing the motion close to the stability limit, where the perturbation is strong. For determining the dynamic aperture, it appears necessary to push the $\theta$ dependence to higher orders in the perturbation by successive canonical transformations, and the analytical treatment becomes more and more difficult. This is why a completely different formalism has been developed.

5. THE SUCCESSIVE LINEARIZATION METHOD

Instead of using the Hamiltonian and transforming the variables by a canonical transformation in order to obtain a new Hamiltonian independent of the variable $s$, we now try to solve the differential equation of betatron motion directly by an iterative process.

Consider an equation

$$L[x(s)] + f(s)x^2(s) = 0 \quad (47)$$

where we define

$$x(0) = x_0, \quad x'(0) = x'_0,$$

and where $L$ is a differential operator defined by

$$L = \frac{d^2}{ds^2} + g(s). \quad (48)$$

In the above case of one-dimensional horizontal betatron motion,\textsuperscript{17} $g(s)$ and $f(s)$ are given, on- and off-momentum, respectively, by

at $\delta = 0$, \hspace{1cm} $g(s) = -k(s)$, i.e., quadrupole strength as a function of $s$,

$$f(s) = -\frac{1}{2} m(s), \hspace{1cm} \text{i.e., sextupole strength as a function of } s, \quad (49)$$

at $\delta \neq 0$, \hspace{1cm} $g(s) = -(1 - \Delta)k(s) - \eta(s)m(s)\Delta$,

$$f(s) = -\Delta (1 - \Delta) m(s), \quad (50)$$
with

\[ \Delta = \frac{\delta}{1 + \delta}. \]  

(51)

The \( \eta \) function is the periodic solution of the equation

\[ \eta'' - (1 - \Delta)k(s)\eta - \frac{3}{2}\Delta m(s)\eta^2 = h(s)(1 - \Delta), \]

(52)

where \( h(s) = 1/\rho(s) \) is the inverse bending radius. Since we deal with circular accelerators, \( k(s), m(s), \) and \( h(s) \) are periodic functions in \( s: k(s) = k(s + L), m(s) = m(s + L), h(s) = h(s + L). \)

In practical applications (e.g., LEP), the sextupole strengths and the amplitude of motion are such that the nonlinear part of Eq. (47) is small (a few percent) compared with the linear one. Therefore, a first approximate solution can be obtained by linearizing the equation:

\[ L[x^{(0)}(s)] = 0. \]

(53)

We call that step “first linearization.”

The solution of Eq. (53) is well known and has been treated extensively in Ref. 1. It describes the betatron motion (on- and off-momentum), taking into account only the linear fields (bending and quadrupoles). Now it is possible to write the complete solution as

\[ x(s) = x^{(0)}(s) + u(s), \]

(54)

and the corresponding equation for \( u(s) \) becomes

\[ L(u) + 2f(s)x^{(0)}(s)u + f(s)u^2 = -f(s)[x^{(0)}(s)]^2. \]

(55)

The idea is now to again drop the quadratic term from the equation for \( u(s) \). This can be called “second linearization.” We then find

\[ L[u^{(0)}] + 2f(s)x^{(0)}(s)u^{(0)} = -f(s)[x^{(0)}(s)]^2. \]

(56)

The approximate solution for \( x \) then becomes

\[ x(s) = x^{(0)}(s) + u^{(0)}(s). \]

(57)

Equation (56), which is again linear, contains the sextupole distribution \( f(s); \) therefore, the approximate solution [Eq. (57)] depends on it. The question remains, how does one distribute the initial conditions \( x^{(0)}(0), u^{(0)}(0), x^{(0)'}(0), u^{(0)'}(0)? \) Basically it is possible to choose any distribution as long as the conditions

\[ x(0) = x_0 = x^{(0)}(0) + u^{(0)}(0) \]

(58)

and

\[ x'(0) = x_0' = x^{(0)'}(0) + u^{(0)'}(0) \]

(59)

are fulfilled. But the best solution seems to be to impose the exact initial conditions completely to the solution of the first linearization:

\[ x^{(0)}(0) = x_0, \quad x^{(0)'}(0) = x_0', \quad u^{(0)}(0) = 0, \quad u^{(0)'}(0) = 0. \]
With this distribution, there must exist a certain interval of \( s \) around \( s = 0 \) where
the nonlinear part of Eq. (55) is really negligible (because of small \( u \)); therefore,
the solution is well described by Eq. (56). Indeed, one can prove\(^{23}\) that with such
a distribution of the initial conditions the approximate solution [Eq. (57)] agrees
up to the fourth-order Taylor term with the exact solution of Eq. (47).

