TOPONIUM PHYSICS

A. Martin

CERN - Geneva

ABSTRACT

After reminding the successes of the potential models for quarkonium, we present new theorems on the order of levels especially well suited to a system with many narrow levels such as toponium. In the second part we discuss the ground state energy of toponium as a function of the mass of mesons with visible top, show that low-lying levels permit a test of QCD and that higher levels allow one to test flavour-independence. Then we describe the peculiarities of toponium decay: the rôle of the Z*, of the W, and the possibility of decay of a single quark.

Lectures given at the
22nd course of the
International School of Subnuclear Physics
Ettore Majorana Centre for Scientific Culture
Erice, 1984

CERN-TH.4060/84

November 1984
TOPONIUM PHYSICS

André Martin

Theoretical Physics Department
CERN
1211 Geneva 23, Switzerland

I. INTRODUCTION

Since the discovery of the J/ψ in 1974, quarkonium physics has had a considerable development, both theoretically and experimentally. Various theoretical approaches have been used to describe the cc, bb and even the ss systems. In 1984, I would not hesitate to say that the most predictive one has been the potential model approach, even if one does not yet fully understand why it is so successful (in particular why the potential is so perfectly flavour-independent). Various potentials have been used, starting with $-4/3 \alpha_s/r + br^4$, then incorporating a smooth $\log r$ part at intermediate distance, then asymptotic freedom and the more QCD-motivated potentials of Richardson, Moxhay and Rosner and Buchmüller. Purely phenomenological potentials have also been proposed such as $V = A + br^a$, $a = 0.17$. All these potentials essentially agree for distances $0.1 < r < 1$ Fermi and this is why they all give a very good fit of all the levels of the J/ψ and T system. Buchmüller even reproduces nicely the P wave splittings. The last successful check of the models was the observation of the first radial excitation of the P states of the T system with mean energies 9.90 and 10.25 GeV, while Buchmüller predicts 9.89 and 10.25 and Martin 9.86 and 10.24 GeV. The 40 MeV discrepancy in Ref. 7 may be due to the absence of singular behaviour at short distances. QCD sum rule predictions for the lowest P states are not so successful: they predict $9.61 \pm 0.03$ GeV. However, the experts say that because of its small size, T is not a good testing ground for QCD sum rules. Whatever the reasons are, we are left with only one predictive model, the potential model.

We have already said that it works too well! This is particularly true when it is applied to ss, as was done in Ref. 7: one has, in GeV units:
\[
V = -8.064 + 6.870 r^{0.1}
\]

\[m_b = 5.174 \quad m_c = 1.8 \quad m_s = 0.518 \quad \cdot \quad (1)
\]

\(m_s\) is adjusted to reproduce the \(\phi\) (remember that these masses have no fundamental meaning!). The spin-spin force is taken to be a Fermi term with strength adjusted to reproduce the \(J/\psi - \eta_c\) splitting. Then one predicts \(m_b - m_\phi = 615\) MeV while experimentally it is 630 MeV. What is more striking is that one can make a parameter-free prediction of the masses of the \(F\) and the \(F^*\):

\[m_F = 1.99\ \text{GeV}, \quad m_{F^*} = 2.11\ \text{GeV}. \quad (2)
\]

At this school, J. Branson reported several experiments giving \(m_\tau = 1.975\ \text{GeV}\) (in contrast with previous experiments which gave 2.01 to 2.03 GeV) and also the \(F^*-F\) separation, 144 MeV according to Argus, giving \(m_{F^*} = 2.12\ \text{GeV}\). The least one can say is that the phenomenological potential \(f\), when used in the non-relativistic Schrödinger equation, gives a nice interpolation recipe to calculate masses of mesons not containing light quarks.

We now come to toponium, the bound \(t\bar{t}\) system which is very heavy. We know with certainty, from Petra experiments, that it is heavier than 42 GeV. There are, however, strong indications that the visible top, i.e., a meson \(t\bar{u}\) or \(t\bar{d}\), might have been seen in the UA1 experiment at the CERN collider, as a decay product of the \(W^0\): \(W^+ + t\bar{b}, \quad t + b + e^+ + \nu\), with a mass \(30 < M_{t\bar{u}} < 50\ \text{GeV}\). For a critical discussion, I send you to the lectures of L. Di Lella at this School. If it is so, the mass of toponium is of the order of 80 GeV and that makes the toponium system a very interesting system. First of all it has many narrow levels with widths certainly less than 100 keV (for more precision on the widths, see the last section). Let us repeat the argument because it is so nice and simple.

According to Zweig's rule, levels are \{narrow/broad\} if \(M_{Q\bar{Q}}(N, l, \text{etc.}) \ll 2M(Q\bar{Q})\), where \(Q\) is a heavy quark and \(q\) a light quark. Flavour independence tells us two things:
1) \(V_{Q\bar{Q}}\) is independent of \(Q\), heavy quark;
2) \(M(Q\bar{Q}) - M_Q\) is independent of \(Q\).

