DYNAMICS OF ELECTRONS IN STORAGE RINGS INCLUDING
NON-LINEAR DAMPING AND QUANTUM EXCITATION EFFECTS

by

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A simple method for incorporating the quantum fluctuations of synchrotron radiation directly into the classical equations of motion of an electron (or positron) in a storage ring is given. It is based on an intuitively appealing, semi-classical representation of the synchrotron radiation power in an arbitrary magnetic field as a Markov process. An equation of Fokker-Planck type follows directly from the equations of motion although it is usually more expedient to make canonical transformations of the dynamical variables first.

The drift and diffusion terms include the linear and non-linear effects of all magnetic elements in a lattice. They are the starting point of systematic approximations and averaging schemes for calculation of higher-order corrections to the distribution function in synchro-betatron phase space. These calculations simplify considerably for separated function lattices. As an illustration of the method, the effect of the amplitude-dependent damping and quantum excitation due to quadrupoles is evaluated.

Besides its utility in evaluation of higher-order effects, the formalism presented here has the advantage of treating the dissipative aspects of electron motion all of a piece with the canonical part. It is hoped that this provides a clear and faithful picture of the basic physics.

With some notable exceptions, most analyses of single-particle motion in electron storage rings have begun by neglecting the effects of synchrotron radiation. In later steps one can superpose the effects of linear damping and diffusion due, respectively, to the interplay between a particle's average energy loss and what it has restored by the RF accelerating system and to the quantum fluctuations. This approach works famously of course: in the first step one enjoys the benefits of working in the analytical framework of classical Hamiltonian dynamics and all terms appearing in the equations of motion can be derived systematically in the course of successive canonical transformations. The price paid is that the effects of photon emissions have to be added linearly and there is an explicit or implicit appeal to the Central Limit Theorem in the conclusion that the particle distribution is Gaussian in each of the three degrees of freedom.

The bigger storage rings get, however, the less this procedure is valid. In particular, the relatively greater strengths of quadrupole and sextupole fields, if not the presence of special magnetic structures, can introduce amplitude dependences of the damping and quantum excitation as well as new coupling mechanisms among the three degrees of freedom.

We outline the principles of a theory of single-particle motion which includes dissipative and fluctuating terms directly in the equations of motion at an early stage so that any non-linear effects may be retained correctly. The instantaneous synchrotron radiation power is represented as a sum of mean and fluctuating parts, each depending only on the local magnetic field and the particle's energy. For clarity we restrict consideration to an ideal planar storage ring whose magnetic guide field is such that a non-radiating particle of energy $E_0$ follows a stable reference orbit $r_0(s)$ when the RF is switched off. In the usual coordinate system $(s,x,y)$ in the neighbourhood of this orbit, the accelerating and guide fields may be described by one component, $A_x(s)$, of the vector potential (with the circumference $2\pi R$ as period)

$$
\text{e}A_x / E_0 = -xG(s)(1 + G(s)x/2) - K(s)(x^2 - y^2)/2
$$

$$
+ \text{e}C_{\text{RF}} / E_0 \delta(s - s_0) \cos \left( \omega_{\text{RF}} t + \phi_0 \right)
$$

(1)

$\delta(s)$ is the periodic extension of the delta-function where the Coulomb gauge is used, electric fields are assumed absent except for the fundamental accelerating mode of the RF cavities at positions $s_0$, and effects of magnets are neglected and no skew quadrupoles, sextupoles or higher multipoles are present. Any or all of these complications can, of course, be included. With $s$ as independent variable, the single-particle Hamiltonian takes the well-known form

$$
\mathcal{H}(x,y,t,p_x,p_y,E) = (E - cp_x^2 - cp_y^2) / (2E) + \mathcal{E}(s)(1 + \mathcal{G}x) - eA_s(s)
$$

(2)

with neglect of terms of order $\mathcal{E}(c^2p_x^2/E)^2$ and $\mathcal{E}(c^2p_y^2/E)^2$. For the purposes of the next section we abbreviate by writing $\mathcal{E} = \mathcal{E}(x,y,t,p_x,p_y,E)$ and also

$$\mathcal{E}(s) = (E_0/\text{ec}) \mathcal{G}(s,x,y),
$$

(3)

for the magnetic field vector at any point $(s_0 = 0)$. Semi-classical Radiation Power