The linearization process described here can be pursued up to any level, giving
a more and more precise solution of the initial equation. From the arguments
given above, it is clear that the function \( u^{(0)}(s) \), defined in Eq. (56), will normally
be small as compared to \( x^{(0)}(s) \):

\[
\frac{u^{(0)}}{x^{(0)}} \ll 1, \quad \text{for all } s. \tag{60}
\]

It follows that a small modification of Eq. (56), changing the result for \( u^{(0)} \) by
only a small fraction, will hardly affect the quality of the approximation for \( x \) in
Eq. (57). Such small modifications may be necessary to allow a closed analytical
treatment of the equation for \( u^{(0)} \). In the particular case of finding the
approximate invariants of the betatron motion, the quantity \( x^{(0)}(s) \) will be
replaced on the left-hand side of Eq. (56) by its average value. This is really a
small modification, since the ratio of the expressions \( f(s)x^{(0)}(s)u^{(0)}(s) \) and
\( f(s)[x^{(0)}(s)]^2 \) in Eq. (56) is small because of Eq. (60).

Looking at the betatron motion equations above, it is clear that for nonzero
momentum deviation the solution depends upon the \( 1 \) function. Therefore,
analytical solutions for the \( 1 \) equation [Eq. (52)] were initially looked for by using
the linearization method precisely. The detailed treatment for \( 1 \) is given in Ref.
23.

5.1. Invariant Curves for the Betatron Motion

It is well known\(^1\) that the solution of the linearized betatron motion [Eq. (53)] is
confined to a regular curve in \( (x, x') \) space, which is given by a quadratic form in
\( x \) and \( x' \) and is an ellipse as long as the condition

\[
|\text{Tr } M| \leq 2 \tag{61}
\]

holds. Here \( M \) is the transfer matrix relating the vectors \( X_{n+1}, X_n \):

\[
\begin{pmatrix}
  x^{(0)} \\ x^{(0)'}
\end{pmatrix}_{n+1} = M
\begin{pmatrix}
  x^{(0)} \\ x^{(0)'}
\end{pmatrix}_n
\tag{62}
\]

where \( n \) numbers the periods of the structure covered by the solution \([x^{(0)}_0 = x^{(0)}(0), x^{(0)'}_0 = x^{(0)'}(0)]\).

The solution of the recurrence relation [Eq. (62)] as a function of \( n \) and the
initial values is:\(^1\)

\[
X^{(0)}_n = \begin{pmatrix}
  x^{(0)}_n \\ x^{(0)'}_n
\end{pmatrix} = X_0 \cos (n\mu) + \frac{\sin (n\mu)}{\sin \mu} (M - I \cos \mu)X_0 \tag{63}
\]

with \( \cos \mu = \frac{1}{2} \text{Tr } M \) and \( \det M = 1 \).
Generalizing from the integer \( n \) to a real number gives a parametric representation of the invariant ellipse in \( (x, x') \) space. If the equation contains nonlinear forces which are not too strong, we expect that the invariants still exist (at least approximately) and that they are distorted with respect to the linear ellipse. These effects will be calculated in this section.

Looking at the horizontal betatron motion at \( \delta = 0 \), let us rewrite Eq. (56) with the corresponding definitions of \( L \) and \( f(s) \) [Eqs. (48) and (49)]:

\[
\dot{u}^{(0)} - [k(s) + m(s)x^{(0)}(s)]u^{(0)} = \frac{1}{2}m(s)[x^{(0)}(s)]^2,
\]

where

\[
u^{(0)}(0) = u^{(0)}(0) = 0.
\]

For the reasons already discussed, let us now replace in the left-hand side of Eq. (64) \( x^{(0)}(s) \) by its average value over an infinite number of periods:

\[
x^{(0)}(s) \rightarrow \lim_{n \to \infty} \frac{1}{n} \int_0^n x^{(0)}(s) \, dn = \langle x^{(0)} \rangle.
\]

