Quantum Yang-Mills Theory
On Arbitrary Surfaces

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Abstract

We study quantum Yang-Mills theory on two-dimensional surfaces. Using path integral methods we derive general and explicit expressions for the partition function and expectation values of homologically trivial and non-trivial Wilson loops on closed surfaces of any genus, as well as for the kernels on manifolds with handles and boundaries.

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1 Introduction and survey of results

The purpose of this paper is to present general and explicit formulae for the partition function of Yang-Mills theory on arbitrary (orientable) two-dimensional surfaces as well as for the expectation values of homologically trivial and non-trivial Wilson loops. Our derivation is based on the ability to compute explicitly the path integral on the disc (and more generally on spheres with boundaries) and has been explained in [1] in the much simpler case of Maxwell theory. The kernels and partition functions on surfaces with any number of handles and boundaries will then be derived from the rules for glueing manifolds and joining boundaries appropriate for two-dimensional Yang-Mills theory.

Previous studies of quantum Yang-Mills theory on topologically non-trivial surfaces have been performed by Rajeev [2] for the cylinder and by Fine [3] for the two-sphere, and in these particular cases our general formulae reproduce the known results.

In order to state our results we have to introduce some notation. \( G \) is a compact connected and simply-connected Lie group and we fix an invariant positive definite scalar product (trace) on the Lie algebra of \( G \) as well as the corresponding normalized Haar measure \( dg, \int_G dg = 1 \). \( \hat{G} \) is the discrete set of equivalence classes of irreducible unitary representations of \( G \), and for \( \lambda \in \hat{G} \) we denote by \( d(\lambda) \) the dimension of the representation \( \lambda \), by \( \chi_\lambda \) the corresponding character (normalized by \( \chi_\lambda(1) = d(\lambda) \)), and by \( c(\lambda) \) the quadratic Casimir invariant of \( \lambda \). For the properties of characters used in this paper see e.g. [4]. \( \Sigma_{g,n} \) is an oriented two-dimensional surface of genus \( g \) with \( n \) boundary components (\( \Sigma_{g,0} \equiv \Sigma_g \)). The action of Yang-Mills theory on a surface \( \Sigma \) is

\[
S = \frac{1}{2e^2} \int_{\Sigma} F_A \ast F_A
\]  

(1)

where \( F_A = dA + \frac{1}{2}[A,A] \) is the curvature of a connection on a (necessarily trivial) \( G \)-bundle over \( \Sigma \), \( \ast \) is the Hodge duality operator with respect to some metric on \( \Sigma \), \( e \) is the coupling constant, and a trace and wedge product are understood in (1). Finally, for \( X \subset \Sigma \) we denote by \( A(X) := \int_X \ast 1 \) the area of \( X \).
With this notation we have the result that the partition function of $G$-Yang-Mills theory on $\Sigma_g$ is

$$Z(\Sigma_g) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} e^{-\epsilon^2 \epsilon(\lambda) A(\Sigma)/2}.$$  \hfill (2)

For a contractible loop $\gamma$ on $\Sigma_g$ the expectation value of the trace of the Wilson loop $P e^{i A}$ in the representation $\mu \in \hat{G}$ is

$$\langle \chi_\mu(P e^{i A}) \rangle_{\Sigma_g} = \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda)^{1-2g} d(\rho) e^{-\epsilon^2 \epsilon(\lambda) A(D) + \epsilon(\rho) A(D')} / 2``` (3)

where $D$ is the disc enclosed by $\gamma$, $\partial D = \gamma$, and $D'$ is its complement in $\Sigma_g$. The formulae for homotopically non-trivial but homologically trivial and homologically non-trivial loops are slightly more complicated and will be given in section 4. Equations (2) and (3) are immediate consequences of a general formula for the kernel $K(\Sigma_{g,n})$ on a surface $\Sigma_{g,n}$ with boundary. $K(\Sigma_{g,n})$ is a functional of the boundary conditions imposed at the $n$ boundary circles of $\Sigma_{g,n}$. As the only gauge invariant degree of freedom of a gauge field $A$ on a circle $S$ is the conjugacy class of its holonomy $P e^{i A} \in G$, the kernel can be considered as a function $K(\Sigma_{g,n})(g_1, \ldots, g_n)$ on $G \times \ldots \times G$ invariant under conjugation in each entry. Explicitly $K(\Sigma_{g,n})$ is

$$K(\Sigma_{g,n})(g_1, \ldots, g_n, A(\Sigma_{g,n})) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g-n} \chi_\lambda(g_1) \ldots \chi_\lambda(g_n) e^{-\epsilon^2 \epsilon(\lambda) A(\Sigma_{g,n})/2}$$  \hfill (4)