These statements are only approximate, especially the second one, because there are recoil corrections. Now we have to choose a specific potential corresponding to a specific choice of quark masses. However, since the positions of high levels are essentially independent of the choice of the potential, the counting of the narrow states will be independent of the potential. With potential (1), we have \(m_b = 5.174\ \text{GeV}\). Hence, using the experimental result of Cleo, \(m_b = 5.274\ \text{GeV}\) we have
\[ M_{Q\bar{Q}} - M_{Q} = 0.1 = \Delta. \]

On the other hand, we have
\[ M_{Q\bar{Q}} = 2M_{Q} + E, \]
where \( E \) is the binding energy. The dissociation threshold will be reached if \( E = 2\Delta \). Now it is a fact that the WKB approximation is excellent for \( \lambda = 0 \) states for the kind of potentials used in quarkonium (the error is less than 2% of the level spacing for the fourth radial excitation). So we have
\[ N - \frac{1}{4} = \frac{1}{\pi} \int \sqrt{M_{Q}} \sqrt{(E-V)}_{+} \, d\nu, \tag{3} \]
\( N \) being the principal quantum number, the symbol \(( \cdot \)\) means that we take the positive part of the function inside the parentheses. At the dissociation threshold \( E = 2\Delta \), independent of the flavour; \( V \) is also independent of the flavour and we get
\[ N_{\text{dissociation}} = \frac{1}{4} + \text{const} \sqrt{M_{Q}}. \tag{4} \]

We determine the constant by using the mass of the \( B \) and by interpolating between the \( T''(10.35 \text{ GeV}) \) and the \( T''(10.58) \). In this way we get
\[ N_{Q\bar{Q}} = \frac{1}{4} + 3.65 \sqrt{\frac{M_{Q}}{M_{B}}}, \tag{5} \]
and if \( m_{u} = 40 \text{ GeV} \), we get \( N_{Q\bar{Q}} \), number of narrow \( \lambda = 0 \) states = 10.4. A direct calculation, with a numerical solution of the Schrödinger equation using potential (1) gives, naturally, exactly the same result. The only place where model dependence enters in (5) is in the value of \( M_{Q} \). This, however, is a very weak dependence. One can show that \( M_{Q} \) is strictly restricted to the interval \( 4.6 < M_{Q} < 5.2 \).

We have drawn on Fig. 1 the spectrum of toponium with potential (1). The striking facts are that the spacing between the two \( \lambda = 0 \) lowest levels is 500 MeV, while the spacing between the 9th and the 10th level is only 80 MeV. We shall see later that it is only the spacing between the lowest levels which is potential-dependent.

Now we want to advocate that in this situation, with so many narrow levels and a potential which is not yet fixed from first principles, it may be of some interest to have some general theorems which allow you to make predictions on the order of levels of various angular momenta without having to solve the Schrödinger equation. This is the pretext we take to discuss this topic, also relevant in other fields of physics, in the next section.
$V = A + Br^{0.1}$

- Figure 1 -
II. THEOREMS ON THE ORDER OF LEVELS IN POTENTIALS

Perhaps I should start by recalling that I spoke first on that subject in Erice in 1977 in the lecture "Why I believe in quarks". In 1974-75, existing potential models all predicted a P state between the J/ψ and the ψ' and a D state above the ψ'. The question which H.A.B. Beg asked me first was to characterize the potentials for which one has such an order of levels, and H. Grosse and myself found two conditions involving the first, second and third derivative of the potential. One guaranteed that the lowest \( L = 1 \) would lie below the first \( L = 0 \) radial excitation, while the second one guaranteed that the lowest \( L = 2 \) level would lie above the first \( L = 0 \) radial excitation. This was not fully satisfactory for two reasons: (i) it looked awkward to have conditions involving the third derivative of the potential; (ii) the theorems could not be generalized to higher excitations because the method of proof made explicit use of the simple nodal structure of the radial wave functions of the low-lying levels. This meant that the theorems were insufficient, already in the case of the T system, and as we have seen in the last section, in the case of toponium.

The problem has now been solved by B. Baumgartner, H. Grosse and myself. Let us first discuss the comparison of the order of energy levels with the energy levels of hydrogen. Let us remember that in a pure Coulomb potential \( V = -1/r \) we have a degeneracy of the levels: if \( N \) designates the Coulomb principal quantum number:

\[
N = m \text{(number of nodes)} + l \text{(orbital angular momentum)} + 1
\]

the energy levels \( E(N,l) \) which, for a general potential would depend on both \( N \) and \( l \) depend only on \( N \) as shown in Fig. 2.

\[
\begin{array}{c}
N=4 \\
N=3 \\
N=2 \\
N=1
\end{array}
\]

\[
\begin{array}{c}
l=1 \\
l=2
\end{array}
\]

- Figure 2 -

Each multiplet contains \( N \) states with \( l < N-1 \).
A characteristic of the Coulomb potential is that it is a solution of the Laplace equation $\Delta V = 0$ for $r \neq 0$. The discovery that we have made is precisely that in the case of a non-Coulomb potential the order of levels with a given $N$ is controlled by the sign of the Laplacian of the potential. We have obtained the following theorems.

Let $V$ be a spherically symmetric potential. Then

**Theorem 1a**

$E(N, \lambda) > E(N, \lambda + 1)$

if $\Delta V(r) > 0$

for all $r > 0$

**Theorem 1b**

$E(N, \lambda) < E(N, \lambda + 1)$

if $\Delta V(r) < 0$

for all $r$ such that $dV/dr > 0$.