Inserting the first component of Eq. (63) into Eq. (66) shows that the integral remains bounded and thus that the whole expression \( \langle x^{(0)} \rangle \) must vanish. So we are left with a modified equation for \( u^{(0)} \):

\[
u^{(0)} - k(s)u^{(0)} = \frac{1}{2}m(s)[x^{(0)}(s)]^2,
\]

with

\[
u^{(0)}(0) = u^{(0)}(0) = 0,
\]

which is Hill's equation with a driving term given by the solution of the linearized betatron motion (first linearization) and the sextupole distribution around the ring. Such an equation can always be reduced to a vector recurrence relation:

\[
U^{(0)}_{n+1} = M U^{(0)}_n + A_n
\]

where \( U^{(0)}_n \) is the vector \( (u^{(0)}_n, u^{(0)}_n') \) and \( n \) again numbers the periods covered by the solution. Given the form of \( x^{(0)}(s) \) [Eq. (63)], the complete mapping must be

\[
U^{(0)}_{n+1} = M U^{(0)}_n + A \cos (2n\mu) + B \sin (2n\mu) + C,
\]

with

\[
u^{(0)}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Again \( M \) represents the transfer matrix associated with the linear betatron motion, while \( A, B, \) and \( C \) are constant vectors uniquely defined by the distribution of quadrupoles and sextupoles in the structure. The particular form \( A \cos (2n\mu) + B \sin (2n\mu) + C \) of the inhomogeneous part in the recurrence relation [Eq. (69)] is due to an expansion of \( x^{(0)}(s) \) [Eq. (67)] in terms of trigonometric functions at the sextupole positions, using Eq. (63). In Appendix B we derive the explicit form of \( A, B, \) and \( C \) for a particular example. Equation (69) is a linear, first-order, inhomogeneous vector recurrence relation. The general theory of such mappings states that the general solution is given by the sum of the homogeneous solution and one particular solution of the
inhomogeneous recurrence relation. In our case that means
\[ U^{(0)}_n = M^n(-F_0) + F_n, \] (70)
where \( F_n \) is the particular solution vector. The form of Eq. (70) implies that \( U^{(0)}_0 = 0 \). The quantity \( M^n \) being already given by Eq. (63), the problem is reduced to finding a particular solution vector \( F_n \) satisfying Eq. (69). Because the operation \( n \rightarrow n + 1 \) applied to the functions \( \cos(2n\mu) \) and \( \sin(2n\mu) \) leads to a linear combination of these same two functions, it becomes clear that the particular solution \( F_n \) must be
\[ F_n = a \cos(2n\mu) + b \sin(2n\mu) + c. \] (71)
If we insert this into the recurrence expression [Eq. (69)] and express everything in three terms, two proportional to \( \cos(2n\mu) \) and \( \sin(2n\mu) \) and one being constant, we may easily compare the coefficients of these three terms on each side of the equations. This leads to a system of three vector equations relating \( a, b, \) and \( c \) to \( A, B, \) and \( C \):
\[
\begin{align*}
    a \cos(2\mu) + b \sin(2\mu) &= Ma + A, \\
    a \sin(2\mu) + b \cos(2\mu) &= Mb + B, \\
    c &= Mc + C.
\end{align*}
\] (72) (73) (74)
The solution for \( a, b, \) and \( c \) is
\[
\begin{align*}
    a &= -[M^2 + I - 2M \cos(2\mu)]^{-1} \{B \sin(2\mu) + [M - I \cos(2\mu)]A\}, \\
    b &= [M^2 + I - 2M \cos(2\mu)]^{-1} \{A \sin(2\mu) - [M - I \cos(2\mu)]B\}, \\
    c &= -(M - I)^{-1} C.
\end{align*}
\] (75) (76) (77)
Finally, using Eq. (70) to obtain \( U^{(0)}_n \) and remembering that
\[ X_n = X^{(0)}_n + U^{(0)}_n, \] (78)
we find an approximate expression for the nonlinear invariants:
\[
X_n = [X_0 - (a + c)] \cos(n\mu) + \frac{\sin(n\mu)}{\sin \mu} (M - I \cos \mu)[X_0 - (a + c)] \\
    + a \cos(2n\mu) + b \sin(2n\mu) + c
\] (79)
Again, generalizing from integer to real \( n \), we find that \((i)\) the invariants are no longer ellipses; \((ii)\) to this order of the approximation, the invariants are closed curves in \((x, x')\) space \([\cos(n\mu), \sin(n\mu), \cos(2n\mu), \) and \( \sin(2n\mu) \) have a common period]; and \((iii)\) the average value of \( X_n \) is \( c \), corresponding to a systematic distortion of the invariants.
Equations (75) and (76) imply that \( a \) and \( b \) are only defined and finite if the matrix
\[ M^2 + I - 2M \cos(2\mu) \] (80)
can be inverted. Using Eq. (63) with \( n = 2 \) for evaluating \( M^2 \), the inverse of Eq. (80) can be written as

\[
\frac{1}{2 \cos \mu - \cos (2\mu)}.
\]