Here we have indicated explicitly the metric (area) dependence of $K(\Sigma_{g,n})$ in our notation. For reasons explained in [1] two-dimensional Yang-Mills theory is almost a topological field theory in the sense that the partition function and correlation functions of metric independent operators (Wilson loops) will depend on the metric only through terms of the form $e^2 A(\Sigma)$, $e^2 A(D)$ (in particular no derivatives of the metric appear).

In section 2 we will calculate explicitly the kernel on the disc $D = \Sigma_{0,1}$ from the path integral. This requires a number of technicalities to be handled and our computation will be based on a spectral representation for the delta function imposing the boundary conditions in the path integral, use of the Schwinger-Fock gauge to have the Nicolai map of [5, 1] at our disposal, and
a fermionic path integral representation for the trace of a Wilson loop. In section 3 we generalize that calculation to the cylinder. At that point it becomes clear that in principle the calculation can be extended further to spheres with three and more boundaries, but as things then become a little murky we proceed in a different and much simpler way. We rederive the result for the cylinder directly from that for the disc, basically by deforming the disc to a rectangle and identifying two opposite sides. This procedure, which is possible due to the almost topological nature of two-dimensional Yang-Mills theory, is the prototype of the operations which then allow us to determine immediately the kernels on surfaces with an arbitrary number of handles and boundaries. In section 4 we derive general formulae for the expectation values of Wilson loops and give some examples. We conclude with additional remarks on possible generalizations and open problems, some of which will be dealt with in [6] to which we also refer for some of the details we are not able to go into in this letter.

2 The wave function on the disc

Let $\gamma = \partial D$ be the boundary of a disc $D$. Choosing the boundary condition to be $Pe^{\int_{\gamma} A} g_1 \in G$ (modulo conjugation, i.e. gauge transformations of $A$) our task is to compute the path integral

$$K(D)(g_1, A(D)) = \int_{\mathcal{A}} e^{-S_s} \delta(Pe^{\int_{\gamma} A} g_1)$$

(5)

where $S_s$ is the BRST invariant quantum action on $D$ (for some gauge condition $G(A) = 0$) and $\int_{\mathcal{A}}$ symbolically denotes the path integral over all $A$. The precise nature of the delta
statistical mechanics. These models are intimately related to the theory of two dimension factorizable $S$-matrices$^{[22]}$ and two-dimensional integrable QFT through the light-cone lattice approach$^{[23]}$. All known integrable QFT are obtained from integrable lattice models as appropriated scaling limits. The Hamiltonian and momentum as well as higher conserved charges follow from the light-cone or diagonal-to-diagonal transfer matrix. In fig. 12 the light-cone transfer matrix is depicted.

Light-cone transfer matrices follow as special cases of row-to-row inhomogeneous transfer matrices [eq. (5.11)]. Therefore the construction of eigenvectors and eigenvalues of row-to-row transfer matrices is the basic step to solve vertex models. QFT in the light-cone lattice approach and spin Hamiltonian generated by row-to-row transfer matrices. The Heisenberg model and its generalizations associated to higher spin representations and to Lie groups other than SU(2) follow from integrable row-to-row transfer matrices through their logarithmic derivatives at zero spectral parameter.

In section III the eigenvectors and eigenvalues of the six-vertex model are constructed in the algebraic Bethe ansatz approach$^{[24]}$. The massive Thirring model and the XXZ Hamiltonian solutions follow from this vertex model with two states per link. It must be stressed that the Bethe Ansatz and its generalizations are not merely Ansatz but they provide the full set of exact eigenvectors and eigenvalues of the transfer matrix.

Section IV contains an exposition of the nested Bethe Ansatz method that provides the solution of vertex models with more than two possible states per link. The $U(1)^q \otimes Z_q$ symmetric $q^{(2q-1)}$ model of ref (25) is solved in some detail. Solutions of other multistate vertex models can be found in refs (26)-(28).