This is illustrated by Figs. 3a and 3b.

---

\[
\begin{array}{cccc}
\lambda = 0 & \lambda = 1 & \lambda = 2 & \lambda = 3 \\
\Delta V > 0 & & & \\
\text{-- Figure 3a --} \\
\end{array}
\]

\[
\begin{array}{cccc}
\lambda = 0 & \lambda = 1 & \lambda = 2 & \lambda = 3 \\
\Delta V < 0 & & & \\
\text{-- Figure 3b --} \\
\end{array}
\]

Before a complete proof of the theorem was obtained, the first indication came from a calculation in the WKB approximation (presumably good for large $n$) by Feldman, Fulton and Devoto\textsuperscript{14}, which gave

\[
E(N, \ell) \leq E(N, \ell + 1) \quad \text{if} \quad \frac{d}{dr} \lambda^2 \frac{dV}{dr} \geq 0
\]

but $1/r^2 \frac{d}{dr} r^2 \frac{dV}{dr}$ is precisely the Laplacian of the potential! The next indication\textsuperscript{15} came from the study of the limiting case.
\[
\nV = -\frac{1}{r} + A \psi, \quad \psi \to 0.
\]

Here the problem seems very simple: one has to calculate
\[
\int \psi^* \left[ (U_{N,\ell} e^{-i})^2 - (U_{N,\ell+1} e^{-i})^2 \right] d\psi,
\]
where \(U_{N,\ell}\) and \(U_{N,\ell+1}\) are Coulomb wave functions, to get the sign of \(E(N,\ell) - E(N,\ell+1)\). The difficulty, however, is that the integrand has many oscillations for \(N\) large. However, the Coulomb degeneracy is due to the \(E\) symmetry of the problem. For this reason the wave functions with same \(N\) but different \(\ell\)'s can be obtained from one another by using raising and lowering operators analogous to the raising and lowering operators of the angular momentum. One has
\[
\begin{align*}
\psi_{N,\ell}^+ = A_{\ell}^{-} U_{N,\ell+1} & \quad \psi_{N,\ell} = A_{\ell}^{+} U_{N,\ell} \\
\end{align*}
\]
with
\[
A_{\ell}^{\pm} = \pm \frac{d}{dr} - \frac{\ell+1}{r} + \frac{1}{2(\ell+1)}
\]
In (7) one can replace \(U_{N,\ell}\) by the expression (8) and eliminate the derivatives of \(U_{N,\ell+1}\) by successive integrations by parts. In the end, one gets exactly the same condition as Feldman et al.

At this point we became convinced that it was the Laplacian which controlled the order of levels, and we managed to find a proof, of which I shall give only the general ideas. The main trick was to generalize the raising and lowering operators. If you take as general form
\[
A_{\ell}^{+} = \frac{d}{dr} + g(r)
\]
and demand that \(A_{\ell}^{+} U_{N,\ell}\) satisfies a Schrödinger equation with angular momentum \(\ell+1\), you find that you have no freedom in the choice and you must take
\[
g(r) = -\frac{U_{\ell+1,\ell}}{U_{\ell+1,\ell}}
\]
\(U_{\ell+1,\ell}\) represents the wave function with \(N = \ell+1\), i.e., the ground state wave function for angular momentum \(\ell\). Hence
\[
U_{N,\ell+1} = A_{\ell}^{+} U_{N,\ell} = \frac{U_{\ell,\ell} e^{-i} U_{\ell+1,\ell} - U_{\ell+1,\ell} e U_{N,\ell}}{U_{\ell+1,\ell}}
\]
except for a normalization factor. It is possible to see that \(U_{N,\ell+1}\) corresponds to a state of angular momentum \(\ell+1\), i.e., behaves at the origin as \(r^{\ell+2}\), and has \(n-1\) nodes if \(U_{N,\ell}\) has \(n\) nodes. Therefore \(U_{N,\ell+1}\) has exactly the same "Coulomb" principal quantum number. However, there is a difference with the case of a pure Coulomb potential, which is that if \(U\) satisfies the Schrödinger equation
\[
\left(- \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \nabla - E\right) U_{N,l}^{(l+1)}(r) = 0
\]  \hspace{1cm} (13)

\( U_{N,l+1}^{(l+1)} \) satisfies a Schrödinger equation with a different potential
\[
\left(- \frac{d^2}{dr^2} + \frac{(l+1)(l+2)}{r^2} + \nabla - E\right) U_{N,l+1}(r) = 0
\]  \hspace{1cm} (14)

with
\[
\nabla - V = 2 \left[ q'(r) - \frac{l+1}{2} \right]
\]
\[
= 2 \left[ \frac{u^2 + (E-V)u}{r^2} + \frac{l-1}{2} \right]
\]
\[
= 2 \left[ \int r^2 u^2 (\frac{dr}{r^2} - 2 \frac{dV}{dr}) - \frac{l+1}{2} \right]
\]  \hspace{1cm} (15)

We thought that we had discovered all this, but it was pointed out to us by K. Chadan that Eq. (12) had been found by Marchenko in 1955.\(^8\) As we said, there was no arbitrariness in the construction of the raising operator and this is why we get exactly the same.