Hence, the condition

\[
\cos \mu - \cos (2\mu) \neq 0
\]

must hold, and this condition is not satisfied for

\[
\mu/2\pi = p \quad \text{and} \quad \mu/2\pi = p/3; \quad p \in \mathbb{Z}.
\]

While the first expression in Eq. (83) is related to the linear stop-band (integer resonance) of the betatron motion, the second one corresponds to the third integer resonance, directly associated with the presence of sextupoles. In both cases, the motion becomes unbounded for any nonzero amplitudes, in complete agreement with physical reality.

As an example, we calculate the invariants of the betatron motion in a FODO-lattice with \( \mu = \pi/3 \) and containing two sextupoles SF and SD near the focusing and defocusing quadrupoles. All the calculations have been done at the exit of the QF. If we subtract from Eq. (79) all the terms produced by the linear invariant ellipse (all terms not containing \( a, b \) or \( c \)), we are finally left with a closed curve representing all the effects caused by the nonlinear elements. Looking at this “Palette curve” (Fig. 10), we see that it is clearly distorted from the center \((0/0)\). In addition, we see that it grows rapidly with increasing amplitude. The crosses belong to numerically obtained results, and they seem to agree well with the analytical formalism.

![FIGURE 10 Nonlinear part of an invariant curve in a FODO lattice.](image-url)
Figure 11 shows the complete invariant curve $X(n)$ in the neighborhood of the third integer resonance. We observe the typical triangular shape, which is well known for this case. Again, the analytical values agree very well with the numerical ones. The results are presented in normalized phase-space coordinates $(x/\sqrt{\beta}, x'\sqrt{\beta})$ at the center of the focusing quadrupole ($\alpha = -\frac{1}{2}\beta' = 0$).

The dynamic aperture $x_{\text{lim}}$ is 2.2 cm.
5.2. Generalization to Two Dimensions

In two dimensions \((x, z)\), the equations of motion for \(\delta = 0\) read as:

\[
x'' - k(s)x - \frac{1}{2}m(s)(x^2 - z^2) = 0, \tag{84}
\]

and

\[
z'' + k(s)z + m(s)xz = 0. \tag{85}
\]

Here we have assumed that there exist no linear coupling terms between the two motions and that the coupling comes only from the nonlinear terms generated by the sextupole fields. We can immediately extend our formalism to this case by applying the first linearization to both equations:

\[
x(0)'' - k(s)x(0) = 0, \quad x(0)(0) = x_0; \quad x(0)'(0) = x_0'. \tag{86}
\]

and

\[
z(0)'' + k(s)z(0) = 0, \quad z(0)(0) = z_0; \quad z(0)'(0) = z_0'. \tag{87}
\]

Then, writing for the two motions

\[
x = x(0) + u,
\]

\[
z = z(0) + v,
\]

the equations for \(u(0)\) and \(v(0)\) become, after the second linearization,

\[
u(0)'' - [k(s) + m(s)x(0)(s)]u(0) + m(s)z(0)(s)v(0) = \frac{1}{2}m(s)\{[x(0)]^2 - [z(0)]^2\} \tag{90}
\]

and

\[
v(0)'' + [k(s) + m(s)x(0)(s)]v(0) + m(s)z(0)(s)u(0) = -m(s)x(0)z(0). \tag{91}
\]

It is again possible to modify these last equations by replacing \(x(0)\) and \(z(0)\) by their average values on the left side of the equations, as in the one-dimensional case. Since \(\langle x(0) \rangle = \langle z(0) \rangle = 0\), this gives

\[
u(0)'' - k(s)u(0) = \frac{1}{2}m(s)\{[x(0)]^2 - [z(0)]^2\} \tag{92}
\]

and

\[
v(0)'' + k(s)v(0) = -m(s)x(0)z(0). \tag{93}
\]

Hence the equations are decoupled with respect to the variables \(u(0)\) and \(v(0)\), and we may treat the problem by just solving each equation separately. Proceeding exactly as in Section 5.1, we may now obtain expressions for \(X(n)\) and \(Z(n)\). Examples will be calculated and published later.