Methods for finite size calculations in integrable theories are reported in section V (29-31). Both regimes, with zero and non-zero gap are considered. When the gap is non-zero physical magnitudes get exponentially small connection in the linear size $N$ of the system. When the gap vanishes these connections turn to be power-like in $N$. It is shown that the long-range regime of gapless lattice models obey conformal laws and provide concrete realizations of conformal field theories. The calculation of the central charge and the conformal dimension is exposed in some detail$^{[30-31]}$.

The YBZF algebras are classified according to their dependence on the spectral parameter $\theta$ in rational, trigonometric/hyperbolic and elliptic types. In table I the connection of this dependence with the symmetry enjoyed by the YBZF algebra is given. Moreover to each YBZF algebra, a simple Lie algebra is associated. In the rational cases it is simply the symmetry algebra. In the trigonometric/hyperbolic cases it is a deformation of the Lie algebra called quantum group that underlies these YBZF algebras (see sec. VII). The same Lie algebra appears in the corresponding BA equations (see sec. V). As it will be discussed in the next section, there are many representations for the generators of a YBZF algebra. Each inequivalent representation defines a class of physical...
models. To start, we have a lattice vertex model. From it, quantum magnetic chains follow (see the Appendix). Moreover trigonometric and rational lattice models (since they are gapless) yield massive QFT and conformal invariant models in appropriate scaling limits (secs. V and VI).

Rational YBZF generators only depend on $\theta$ besides the Lie algebra $G$ and its representation. Trigonometric/hyperbolic YBZF generators depend in addition upon a continuous parameter $\gamma$ (the anisotropy). In elliptic solutions an elliptic modulus $k$ appears too.

Integrable lattice models can be also formulated on faces instead of vertices (IRF or SOS models). They are connected with the vertex language through the intertwining vectors [66,59]. In this SOS language a series of new models follow when $\pi/\gamma$ is a rational number as restrictions of the preceding ones. These models are called RSOS [58] and have a precise characterization in the BA solution of the initial vertex model [59].

In section V the light-cone lattice approach to integrable QFT on the lattice is reported. Section VII contains a survey on quantum groups and the connection of YBZF algebras with Kac-Moody algebras.

II. VERTEX MODELS AND YBZF ALGEBRAS.

Two dimensional vertex models appear in different physical contexts like ferroelectric systems, spin models, crystal models (like the ice crystal and KDP) etc. [2,32]. Some purely mathematical colouring problems can also be related to vertex models[33].

Let us consider a two-dimensional square lattice (fig. 1) where each link can take different states. Horizontal and vertical links can be in local states belonging to the vector spaces $V$ and $V'$ respectively. More general lattices formed by a set of straight lines intersecting at arbitrary angles can also be considered [34].

We associate a statistical weight $w$ to each vertex configuration of the lattice. The configuration is defined by the states of the four links joining together at the vertex (fig. 2).

\[ w = w(\alpha \beta | \alpha \beta) \quad (2.1) \]

\[ v = \dim V', \quad h = \dim V \]

Now, the statistical weight for a given configuration of the whole lattice reads

\[ \prod_{\text{all vertices}} w(\alpha \rho | \alpha \beta) \quad (2.2) \]

Then, the partition function of the system writes

\[ Z = \sum_{\text{all configurations}} \prod_{\text{all links}} w(\alpha \rho | \alpha \beta) \quad (2.3) \]
where one sums over all possible configurations compatible with the given weights (2.1). The free energy is defined as usual in the thermodynamic limit as

$$ f = - \lim_{N \to \infty} \frac{1}{N} \log Z $$

(2.4)

Let us now introduce the monodromy operator $T_{ab}(\theta)$. It is associated to a line of the lattice, say a horizontal line. It reads [see fig. 3]

$$ T_{ab}^{(\nu)}(\theta) = \sum_{a_1, \ldots, a_{\nu} = 1} T_{a_{a_1}}^{(1)}(\theta) \otimes T_{a_{a_2}}^{(2)}(\theta) \otimes \cdots \otimes T_{a_{a_{\nu}}}^{(\nu)}(\theta) $$

(2.5)

Here, $[\kappa_{ab}(\theta)]_{a,b} = \omega(\alpha \beta / \alpha b)$. The $t_{ab}^{(k)}$ acts in the $\nu$ dimensional vertical space $\gamma$ associated to the $k$th column of the lattice. So, $T_{ab}(\theta)$ is an operator in the tensor product space

$$ \bigotimes_{i=1}^{N} \gamma^{(i)} = \gamma_1 \otimes \cdots \otimes \gamma_N $$

(2.6)

The variable $\theta$ can be considered as a coupling constant also depending on the temperature. The weights $\omega(\alpha \beta / \alpha b)$ may depend in other variables besides the spectral parameter $\theta$ that will play a prominent role.