Now we have made a decisive step forward: assume that we know that
\[
\nabla > V
\]  \hspace{1cm} (16)

Then from Eqs. (13) and (14), we have
\[
E(N, l+1, \nabla) = E(N, l, V)
\]  \hspace{1cm} (17)

But the energies are monotonous functions of the potential (this applies to all levels, not only the ground states). Hence
\[
E(N, l+1, V) < E(N, l+1, \nabla) = E(N, l, V)
\]  \hspace{1cm} (18)

Conversely if \( \nabla < V \), we get
\[
E(N, l+1, \nabla) > E(N, l, V)
\]

We are left now with a relatively tough technical problem: prove that
\[
q'(r) - \frac{l+1}{2} \geq 0 \quad \text{if} \quad \Delta V(r) \geq 0.
\]
It is impossible to give the details of the proof here. Let me say, however, that the proof is using only elementary means such as comparing the ground state wave function \( u \) in an interval \( 0 \leq r \leq r \) to a Coulomb wave function and noticing that the conditions \( \Delta V \gg 0 \) are convexity (concavity) properties, not with respect to straight lines but to hyperbolae \( A + B/r \). Specifically:

i) if \( \Delta V > 0 \), i.e., \( \frac{d}{dr} r \frac{dV}{dr} > 0 \), any hyperbola tangent to \( r = R \) to \( V(r) \) is below \( V(r) \), or explicitly:

\[
\forall R, \quad V(r) > V(R) + R \frac{dV}{dr} - \frac{r^2 \frac{dV}{dr}}{R}
\]

ii) if \( \Delta V < 0 \), it is the opposite:

\[
\forall R, \quad V(r) < V(R) + R \frac{dV}{dr} - \frac{r^2 \frac{dV}{dr}}{R}
\]

This is illustrated in Figs. 4a and 4b.

- Figure 4a -

- Figure 4b -

In fact, inequality

\[
\mathcal{E}(N, l) < \mathcal{E}(N, l+1),
\]

can be established under a slightly weaker condition because it suffices to have \( g' < (l+1/r^2) \), but since
\[ g'(r) = \int_{\infty}^{\infty} u^2 \left( \frac{dV}{dr} - \frac{2 (\ell + 1) \ell}{r^3} \right) dr \]

we see that if \( dV/dr \) is negative beyond a certain distance, \( g' \) is negative and hence the inequality is satisfied.

We now turn to applications of Theorems Ia and Ib. The first application is to quarkonium. All quarkonium potentials have the property \( \Delta V > 0 \). At short distances this is clear, because asymptotic freedom is equivalent to antiscreening, but the fact is that in all potentials used this is true at all distances:

\[ V = - \frac{\alpha}{r} + b r \]
\[ V = - \log r \]
\[ V = A + B r^{0.1} \]

The Richardson and Buchmüller potentials satisfy this condition and analytical fits to lattice QCD potentials have also this property. Therefore we believe that between two successive \( \ell = 0 \) levels of toponium we shall find an \( \ell = 1 \) level, higher than the \( \ell = 0 \) level with the same number of nodes (because of the positivity of centrifugal energy), lower than the \( \ell = 0 \) level with one more node.

There are, however, other applications. One is mesic atoms. If a negative muon turns around a uranium nucleus, the electrostatic potential of the nucleus cannot be approximated any more by a point charge. This has been discussed long ago by Wheeler. However, we want to point out that irrespective of a detailed model of the nucleus, we have

\[ V = - e^2 \int \frac{\rho(x') d^3 x'}{|x - x'|} \quad \text{where} \quad \rho(x) > 0, \]

since the protons have positive charges, and hence \( \Delta V > 0 \). Hence

\[ E(N, \ell) > E(N, \ell + 1). \]

Another application is very old: the justification of the Mendeleev classification. Take for simplicity an alcaline atom. The outer electron sees the potential created by the nucleus (\( \Delta V = 0 \)) and the one created by the negatively charged electron cloud (\( \Delta V < 0 \)). Hence

\[ E(N, \ell) < E(N, \ell + 1). \]

This is visible on the spectrum of sodium (Fig. 5) and explains why, for instance, the third electron of lithium is 3S instead of 3P! Physicists above 60 generally tell me that this was obvious anyway, because electrons with large \( \ell \) are more often far away from the nucleus. This, however, does not dispense us from giving a clean proof.
Energy level diagram for sodium. The $S$ series is known to $n = 14$, the $P$ to $n = 59$, the $D$ to $n = 15$, and the $F$ to $n = 5$. The only doublet terms which have been resolved are the first seven of the $P$ series.