5.3. The Dynamic Aperture

As explained above, a typical feature of the solutions for the nonlinear equation of betatron motion is to remain bounded up to a certain amplitude and then to become suddenly unbounded. The threshold amplitude is called the dynamic aperture of the system. The purpose of this section is to improve our understanding of the mechanisms driving this effect and to give a semianalytic
tool for finding the approximate initial conditions above which the motion becomes unbounded (i.e., the stability limit). All the calculations are done for a one-dimensional horizontal motion but can as a rule be extended to two dimensions, although this has not yet been done.

The basic effect can easily be explained qualitatively by inspecting the equation of motion:  

$$x'' - k(s)x - \frac{1}{2}m(s)x^2 = 0.$$  \hspace{1cm} (94)

From a certain amplitude onwards, the nonlinear term will become the dominant contribution to the behavior of the solution and a self-amplification will take place. This can be demonstrated by considering a simple example of a nonlinear recurrence relation:

$$x_{n+1} = \lambda x_n^2.$$  \hspace{1cm} (95)

Its exact solution is

$$x_n = x_0(\lambda x_0)^{2^n-1},$$  \hspace{1cm} (96)

$x_0$ being the initial value. It follows clearly from Eq. (96) that $x_n$ will remain bounded if

$$|\lambda x_0| < 1.$$  \hspace{1cm} (97)

In other words, for $\lambda, x_0 > 0$, the dynamic aperture of the system described by Eq. (95) is

$$(x_0)_{\text{lim}} = \frac{1}{\lambda}.$$  \hspace{1cm} (98)

If $x_0$ exceeds that critical value, the nonlinear term of Eq. (95) dominates, and a rapid self-amplification takes place. So far, it seems that the effect of dynamic aperture is a pure “large amplitude effect,” i.e., there is no hope of describing its occurrence by perturbative methods. In real systems, however, e.g., the one described by Eq. (94), we normally watch the occurrence of the stability limit at small amplitudes, when the nonlinear term is not yet dominant; consequently perturbative methods should be applicable. The presence of an instability at small amplitudes shows the existence of a second mechanism not described by the simple model above a mechanism that drives the solution towards the self-amplification regime. In order to look for such a mechanism, we use again the successive linearization method. The equation valid for the function $u^{(0)}$ after the second linearization has already been given, i.e., Eq. (64), which we repeat here for convenience:

$$u^{(0)}'' - [k(s) + m(s)x^{(0)}(s)]u^{(0)} = \frac{1}{2}m(s)[x^{(0)}(s)]^2,$$  \hspace{1cm} (64)

with

$$u^{(0)}(0) = u^{(0)'}(0) = 0.$$  \hspace{1cm} (65)

It is proved in Appendix A that the unboundedness of Eq. (64) is ensured by the unboundedness of only the associated homogeneous part of the equation:

$$u^{(0)}'' - [k(s) + m(s)x^{(0)}(s)]u^{(0)} = 0.$$  \hspace{1cm} (99)
On the other hand, we know from the previous approach for the invariant curves (Section 5.1) that the inhomogeneous term produces an unbounded motion only for distinct \( J_1 \) values (resonances). So, to that order of approximation, the complete information about the dynamic aperture is, in general, contained in the homogeneous part [Eq. (99)]. As we can see, the solution of the linear betatron motion \( x^{(0)}(s) \) is contained in the coefficient of Eq. (99), and this introduces a dependence of the solution \( u^{(0)} \) on the linear amplitude.

Now, since \( u^{(0)}(s) \) is generally not periodic over one period of the magnetic structure described by \( m(s) \) and \( k(s) \), Eq. (99) can only be reduced to a vector recurrence relation containing a nonconstant transfer matrix:

\[
\begin{pmatrix}
  u^{(0)}_{n+1} \\
  u^{(0)}_n
\end{pmatrix} = M_n \begin{pmatrix}
  u^{(0)}_n \\
  u^{(0)}_{n-1}
\end{pmatrix},
\]

(100)

However, if the linear tune \( Q \) of the structure is rational, i.e., if

\[
Q = \frac{p}{q}, \quad p \text{ and } q \text{ being integers},
\]

(101)

the theory developed in Ref. 1 and the concomitant Eq. (63) prove that \( x^{(0)}(s) \) becomes periodic with a period equal to \( q \) times the magnetic period of the structure. Hence Eq. (99) becomes Hill’s equation with an associated matrix (independent of \( n \)):

\[
R = \prod_{i=q-1}^{0} M_i
\]

(102)

\( M_i \) being the transfer matrix over the \( i \)th magnetic period as given in Eq. (99). The linear theory tells us that the solution \( u^{(0)} \) will be bounded if the condition

\[
|\text{Tr } R| < 2
\]

(103)

is fulfilled.