Let us define the row-to-row transfer matrix as the trace of the monodromy matrix over the horizontal indices

$$ \tau^{(\nu)}(\theta) = \text{Tr}_{\nu} \left[ T_{ab}^{(\nu)}(\theta) \right] = \sum_{a=1}^{N} T_{a}^{(\nu)}(\theta) $$. 

(2.7)

Assuming periodic boundary conditions in horizontal and vertical directions $Z$ can be written in terms of $\tau^{[N]}(\theta)$ as

$$ Z = \text{Tr}_{\nu} \left[ \tau^{(\nu)}(\theta)^{M} \right] $$

(2.8)

Where $\text{Tr}_{\nu}$ means trace over the space (2.6). The preceding discussion holds irrespective of the integrability of the model.

Eq. (2.8) shows how important is the knowledge of the eigenvalue of $\tau^{[N]}(\theta)$. Actually, just the largest eigenvalue $\lambda^{(M)}_{\text{max}}(\theta)$ gives the free energy in the thermodynamic limit.

$$ f = - \lim_{N \to \infty} \frac{1}{N} \log \lambda^{(\nu)}_{\text{max}}(\theta) $$

(2.6)

Our discussion up to now holds irrespective of the integrability of the model. Let us consider now the sufficient conditions for integrability, namely the Yang-Baxter equations.

Let us assume that there exists a non singular matrix $R_{ab}(\theta, \theta')$
such that

\[ R(\theta, \theta') \left[ \tau(e) \otimes \tau(e') \right] = \left[ \tau(e) \otimes \tau(e') \right] R(\theta, \theta') \]

(2.9)

Here, \( \otimes \) means tensor product of matrices acting on \( h \)-dimensional horizontal spaces \( \mathcal{A} \). \( R \) acts on \( \mathcal{A} \otimes \mathcal{A} \). More explicitly eq. (2.9) reads

\[ R_{\alpha \beta}^{\epsilon \delta}(\theta, \theta') \left[ t_{a \epsilon}(\theta) \right]_{\alpha \gamma} \left[ t_{\beta \delta}(\theta') \right]_{\gamma \delta} = \left[ t_{a \epsilon}(\theta) \right]_{\alpha \gamma} \left[ t_{\beta \delta}(\theta') \right]_{\gamma \delta} R_{\alpha \beta}^{\epsilon \delta}(\theta, \theta') \]

(2.10)

In both eqs. (2.9) and (2.10) there is a matrix product on the vertical spaces \( \mathcal{V} \). We can associate a new vertex configuration to \( R_{ab}^{cd}(\theta) \) as depicted in fig. 4. All bonds joining in this type of vertex take values in the space \( \mathcal{A} \). Notice the correspondence between the angles into the intersecting lines and the vertex argument. Now, eqs. (2.9)-(2.10) can be naturally interpreted as in fig. 5. The trilinear relation (2.9)-(2.10) tell us that one can shift a lattice line of type \( \mathcal{V} \) through a vertex of the type of fig. 4 provided the angles between the lines \( \theta, \theta' \) and \( \theta, \theta' \) are kept constant.

A relation analogous to eq. (2.9) holds for \( T_{ab}^{[N]}(\theta) \). It follows from the definition (2.5) of \( T_{ab}(\theta) \) and eq. (2.9) together with the tensor product properties that

\[ R(\theta, \theta') \left[ T(e) \otimes T(e') \right] = \left[ T(e) \otimes T(e') \right] R(\theta, \theta') \]

(2.11)

We have obtained here a global equation valid for \( T^{[N]}(\theta) \) from the local equations (2.9) valid for each vertex. Eq. (2.11) holds in \( \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{V} \).

Eq. (2.11) has also a natural graphical interpretation [fig. 6]. Left-multiplying eq. (2.11) by \( R(\theta, \theta')^{-1} \) and taking trace on \( \mathcal{A} \otimes \mathcal{A} \) yields

\[ \text{tr} \left[ T^{[N]}(\theta) \right] \text{tr} \left[ T^{[N]}(\theta') \right] = \text{tr} \left[ T^{[N]}(\theta) \right] \text{tr} \left[ T^{[N]}(\theta') \right] \]

(2.12)

since \( \text{tr} \left( A \otimes B \right) = \text{tr} \ A \cdot \text{tr} \ B \). We have then a one-parameter family of commuting transfer matrices.