An individual photon emission in a bunding magnet takes place within an azimuthal angle $\Delta \phi = (mc^2/E_0)\Delta \phi$ in high energy machines. It is reasonable to regard it as instantaneous. We therefore write the instantaneous radiation power of a particle as a stochastic function of its coordinates through its energy and the local magnetic field

$$
\mathcal{P}(x,y) = \int du \, \mathcal{E}(x,y) \mathcal{G}(x,y) / (2E)
$$

(4)

where $u$ is the energy of a photon emitted at $s_i$ and $\mathcal{G}(s,y)$ is the density of the two-dimensional stochastic process $(s,y)$:

$$
\mathcal{G}(s,y) = \int \delta(s - s_j) \mathcal{G}(s,y) / (2E)
$$

(5)

with expectation value

$$
\langle \mathcal{G}(x,y) \rangle = \mathcal{N}(x,y) \langle \mathcal{G}(x,y) \rangle / c
$$

(6)

where

$$
\mathcal{N}(x,y) = (5/3) \left( e^2 / h \right) (E_0/mc^2)
$$

(7)

is the average rate of photon emission and is independent of the particle's energy $E$. The distribution of photon energies $\mathcal{N}(x,y)$ is well known but we shall only need its first two moments

$$\langle \mathcal{P}(x,y) \rangle = \mathcal{P}(x,y) \langle \mathcal{G}(x,y) \rangle / (2E)
$$

(8)

$$\langle \mathcal{G}^2(x,y) \rangle = \left(11/12\right) \langle \mathcal{G}(x,y) \rangle^2 / (2E)
$$

(9)

The expectation value of the power is then

$$\mathcal{P}(x,y) = \mathcal{P}(x,y) \langle \mathcal{G}(x,y) \rangle / (2E)
$$

(10)
and the two-point correlation function of the fluctuating part

\[ \hat{P}(x,s) = P(x,s) - \langle \hat{P}(x,s) \rangle \]

is

\[ \langle \hat{P}(x,s) \hat{P}(y,s') \rangle = cN(x,s)c^2 \delta(x-s) \]

\[ = \frac{55}{24/3} e^{-\frac{E_0}{kT}} k \left| x, y, \lambda \right|^2 \delta(x-s) \]

(12)

This leads us to adopt the formal representation

\[ P(x,s) = E^2 [c_k^2 \delta(s-s')] \]

\[ \left( \frac{1}{2} I_{x,y} \right) \]

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where

\[ c_k = \frac{(2/3) e^{-\frac{E_0}{kT}} k^2}{c_k} \]

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and \( s \) is a centered Markov process with the properties

\[ \langle s \rangle = 0 \]

\[ \langle s(s') \rangle = \delta(s-s') \]

(15)

The correction to the classical formula is of order \( h^{1/2} \). Further quantum corrections start with a (deterministic) term of order \( h \).

Equations of Motion

With the canonical coordinates \( X \), the equations of motion have a particularly simple form (primes denote derivatives w. r. t. \( s \))

\[ z' = \frac{\partial H}{\partial \dot{z}} \]

\[ p_z' = -\frac{\partial H}{\partial z} - P(x,s) \frac{\partial H}{\partial \dot{z}} \]

\[ z = x, y \]

(13)

\[ t' = \frac{\partial H}{\partial \dot{t}} \]

\[ -e' = \frac{\partial H}{\partial t} - P(x,s) \frac{\partial H}{\partial \dot{t}} \]

(16)

where its physical meaning is transparent: besides its canonical evolution the energy is damped by the deterministic part of the synchrotron radiation power but fluctuates in addition. The other coordinates fluctuate and are damped through the terms in the Hamiltonian which couple them to the energy. The small transverse components of the photon recoil are responsible for the damping of transverse oscillations.

From (13) and (16) follows a Fokker-Planck equation for the (averaged) phase-space distribution function \( F(X,s) \):

\[ F' = L(X,s)F(X,s) \]

(17)

which could, in principle, be used to "track" the phase space distribution around the storage ring: \( L(X,s) \) is a second order linear differential operator. Exact solutions are easily written down to describe the traversal of straight sections or point-like RF cavities. Magnets can be treated in thin lens approximation. Approximate versions of (17) (averaged over a revolution time) supply moment equations to describe the evolution of beam dimensions.

Although the effects of synchrotron radiation find their simplest expression in terms of the coordinates \( X \), it is convenient to make canonical transformations to other sets of coordinates. The way in which the conservative terms in the equations of motion transform is well-known. To transform the dissipative and fluctuating terms we make use of the following results.