It must be noted that \( R \) explicitly contains the initial conditions \( x_0 \) and \( x'_0 \), as can be seen from Eq. (99). Consequently, the condition for boundedness of the motion becomes a function of \( x_0 \) and \( x'_0 \), and can be used directly for an estimation of the stability limit.

It now becomes clear that the second mechanism driving nonlinear instabilities is that of a parametric resonance caused by “fluctuating transfer matrices” in the perturbing equation for \( u^{(0)} \) [Eq. (99)]. This resonance may occur at small amplitudes, driving the solution towards higher amplitudes, where the self-amplification effect then takes over.

Let us briefly summarize the algorithm for finding the approximate stability limit for a given tune \( Q \) and initial values \( x_0 \) and \( x'_0 \):

(i) Approximate \( Q \) by a rational number \( p/q \), with \( q \) as small as possible.

(ii) Evaluate the matrix

\[
R = \prod_{i=q-1}^{0} M_i
\]

(104)

(iii) Calculate \(|\text{Tr } R|\) and satisfy \(|\text{Tr } R| < 2\).
For a complicated structure and large $q$, it is more and more difficult and tedious to get closed expressions for the matrix product in Eq. (104). Nevertheless, this method, which implies evaluating the product of a finite number of matrices with $q \approx 100$, offers a big advantage over tracking many particles over a very large number of periods. This is clear for applications to proton beams (no damping), and means in all cases a reduction of computer time.

As an example of an application, we use a LEP-type FODO lattice with two families of sextupoles $SF$ and $SD$ near the $F$ and $D$ quadrupoles. We vary $Q = \mu_{cell}/2\pi$ from 0.25 (= 30/120) to 0.417 (= 50/120). According to Eq. (102) the matrices $M_i$ are then

$$M_i = \left( \begin{array}{cc} 1 & 1 \\ -a - 2b_F x^{(0)F}_{i+1} & 1 - a - 2b_F x^{(0)F}_{i+1} \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ a + 2b_D x^{(0)D}_i & 1 + a + 2b_D x^{(0)D}_i \end{array} \right),$$

(105)

where

$$b_{F,D} = \frac{L}{4} (K_F'_{F,D} \cdot F_D)$$

(106)

and

$$a = 2 \sin (\pi Q),$$

(107)

while $L$ is the magnetic period length and $x^{(0)D}_i$ and $x^{(0)F}_i$ are the linear betatron values at the $F$ and $D$ quadrupoles after the $i$th revolution. The matrices $M_i$ are obtained from the homogeneous Eq. (99) and have exactly the form of the transfer matrix for a FODO lattice when we replace $k(s)$ by $k(s) + m(s)x^{(0)}(s)$. In addition, we transform the independent coordinate $s$ into $s^* = (2/L)s$. Now, applying the condition

$$\left| \text{Tr} \prod_{i=119}^{0} M_i \right| < 2$$

(108)

leads to a curve $(x_0)_{\text{lim}} = F(Q)$ being the maximum initial value for which $u^{(0)}$ is still bounded. Figure 12 shows $\text{Tr} R(x_0)$ for $Q = 42/120 = 7/20$. As can be seen, the curve crosses the line $\text{Tr} R = -2$ at $x_0 = 0.015$ m, while the expected limit (numerical computation) is $x_0 = 0.013$ m. In Fig. 13, the complete resulting curve $(x_0)_{\text{lim}} = F(Q)$ is compared to the expected limit and to the resonance fixed-point limit calculable near the third integer resonance ($Q = \frac{1}{3}$). The expected limit was found by computing the exact nonlinear transformation related to this problem over $10^5$ periods and increasing $x_0$ in small steps of 1 mm until the limit of unboundedness is reached (overflow). Stability means then $x(10^5) - x(10^4) < 1$ mm.

We see clearly the occurrence of the fourth- and third-order resonances (zero stability limit). The agreement between the expected limit and the limit found by the trace condition [Eq. (108)] is very good in the range of $Q$ from 36/120 to 50/120. For $Q = 30/120 \div 35/120$, the agreement is not so good. We suspect that this is due to the linearization of the $u$ equation being equivalent to neglecting cross terms between the sextupole strengths $SF$ and $SD$ which are important for describing higher-order resonances. As expected the third-order fixed-point treatment (to our knowledge the only analytic treatment for the dynamic aperture up to now) breaks down quickly if the actual $Q$ value is too far from $Q = 1/3$. 