Let us now discuss the properties of the \( R \)-matrix \( R_{ab}^{cd}(\theta, \theta') \) that plays a crucial role in the study of integrable theories. \( R \) defines the basic algebraic structure of the theory through eq. (2.11). This is sometimes called a Yang-Baxter-Zamolodchikov-Faddeev algebra (YBZF).

Let us consider the consistency of the YBZF algebra (2.11) for products of three operators \( T \). The product
\begin{equation}
T(\theta_1) \otimes T(\theta_2) \otimes T(\theta_3)
\end{equation}

(2.13)

can be reordered with the help of the YBZF algebra (2.11). However, we have two inequivalent paths to relate (2.13) to

\begin{equation}
T(\theta_2) \otimes T(\theta_3) \otimes T(\theta_1)
\end{equation}

(2.14)

That is

\begin{align*}
(123) & \rightarrow (321) & (312) & \rightarrow (321) \\
(132) & \rightarrow (123) & (312) & \rightarrow (321)
\end{align*}

(2.15)

Both must lead to the same result (2.14). One finds in this way the condition

\begin{equation}
S_{42}^{-1} S_{43}^{-1} S_{23}^{-1} \left[ T(\theta_1) \otimes T(\theta_2) \otimes T(\theta_3) \right] S_{23} S_{42} S_{43} = \frac{1}{2} \sum_{i=1}^{3} \left[ T(\theta_1) \otimes T(\theta_2) \otimes T(\theta_3) \right] S_{42} S_{43} S_{23}
\end{equation}

(2.16)

where the matrices \(R_{ij}\) \((i,j = 1,2,3, i \neq j)\) act in the space \(\mathbb{A}_1 \otimes \mathbb{A}_2 \otimes \mathbb{A}_3 \). \(S_{ij}\)
eq PR(\theta_1 - \theta_j) in the space \(\mathbb{A}_i \otimes \mathbb{A}_j\) and it is the unit matrix in the space \(\mathbb{A}_k\) \(j \neq k \neq i\). That is \(S_{ij}(\theta) = R_{ij}(\theta)\delta_{ij}\). We denote by \(P\) the matrix

\begin{equation}
P_{a}^{c} \delta_{b}^{d} = \delta_{a}^{d} \delta_{b}^{c}
\end{equation}

(2.17)

It follows from eq. (2.16) that

\begin{equation}
\left[ S_{23} S_{42} S_{43}^{-1} S_{13}^{-1} S_{12}^{-1} T(\theta_1) \otimes T(\theta_2) \otimes T(\theta_3) \right] = 0
\end{equation}

(2.18)

The left entry of this commutator is a scalar in \(\mathcal{V}\) whereas the right entry is a complicated operator in this space. Since eq. (2.18) holds in the full space

\[ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{V} \]

the more plausible solution is that the left entry is proportional to the unit matrix in \(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}\). We can then write

\begin{equation}
S_{42} S_{43} S_{23} = S_{23} S_{41} S_{43}
\end{equation}

(2.19)

since the determinant of the left entry of (2.18) equals one. This is a sufficient condition for the consistency (associativity) of the YBZF algebra (2.11). Eq. (2.19) is the so-called Yang-Baxter equation (YBE). More explicitly, it easily follows from (2.19) that

\begin{equation}
R_{a \theta_1 \theta_2}^{b \theta_3 \theta_4} R_{e \theta_3 \theta_4}^{b \theta_2 \theta_3} R_{e \theta_2 \theta_3}^{b \theta_1 \theta_2} (\theta_1 - \theta_2)(\theta_2 - \theta_3) = 1
\end{equation}
\begin{equation}
R_{a s a t z} \eta_{s t} \cdot R_{m a t s} \eta_{a t} \cdot R_{m a t r} \eta_{a r} \tag{2.20}
\end{equation}

Eqs. (2.19) or (2.20) guarantee that the algebra of three $T(\Theta)$ fulfills the associativity property. This graphical representation makes easy to show that eqs. (2.19)-(2.20) implies associativity for the multiplication of any number of $T(\Theta)$. It is a question of pushing lines repeatedly through vertices.