Let a conservative system be described by canonical coordinates \( q_1, p_1, \ldots, q_n, p_n \) and Hamiltonian function \( H(q,p,s) \). A canonical transformation to coordinates \( Q_1, P_1 \) and Hamiltonian \( K(Q,P,s) \) is defined by a generating function \( F_0(Q,P,s) \). If an associated dissipative system is described by the (possibly stochastic) differential equations

\[ q' = \frac{\partial H}{\partial p} + a_i(q,p,s) \]

\[ p' = -\frac{\partial H}{\partial q} + b_i(q,p,s) \]

or, equivalently, by

\[ q' = \frac{\partial K}{\partial p} + A_i(Q,P,s) \]

\[ p' = -\frac{\partial K}{\partial q} + B_i(Q,P,s) \]

then the \( A_i \) and \( B_i \) are linearly related to \( a_i \) and \( b_i \) by the matrix equations

\[ A = M \cdot A + N \cdot \hat{M}^{-1}(\hat{p} - L) \]

\[ B = M^{-1}(B - L) \]

(20)

where \( MT \) is the transpose of \( M \) and

\[ L_{ij} = \frac{\partial^2 K}{\partial q_i \partial p_j}, \]

\[ M_{ij} = \frac{\partial^2 K}{\partial q_i \partial q_j}, \]

\[ N_{ij} = \frac{\partial^2 K}{\partial p_i \partial p_j} \]

(21)

The existence of the canonical transformation guarantees that \( M \) is invertible. Similar rules apply for the other kinds of generating function.

To illustrate the procedure we study the radial betatron motion. We neglect the vertical coordinate \( y \) and split off the part of \( x \) due to synchrotron motion. The generating function for this canonical transformation to variables \( x_0, T, p_x, \dot{p}_x \) is

\[ F_0 = \frac{\partial K}{\partial x_0} + \epsilon_n \dot{x}/c + (E_0 - \epsilon_n)^2 \]

so that

\[ x = x_0 + \epsilon_n \dot{x}/c, \]

\[ E = E_0 + \epsilon_n \dot{x}/c \]

(23)

Choosing the dispersion \( \epsilon_n \) in the usual way eliminates terms linear in \( x_0 \) and \( \dot{x}_0 \) from the new Hamiltonian. It is convenient to rescale phase space and use \( H_2 = H_2(x_0, \dot{x}_0) \) expressed in terms of canonical variables \( (x_0, T, p_x, \dot{p}_x, 0, 0) \) to assume that \( \epsilon_n = \epsilon_n = 0 \) at the cavities. We keep only quadratic terms in \( H_2 \) in order to exhibit the effects of non-linear dissipative terms on the linear motion.

Evaluation of the matrices (21) is straightforward and leads to equations of motion

\[ x_0' = \frac{\partial H_2}{\partial p_x} + \epsilon_n \frac{\partial H_2}{\partial x_0} \]

\[ p_x' = -\frac{\partial H_2}{\partial x_0} + \epsilon_n \frac{\partial H_2}{\partial p_x} \]

(24)

\[ z' = \frac{\partial H_2}{\partial \dot{z}} + \epsilon_n \frac{\partial H_2}{\partial z} \]

\[ -e' = \frac{\partial H_2}{\partial \dot{e}} + \epsilon_n \frac{\partial H_2}{\partial e} \]

where \( \Pi_1 \) and \( \Pi_2 \) are defined in terms of (2) and (13) as

\[ \Pi_1 = \frac{\partial p_x}{\partial x_0} + \epsilon_n \frac{\partial p_x}{\partial x_0} \]

\[ \Pi_2 = \frac{\partial p_x}{\partial \dot{z}} + \epsilon_n \frac{\partial p_x}{\partial \dot{z}} \]

(25)

[At each step we shall understand \( \Pi_1, \Pi_2 \) etc. to be expressed in terms of the current set of canonical variables.] Because we work with transverse momenta rather than slopes, the cavities appear to play no role in damping the betatron motion. Otherwise the physical interpretation of (24) is familiar.

From this point on we consider only the radial betatron motion and transform to action-angle variables (3, 6) via the generating function

\[ F_1(q, x_0) = -\left( \frac{\partial K}{\partial x_0} / 2 \right) + g(q, x_0) \]

\[ g(q, x_0) = q + \epsilon_n \frac{\partial g}{\partial x_0} \]

(26)

Then
\( x_{\phi} = (2\mathcal{R})^{1/2} \cos \phi \), \( \nu_{\phi} = (2\mathcal{R})^{1/2} (\alpha \cos \phi + \sin \phi) \) \( (27) \)

and, provided the lattice functions \( a(s) \) and \( b(s) \) are chosen as usual, the Hamiltonian reduces to \( Q/2 \).