FIGURE 12 Stability limit for FODO lattice with $Q = 7/20 = 0.35$:

$$L_{\text{cell}} = 79 \text{ m}, \ K'_p = -0.127 \text{ m}^3, \ K'_D = 0.133 \text{ m}^{-3}$$

FIGURE 13 Stability limit for the betatron motion as function of $Q$, $Q = p/120$. LEP-type FODO lattice:

$$Q = 0.25 - 0.417,$$

$$L_{\text{cell}} = 79 \text{ m}, \ K'_p = -0.127 \text{ m}^{-3}, \ K'_D = 0.133 \text{ m}^{-3}.$$
REFERENCES


APPENDIX A

Here we prove the following corollary: If all the homogeneous solutions belonging to the equation

\[ u'' + a(s)u = b(s) \]  

(A-1)
are unbounded, then at least the solution of the inhomogeneous equation that satisfies the initial conditions \( u(0) = u'(0) = 0 \) will be unbounded.

**Proof:** The solution of Eq. (A-1) with the initial conditions mentioned above can be written in an integral form

\[
u(s) = v_{HC}(s) \int_0^s v_{HS}(s) b(s) \, ds - v_{HS}(s) \int_0^s v_{HC}(s) b(s) \, ds,
\]

where \( v_{HC}(s) \) and \( v_{HS}(s) \) are solutions of the inhomogeneous Eq. (A-1), verifying the initial conditions

\[
\begin{align*}
v_{HC}(0) &= 1, & v'_{HC}(0) &= 0, \\
v_{HS}(0) &= 0, & v'_{HS}(0) &= 1.
\end{align*}
\]

The functions \( v_{HC} \) and \( v_{HS} \) are often called sinelike and cosinelike functions.

Equation (A-2) can be written in vector notation:

\[
\begin{pmatrix} v_{HC} \\ -v_{HS} \end{pmatrix} = \begin{pmatrix} u_{HC} \\ u_{HS} \end{pmatrix} \int_0^s b(s) \begin{pmatrix} u_{HS} \\ u_{HC} \end{pmatrix} \, ds = \mathbf{V}_1 \cdot \mathbf{V}_2.
\]

If we assume that \( u(s) \) is bounded while \( v_{HS} \) and \( v_{HC} \) are not, the two vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) in Eq. (A-4) have to be orthogonal so that their scalar product remains finite.

Since the initial vectors

\[
\begin{pmatrix} v_{HC} \\ -v_{HS} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_{HC} \\ u_{HS} \end{pmatrix}
\]

are perpendicular, the above statement means that the integral operation in Eq. (A-4) must not change the direction of the vector contained in it. Hence,

\[
\begin{pmatrix} u_{HS} \\ u_{HC} \end{pmatrix} = \lambda \begin{pmatrix} u_{HS} \\ u_{HC} \end{pmatrix},
\]

where \( \lambda \) is an arbitrary real constant.

Now by differentiating Eq. (A-5) once with respect to \( s \), we obtain two first-order differential equations for \( v_{HS} \) and \( v_{HC} \). But, given the form of Eq. (A-5), it is clear that \( v_{HS} \) and \( v_{HC} \) do not depend on the coefficient function \( a(s) \). This is clearly in contradiction to the homogeneous part of Eq. (A-1), whose solutions are by definition the functions \( v_{HS} \) and \( v_{HC} \). Therefore, Eq. (A-5) cannot be fulfilled in general by \( v_{HS} \) and \( v_{HC} \), and \( u(s) \) will necessarily become unbounded if \( v_{HS} \) and \( v_{HC} \) are unbounded. So, the corollary is proved.