- Figure 5 -

Before ending this section, let me remind you that there was another problem of ordering of levels, which was the relative position of the lowest $\lambda = 2$ state and of the first $\lambda = 0$ radial excitation. More generally, if $\varepsilon(n, \lambda)$ designates the energy of the level with $n$ nodes, and angular momentum $\lambda$, how do we compare $\varepsilon(n, \lambda)$ and $\varepsilon(n-1, \lambda+2)$? Taking $V = r^2 + \lambda V$, $\lambda$ small, we have obtained the condition [15]

$$\varepsilon(n, \lambda) \gtrsim \varepsilon(n-1, \lambda+2)$$

if

$$\frac{d}{dr} \frac{1}{r} \frac{dV}{dr} \geq 0.$$ (19)

Naturally if

$$V = r^2, \quad \varepsilon(n, \lambda) = \varepsilon(n-1, \lambda+2).$$

Recently (after the lectures were given!) we (B. R., H.G. and A.M.) have been able to prove (19) in full generality [17], without assuming $V$ small, i.e.:

$$\varepsilon(n, \lambda) \gtrsim \varepsilon(n-1, \lambda+2)$$

if

$$\frac{d}{dr} \frac{1}{r} \frac{dV}{dr} \geq 0.$$ (20)
The proof is based on the change of variable $\rho = r^2$ which transforms, for instance, the harmonic oscillator into a Coulomb-Schrödinger equation, provided the orbital angular momentum is rescaled. In fact, we went further than that and proved, for instance:

$$\mathcal{E}(n', l') \leq \mathcal{E}(n-1, l+3)$$

if

$$r V''(r) \leq 3 V'(r)$$

which is the case for $V = r^4$, for instance.

III. TOPONIUM: THE ENERGY LEVELS

We have seen that the number of narrow levels $l = 0$ is $N = 10$ if $m_\pi = 40$ GeV. In any case, with the lower limit on toponium mass from PETRA we know that there are at least six narrow levels.

We expect that knowing the energies of 10 narrow levels will allow a better exploration of the potential. However, this does not increase our information on the long-range part of the potential because, as can be seen by looking at the WKB derivation of $N$, the $N = 10$ level of toponium has the same extension in space as the $N = 3$ level of $\Upsilon$, or $N = 2$ level of $J/\psi$. On the other hand, we shall know more about the short-range part by looking at the low-lying levels. However, only the two lowest-lying levels, 1S and 2S are really sensitive to the choice of the potential. Let us repeat the possible candidates for $V$:

(i) $-a/r + br^{1/3}$;

(ii) potentials incorporating asymptotic freedom

$$V \sim -\frac{4}{3} \frac{\alpha_s^3}{r} \ln r \to 0$$

with

$$\alpha_s = 12\pi/[25 \ln (1/r^2 \Lambda_{\text{MS}}^2)]$$

in this category, we have: Krasemann-Qiao, Richardson, Moxhay-Rosner, Buchmüller et al.;

(iii) purely phenomenological potentials such as $V = -A + Br^{0.1}$.

Naturally besides the potential approach, one could envisage, for a very heavy system with a very small extension in space a direct calculation of the spacing of the lowest-lying levels from
QCD as was proposed for instance by Leutwyler, but we shall not speak about it here.

All good potentials are very close to one another in the 0.1 + 1 Fermi range.

--- Figure 6 ---

We see, however, that below 0.1 Fermi, potentials begin to differ appreciably. To get a feeling, let us indicate the asymptotic behaviour, for $\Lambda_\nu \to \infty$, of the level spacing $^{21)}$.

If

$$V = \pm r^\lambda \begin{cases} + \frac{p_\lambda}{2} \lambda > 0 \\ - \frac{p_\lambda}{2} \lambda < 0 \end{cases}$$

$$E_2 - E_1 \sim M_\nu - \frac{\nu}{2 + \nu}$$

(22)
In particular
\[ E_2 - E_1 = \ln r \quad \text{for} \quad V = \log r. \]

In this way, we understand that \( V = r^{0.1} \) explains
\[ \frac{M_{\gamma'} - M_{\gamma}}{M_{\psi'} - M_{\psi}} \approx \frac{560}{590} < 1 \]

however, such a trend will not continue for ever because, as the quark mass increases we are sensitive to shorter and shorter distances. Without asymptotic freedom, we would expect a Coulomb-like potential, i.e., \( \alpha = -1 \), which would give
\[ E_2 - E_1 \sim M_Q. \]

With asymptotic freedom, i.e., \( V \sim -1/(r \log(1/r)) \), we expect
\[ E_2 - E_1 \sim \frac{M_Q}{\log(M_Q)} \quad (23) \]

To know (by trial and error) which part of the potential is relevant, it is useful to remember that for pure power potentials the radius of a state with fixed quantum numbers behaves like
\[ \sqrt{<R^2>} \sim M_Q^{-1 - \frac{2}{2+\alpha}} \quad (24) \]

For a 40 GeV top quark mass, we get in this way
\[ R_{1S} < 0.07 \quad \text{Fermi}, \]
\[ R_{2S} \approx 0.18 \quad \text{Fermi}, \]
\[ R_{3S} \approx 0.30 \quad \text{Fermi}, \]

and, since the Q\overline{Q} potential is essentially fixed for distances larger than 0.1 Fermi we do not expect any spectacular spread of the prediction for the 3S-2S separation. This is indeed what exact solutions of the Schrödinger equation confirm as indicated by the Table.
<table>
<thead>
<tr>
<th>$A + Br^{0.1}$</th>
<th>Krasemann</th>
<th>Buchmüller</th>
<th>Moxhay-Rosner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ono</td>
<td>$\Lambda = 0.2$</td>
<td>$\Lambda = 0.5$</td>
</tr>
<tr>
<td>$E_2 - E_1$</td>
<td>520 MeV</td>
<td>661</td>
<td>625</td>
</tr>
<tr>
<td>$E_3 - E_2$</td>
<td>302</td>
<td>322</td>
<td>310</td>
</tr>
<tr>
<td>$E_4 - E_3$</td>
<td>217</td>
<td>228</td>
<td>209</td>
</tr>
<tr>
<td>$E_{10} - E_9$</td>
<td>84</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