Following the rule for transformation of the dissipative terms \( \psi \) is a simple matter and leads to the equations of motion

\[
\dot{\psi} = Q/2 - (2\mathcal{R})^{1/2} \left( \left[ b - a \right] \cos \phi - \sin \phi \right) \Pi_1 + \frac{1}{2} \left( 3/2 \right) \cos \phi \Pi_2,
\]

\( (28) \)

\[
J = (2\mathcal{R}) \left( \left[ b - a \right] \sin \phi + \alpha \cos \phi \right) \Pi_1 + \left( 2\mathcal{R} \right) \sin \phi \Pi_2.
\]

\( (29) \)

In the absence of low-order resonances, the distribution in phase \( \psi \) rapidly becomes uniform. If the betatron motion is stable any island structure due to a low-order non-linear resonance at some amplitude will be reflected in the form of the distribution function. Although it is somewhat unrealistic to keep non-linear dissipative terms when the conservative ones have been dropped, let us assume that the dependence of the tune on \( J \) is such that no resonant effects need be taken into account. Then the averaging method may be applied immediately to \( (28) \) and \( (29) \) to obtain a one-dimensional Fokker-Planck equation for \( F(J,s) \), the distribution in \( J \). This, and similar calculations in the context of the present formalism, are greatly simplified for separated-function lattices in which products of different gradient functions \( G(s), k(s), \ldots \) vanish. The algebra involved readily lends itself to computer execution.

Separating \( (28) \) and \( (29) \) into mean and fluctuating parts,

\[
\psi' = D(J,s)g(J,s)s\{1+\epsilon(J,s)\}\{s\},
\]

\( (30) \)

we must evaluate

\[
D(J) = (2\mathcal{R})^{-1} \int ds \int ds' \frac{\partial}{\partial \psi} \frac{\partial}{\partial \psi'} \frac{F(J,s)}{F(J,s')},
\]

\( (31) \)

\[
Q(J) = (2\mathcal{R})^{-1} \int ds \int ds' \frac{\partial}{\partial \psi} \frac{\partial}{\partial \psi'} \frac{F(J,s)}{F(J,s')},
\]

\( (32) \)

where

\[
S(J) = (2\mathcal{R})^{-1} \int ds \int ds' \frac{\partial}{\partial \psi} \frac{\partial}{\partial \psi'} \frac{F(J,s)}{F(J,s')},
\]

\( (33) \)

and \( I_{1s}, I_{2s}, I_{1s} \) and \( H(s) \) have their familiar definitions; \( \alpha_x \) and \( \epsilon_x \) are the linear damping rate and emittance, and \( \kappa_x \) is the equivalent of the storage ring constants.

[ QE: Fokker-Planck equation is

\[
\frac{df}{ds} = -\frac{3}{8} \left( D + \frac{180}{24} + \frac{3}{2} S \right) F + \frac{1}{2 \sigma_x^2} \left( QF \right)
\]

\( (38) \)

and has the equilibrium solution

\[
F_0(J) = \frac{1}{1 + \left( 2\mathcal{R} \right)^{1/2}} \left( 1 + 2 \frac{3}{2} \int ds \int ds' \frac{F(J,s)}{F(J,s')} \right)
\]

\( (36) \)

where \( Q \) is a suitable normalisation constant. It is messy, but not difficult, to do the integral in \( (36) \) and a careful analysis of the solution is necessary for a practical calculation of the quantum lifetime. Setting \( k = 0 \) in \( (36) \), we recover \( (2/\mathcal{R}) \exp (-2\mathcal{R} \kappa_x) \), corresponding to a Gaussian distribution of \( x_0 \) with standard deviation \( \alpha_x = (e \epsilon_x)^{1/2} \). For very small or large amplitudes, the distribution has quite different asymptotic forms

\[
F_0(J) \sim \frac{1}{Z} \exp \left[ -2\mathcal{R} \kappa_x + (2\mathcal{R}) \lambda - 2 \sigma_x^2 / \epsilon_x \right] \] (J=0)

\( (37) \)

Evaluating the synchrotron integrals for a LEP lattice with 90° phase advance per cell shows that the damping due to quadrupoles becomes equal to that due to dipoles at a value \( (2\mathcal{R})^2 \kappa_x \approx \epsilon_x^2 / 4 \approx 9 \times 10^{-3} m^2 / 2 \). The diffusion enhancement due to the quadrupoles doubles the linear diffusion rate at \( (2\mathcal{R})^2 \kappa_x \approx \epsilon_x^2 / 4 \approx 9 \times 10^{-3} m^2 / 2 \). Typically, the latter value is inside the part of phase space which it is intended to exploit (the dynamic aperture). Since \( \kappa_x \) is the only adjustable parameter in \( (36) \) it follows that effects of this kind must be considered a performance limitation for the largest \( \epsilon_x \) storage rings.

Conclusion

By means of a one-dimensional, but non-trivial, example we have illustrated the utility of the picture of electron motion embodied in \( (13) \) and \( (16) \). The tails of the radial betatron distribution have been shown to be substantially non-Gaussian because of the enhancement of quantum diffusion by the strong lattice quadrupoles in LEP. Numerous problems of beam dynamics (see, e.g. Ref. 2) whose correct formulation has proved elusive can be treated systematically in this way.

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References