**APPENDIX B**

Here we derive \( A, B, \) and \( C \), the vectors occurring in the recurrence relation [Eq. (69)] for the particular example of a structure containing drift spaces and quadrupoles in arbitrary order and one thin sextupole of strength \( K' \) at the
position \( s = j \). As indicated, the basic equation [Eq. (67)] is

\[
 u'' - k(s)u = \frac{1}{2}m(s)[x^{(0)}(s)]^2. \tag{B-1}
\]

Except at the point \( s = j \), this equation is exactly the one for the linear betatron motion, and the transformation of the vector \( \begin{pmatrix} u \\ u' \end{pmatrix} \) between two points not containing \( s = j \) is given by the linear transfer matrices. The transformation from the entrance to the exit of the thin sextupole can be obtained by integrating Eq. (B-1) over the interval \((j - \varepsilon, j + \varepsilon)\), where \( \varepsilon \) tends to zero:

\[
 u'_{\text{exit}} - u'_{\text{ent}} = \frac{1}{2} \int_{j-\varepsilon}^{j+\varepsilon} m(s)[x^{(0)}(s)]^2 \, ds = \frac{1}{2}(K'')(x_j^{(0)})^2. \tag{B-2}
\]

Here \( (K'') \) is the integrated strength of the sextupole. In addition, the continuity condition

\[
 u_{\text{exit}} = u_{\text{ent}} \tag{B-3}
\]

must hold, so the complete transformation is

\[
 \begin{pmatrix} u' \\ u \end{pmatrix}_{\text{exit}} = \begin{pmatrix} u' \\ u \end{pmatrix}_{\text{ent}} + \frac{1}{2}(K'')[x_j^{(0)}]^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{B-4}
\]

Let us now consider the transformation of the vector

\[
 U = \begin{pmatrix} u \\ u' \end{pmatrix} \tag{B-5}
\]

over the complete structure from revolution \( n \) to revolution \( n + 1 \). We define the matrices \( M_{0,j} \) and \( M_{j,1} \) as being the constant (independent of \( n \)) matrices from the beginning of the structure to the sextupole position \( s = j \) and from \( s = j \) to the end of the structure, respectively. Of course \( M_{0,1} \) is then defined as

\[
 M_{0,1} = M_{0,j} \cdot M_{j,1} = M, \tag{B-6}
\]

where \( M \) is the transfer matrix of the complete structure. The following successive transformations have to be applied:

(i) Transformation from the beginning of the structure to the entrance of the sextupole:

\[
 U_{j-\varepsilon} = M_{0,j} U_n. \tag{B-7}
\]

(ii) Transformation through the sextupole:

\[
 U_{j+\varepsilon} = U_{j-\varepsilon} + \frac{1}{2}(K'')[x_j^{(0)}]^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{B-8}
\]

(iii) Transformation from the exit of the sextupole to the end of the structure:

\[
 U_{n+1} = M_{j,1} U_{j+\varepsilon}. \tag{B-9}
\]
From (i) to (iii) we then find the complete transformation, which reads as

\[ U_{n+1} = M U_n + \frac{1}{2} (K'I)[x_j^{(0)}]^2 M_j,1 \]

(B-10)

The quantity \( x_j^{(0)} \) in Eq. (B-10) can be expressed as

\[ x_j^{(0)} = \{M_0, 0, X_0^{(0)}\}_1, \]

(B-11)

where the notation \( \{ \} \) means the first component of the vector enclosed in the brackets. Now, using Eq. (63) for \( X_0^{(0)} \), we find for \( [x_j^{(0)}]^2 \)

\[ [x_j^{(0)}]^2 = \alpha \cos (2n\mu) + \beta \sin (2n\mu) + \gamma, \]

(B-12)

with

\[ \alpha = \frac{1}{2} \left| \{M_0, 0, X_0\}_1^2 - \frac{\{M_0, 0, (M - I \cos \mu)X_0\}_1^2}{\sin^2 \mu} \right|, \]

(B-13)

\[ \beta = \frac{1}{2} \left| \{M_0, 0, X_0\}_1 \frac{\{M_0, 0, (M - I \cos \mu)X_0\}_1}{\sin \mu} \right|, \]

(B-14)

and

\[ \gamma = \frac{1}{2} \left| \{M_0, 0, X_0\}_1^2 + \frac{\{M_0, 0, (M - I \cos \mu)X_0\}_1^2}{\sin^2 \mu} \right|. \]

(B-15)

Inserting Eqs. (B-13) through (B-15) into Eq. (B-10), we finally get

\[ U_{n+1} = M U_n + A \cos (2n\mu) + B \sin (2n\mu) + C, \]

(B-16)

where

\[ A = \frac{\alpha}{2} (K'I)M_j,1 \]

(B-17)

\[ B = \frac{\beta}{2} (K'I)M_j,1 \]

(B-18)

and

\[ C = \frac{\gamma}{2} (K'I)M_j,1. \]

(B-19)