We conclude that the observation of the 1S and 2S levels of toponium will constitute a sensitive test of QCD. However, the effect is not as spectacular as one might have expected! A much larger separation would be an indication of the absence of asymptotic freedom: the Bhanot-Rudaz potential \(^2\) predicts $E_2 - E_1 \sim \sim 1300$ MeV.

Is the observation of the higher levels useless? Certainly not because it constitutes a crucial test of flavour independence: if the $tt$ potential differs from the $bb$ potential in the range 0.1 to 1 Fermi, the experiments will give a result which differs from the common predictions of all models.

Now we come to an important prediction which is the difference $\Delta E$ between two times the mass of the T meson and the ground state energy of toponium:

$$\Delta E = 2M_{\phi\bar{q}} - M_{\phi\bar{q}}. \tag{25}$$

Clearly, getting the best possible information on this quantity will allow us to reduce the search time at LEP and the SLC if the mass of the top meson is already known with sufficient accuracy from $pp$ collider experiments.

First there are general inequalities obtained by Bertlmann and myself\(^22\):

(i) for any potential, $M_Q \Delta E_Q$ is a convex function of $M_Q$; From the $bb$ and $cc$ data one deduces, for $M_t = 40$ GeV, $\Delta E_t > 0.9$ GeV;

(ii) if the potential is concave, which all potentials are (in particular they grow at most linearly), we have:
\[ \Delta E_T > 3 \frac{1}{2} \left( \frac{M_b}{M_T} \right)^{1/2} + \Delta E_b, \]  

where \( T_{b\bar{b}} \) is the expectation value of the kinetic energy in the \( b\bar{b} \) system which, itself, satisfies the inequality \[ T > \frac{3}{4} (E_P - E_{1S}); \]  
in this way we get \( \Delta E_T > 1.57 \text{ GeV} \). However, we have an estimate of \( T \) which improves (27), which is \[ T \approx \frac{3}{4} [E_P - E_{1S}] \left[ 1 + \frac{7}{9} \left( \frac{E_{2S} + E_{1S} - 2E_P}{E_{2S} - E_{1S}} \right)^2 \right]; \]  
this gives \( T_{b\bar{b}} = 0.41 \text{ GeV} \) and \( \Delta E_T > 1.69 \text{ GeV} \);  

(iii) if the potential satisfies \( \Delta V > 0 \) (which we believe it does!), we have \[ \Delta E_T < \Delta E_b + T_{b\bar{b}} \left( \frac{M_T}{M_L} - 1 \right), \]  
which leads to \( \Delta E_T < 3.85 \text{ GeV} \).  

Now, if we turn to models, we get the following predictions:  

\( \Delta E_T = 1.77 \text{ GeV} \) (A+Br0.1)  

\( \Delta E_T = 1.88 \text{ GeV} \) (Buchmüller and Krasemann-Ono)  

\( \Delta E_T = 2.2 \text{ GeV} \) (Moxhay-Rosner),  

and finally the rather extreme prediction of a model with a pure Coulomb singularity:  

\( \Delta E_T = 2.6 \text{ GeV} \) (Bhanot-Rudaz).  

We conclude that the search for the toponium ground state should be concentrated in an interval 1.7 to 2.5 GeV below two \( T \) masses.  

Now I would like to say a few words on the detectability of the levels of toponium. This question has already been studied by Jackson, Olsen and Tye27). What one finds is that the signal to background ratio depends very much on the mass of toponium, because the \( Z^0 \) intermediate state competes with the photon intermediate state, both in the background and in toponium prediction. For \( M_{t\bar{t}} = 75 \text{ GeV} \), Jackson et al get a signal/background of about 1.
For $M_{t\bar{t}} = 80$ GeV it would be four times less, but this is based on an energy resolution perhaps too pessimistic (60 MeV on the total energy). If we take 30 MeV we get: signal/background $= 0.5$.

Now, with a luminosity of $10^{31}$ cm$^{-2}$ sec$^{-1}$, and assuming that open top has a mass known with an accuracy of ±2 GeV, and taking into account the uncertainty of 1 GeV maximum on the binding energy of toponium, one finds that one has to scan a 5 GeV interval, and one ends up with a machine time (based on Jackson's estimate) which is ~100 days. The search for the 2S state is restricted to a 400 MeV interval and should take $(1/0.55) \times 100 \times (4/50) = 15$ days. 0.55 is the relative leptonic width of the 2S state. The higher states are much more precisely located but there are two major difficulties: the relative leptonic widths are going down, for instance $|\psi_{10}(0)|^2/|\psi_0(0)|^2 = 0.15$, and the distance between the levels is becoming of the order of the energy resolution. For instance, $E_{10} - E_0 \approx 80$ MeV. If the energy resolution is 60 MeV, it is hopeless to separate these levels. If it can be made to be 30 MeV, the time needed will be at least 30 days.

All these numbers are, of course, very uncertain because: (i) we do not know exactly the toponium mass and everything is terribly sensitive to the mass difference between the (tℓ) and the $Z'$; (ii) we do not know what will be the true luminosity; (iii) we do not know how clever will be the machine builders to improve the energy resolution (use of polarization might help!); (iv) the open top mass will possibly be known with better accuracy than anticipated; (v) extra selection criteria for "good" events could be used. Sphericity is one, but another very favourable feature is the asymmetry in lepton pair decays of toponium (see Section IV).

Finally, let me just say a word on the observability of the P states. We shall just consider the lowest P state which is reached by

$$ (N=2, l=0) \rightarrow \gamma + (N=2, l=1). $$

Numerical experiments$^{24}$ show that the Thomas-Reiche-Kühn sum rules are pretty well saturated by the lowest P states. Hence we get

$$ \Gamma(2S \rightarrow \gamma + 3P_2) \approx \frac{40}{2\pi} \alpha e^2 \frac{(E_{1S} - E_P)^2}{M_{1S}}. $$

With $E_{2P} - E_P \sim 150$ MeV, we get $\Gamma \sim 2.7$ keV, and if, anticipating on the next section, we take $\Gamma_{tot}(2S) = 80$ keV, we get a branching ratio of 3.5%, which is not unacceptable.
IV. DECAYS OF TOPONIUM

Let me indicate right away that, on this subject, there are excellent articles by Kühn\textsuperscript{25} and by Seghal\textsuperscript{26} who gave lectures on the subject, in Erice, at a recent Europhysics conference. Toponium decays are extremely interesting for four reasons.

(i) Strong interactions get weaker because of asymptotic freedom effects. So the three-gluon decay becomes less important: one can take, neglecting QCD radiative corrections:

\[
\frac{\Gamma(t\bar{t} \to 3g)}{\Gamma(\gamma \to 3g)} \sim \left[ \frac{\ln (M_{\gamma}/0.12)}{\ln (M_{t\bar{t}}/0.12)} \right]^3
\]

using the fact that $|\psi(0)|^2/M_0^2$ is practically constant for the most favoured potentials. Then one gets, if $\Gamma(\Upsilon \to 3g) = 27$ keV\textsuperscript{27}, $\Gamma((t\bar{t}) \to 3g) \approx 8$ keV.

This is only an order of magnitude, for we know that QCD radiative corrections are not small. For instance, if one tries to fit the $\Upsilon$ and $J/\psi$ gluonic width by this naive formula, one finds that the scale parameter 0.12 has to be replaced by 0.01 (crazy!). Then one gets

\[
\Gamma(t\bar{t} \to 3g) \sim 12 \text{ keV}
\]

It has been pointed out by Kühn\textsuperscript{26} that also the $P$ states get narrow, because of the decrease of $\alpha$ and because of the decrease of the size of the system, and have widths less than 100 keV.

(ii) The $Z^0$ can be an important intermediate state in decays (as well as in production) into $q\bar{q}$, $l^+l^-$, it will interfere with the photon, but also new decays in neutrino pairs will be possible:
In the $e^+e^-$ and $q\bar{q}$, the rates will be modified and parity violating effects will take place; one observes a forward-backward asymmetry in reactions $e^-e^+ \rightarrow \ell\bar{\ell}$ and also, in case the electrons are longitudinally polarized, a difference between cross-sections for left-handed and right-handed electrons:

$$\alpha_{RL} = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L}$$

$$\frac{d\sigma}{d\cos\theta} = \text{const} \left[ 1 + \cos^2\theta + 2\alpha_{FB}\cos\theta \right].$$

In the case $\ell\bar{\ell} = \ell\bar{\ell}$, $\alpha_{FB}$ is just $\alpha_{FB} = (\alpha_{RL})^2$. This is because the intermediate state is a bound ($t\bar{t}$) vector particle. $\alpha_{RL}$ is given by

$$\alpha_{RL} = -\frac{2}{\Lambda^2 + \Lambda'^2} \Lambda_e \Lambda_{e'}'$$

where $\Lambda$ is proportional to the vector coupling of $(t\bar{t})$ to $e^+e^-$, which contains contributions from $\gamma$ and $Z^0$ exchange, while $\Lambda'$ is proportional to the axial vector coupling; with $\sin^2\theta_W = 0.215$, and $M_{Z^0} = 93$ GeV, we get

$$\Lambda_e = -\frac{2}{3} \frac{0.0227}{8649 - M_V^2}$$

$$\Lambda_e' = \frac{0.1623}{8649 - M_V^2}$$

where $M_V$ is the $(t\bar{t})$ mass. We see that there is a value of $M_V$ for which $\Lambda_e = -\Lambda_e'$, i.e., where the $(t\bar{t})$ system has an exactly $V+A$ interaction with the $e^+e^-$ pair. Then the left-handed cross-section vanishes and the asymmetry for the decay into lepton pairs is maximum, i.e., unity. This value is $M_{t\bar{t}} = 82$ GeV. But what is extremely interesting is that the asymmetry is larger than $+50\%$ for $74 < M_{t\bar{t}} < 87$ GeV, while the asymmetry of the background in the same interval is algebraically less than $-50\%$. Since there is a non-zero probability that the $t\bar{t}$ mass lies in this interval, this means that Nature might have decided to be extremely kind to physicists. For more details, we send the reader to the Erice talk of Sehgal in 1983 (notice, however, that there is a misprint in Fig. 2). If we had an object made of quarks with different charge or different weak isospin attributions, the predictions would be drastically different.

The neutrino-antineutrino decay is especially interesting for it gives a new method of counting neutrino species. For a 80 GeV toponium, one has
\[ \frac{BR(\nu \bar{\nu}) \text{ per species}}{BR(\mu \bar{\mu})} = 0.7 \]

The question is how to see it. Two methods have been proposed:

a) produce the 2S state; we have

\[ (2S) \rightarrow (1S) + \pi \pi \]

with a rate calculable from QCD\(^2\)

\[ \frac{\Gamma_{\pi \pi}(t \bar{t}, 2S)}{\Gamma_{\pi \pi}(b \bar{b}, 2S)} = \left( \frac{<r^{2}_{t \bar{t}} - 16>}{<r^{2}_{b \bar{b}} - 16>} \right)^{2} \approx 10^{-2} \]

i.e., \( \Gamma_{\pi \pi}(2S) \approx 1 \text{ keV} \); then look at the events \( \pi \pi + \text{missing mass, } \pi \pi + \ell^{+} \ell^{-} \), with the constraint that the missing mass or the lepton pair mass agree with the 1S mass;

b) take an \( e^{+}e^{-} \) energy slightly above the \( (1S) \) mass and look at

\[ e^{+}e^{-} \rightarrow \gamma + X \rightarrow \mu^{+} \mu^{-} \text{ invisible} \]

and constrain the photon energy. The latter method is a variant of one proposed using the \( Z^{0} \) itself.

(iii) Decay can take place via crossed channel \( W \)'s instead of direct channel \( Z^{0} \)'s, producing \( b \bar{b} \) pairs:

![Diagram of t\bar{t} and W decay](attachment:diagram.png)

If the \( t \bar{t} \) was much higher in mass, this process, which depends critically on the number of colours, would become very important if the mass of toponium was above 100 GeV.

(iv) Finally, a most remarkable decay is that of a single quark \( t \) or \( \bar{t} \), in flight, without annihilation of the \( t \bar{t} \) pair. This process, which is not different in Nature from the open top decay for which the UAL group has given preliminary evidence\(^10\), becomes very important because of the large mass difference of the \( t \) and the \( b \). In the limit

\[ \left( \frac{m_{t}^{2}}{m_{W}^{2}} \right)^{2} \ll 1 \quad \text{and} \quad \left( \frac{m_{b}^{2}}{m_{t}^{2}} \right)^{2} \ll 1 \]

we have

\[ \Gamma(t \rightarrow b + l^{+} + \nu) \approx \frac{G_{F}^{2} m_{t}^{5}}{192 \pi^{3}} \]

which is a good enough approximation with the actual masses. This should be multiplied by \( 18 \) to take into account the various leptons and quark final states and the possibility of having a decay of \( t \) or \( \bar{t} \).
This mode contributes 44 keV to the total width of $t\bar{t}$. One important point is that it is an incompressible quantity when one goes to higher radial excitations, while contributions arising from (i), (ii) and (iii) are all proportional to the square of the wave function at the origin and decrease for higher levels.

If we use Jackson's or Kühn's figures, we get for the $1S$ state a total width of 127 keV if $M_{t\bar{t}} = 80$ GeV, out of which the single quark decay contributes 32% and the neutrino pair decay contributes 10%. For higher states we get

$$\Gamma_{tot} = 44 + \frac{\Gamma_{e}(nS)}{\Gamma_{e} (1S)} \times 83 \text{ keV}$$

which means that the 10th-state has a width of about 56 keV.

Figure 7 shows the dependence on the toponium mass of the various channels.

Finally, let us indicate another very exciting decay. If the Higgs exists and has a mass which is relatively small compared to the mass of toponium, then toponium has an appreciable branching ratio to Higgs+photon in the simple standard model.
\[ \frac{R(1S \rightarrow Higgs + \ell)}{R(1S \rightarrow \gamma^* \rightarrow e^+e^-)} = \frac{G_F}{4\sqrt{2}\pi\alpha} \left( \frac{M_{t\bar{t}}^2}{M_{t\bar{t}}^2} \right) \]

with 5 keV for the conventional leptonic width we get a partial width of 3 keV, i.e., a few per cent branching ratio.

A last remark: if the (t\bar{t}) bound was at a distance of the Z^0 of the order of the Z^0 width, i.e., \(3\) GeV, one would have to consider, as pointed out by Renard\(^4\), mixings of the Z^0 and the (t\bar{t}) which would no longer be the eigenstates of the mass matrix.

REFERENCES

17) J. Stack - ITP-Santa Barbara Preprint N° 136 (1983);
19) B. Baumgartner, H. Grosse and A. Martin - in preparation (University of Vienna and CERN).