It must be stressed that the YBE (2.20) are a heavily overdetermined set of algebraic equations. They contain a priori $h^4$ unknowns, that is the elements of $R(\Theta)$ and $h^6$ equations. Despite of this fact a rich set of solutions of the YBE equations is known. All of them possess symmetries that reduce the number of independent equations such as discrete $Z_h$ symmetry or continuous symmetries. The elements of the solutions $R(\Theta)$ may be written in terms of elliptic functions or degenerate forms (trigonometric, hyperbolic or rational functions). Moreover, there exist a parametrization such that $R(\Theta, \Theta')$ depends only on the difference $\Theta - \Theta'$. The star-triangle solutions found in ref. [35] involve algebraic curves of genus larger than one and cannot be parametrized as functions of $\Theta - \Theta'$.

The YB eqs. (2.13) admit the natural graphical representation given in fig. 7. As for eqs. (2.9) [fig. 5] they express the freedom to shift lines of $\mathcal{A}$ type through vertices of the type of fig. 4. When $\theta = \Theta'$, eq. (2.11) suggest naturally that $R(\Theta)$ is a multiple of the unit matrix in $\mathcal{A} \otimes \mathcal{A}$. More precisely, a solution of the YBE is called regular if

\begin{equation}
R(\Theta) = c \cdot \mathbb{1} \tag{2.21}
\end{equation}

where $c$ is a numerical constant and $\mathbb{1}$ the unit matrix in $\mathcal{A} \otimes \mathcal{A}$.

Setting $\Theta_1 = \Theta_3$ in eq. (2.20) gives with the help of eq. (2.21)

\begin{equation}
M_{a t a v} \cdot \delta_{a t} \cdot \delta_{a v} = \delta_{a t} \cdot M_{a t a v} \cdot \delta_{a v} \tag{2.22}
\end{equation}

where $\Theta = \Theta_3 - \Theta_2$ and

\begin{equation}
M_{a t a v} \cdot \delta_{a t} \cdot \delta_{a v} = \mathbb{1} \cdot M_{a t a v} \cdot \delta_{a t} \cdot \delta_{a v} \tag{2.22}
\end{equation}

Then, we see that $M_{a t a v}$ must have the index structure $M_{a t a v} = \delta_c \cdot \delta_{a t} \cdot \delta_{a v} f(a)$. This can be written as

\begin{equation}
R(\Theta) R(-\Theta) = f(\Theta) \cdot \mathbb{1} \tag{2.22}
\end{equation}

where $\rho(\Theta)$ is a c-number function. It follows from eq. (2.21) that

$\rho(-\Theta) = \rho(\Theta)$. Eq. (2.22) has the graphical interpretation given in fig. 8.

The YBE (2.20) are invariant under the exchange $R \rightarrow R^T$ [that is $R_{cd} \rightarrow R_{bd}$] and $\theta_2 \rightarrow \theta_1 - \theta_2, \theta_1 \rightarrow \theta_1$. So, if $R$ fulfills the YBE, $R^T$ also will fulfill them. Usually one finds
\[ R(\theta) = R(\theta)^T \] (2.23)

One can consider the \( T(\theta) \) as "generators" of the YBZF algebra whereas \( R(\theta-\theta') \) plays the role of "structure constants". One can have many different operators \( T(\theta) \) acting in different vertical spaces for a given \( R \)-matrix. In addition, the family of commuting transfer matrices \( \tau(\theta) \) [eq. (2.7)] plays the role of "Cartan subalgebra". Any \( R \)-matrix solution of the YBE provides by itself a representation of the YBZF algebra (2.9)-(2.11). Setting

\[ \left[ t_{ab}^A(\theta) \right]_{cd} = R_{c a}^{b d}(\theta) \] (2.24)

or

\[ \left[ \bar{t}_{ab}^A(\theta) \right]_{cd} = R_{c b}^{a d}(-\theta) \] (2.25)

eq. (2.9) follows from eq. (2.19). So \( t^A \) and \( \bar{t}^A \) can be considered as "adjoint" representations of the YBZF algebra. In the same way as for Lie algebras, higher dimensional representation (higher dimensional \( \Psi' \)) can be obtained by tensor product and "fusion" of known \( T(\theta) \) and by direct methods \([1,36]\). Infinite dimensional representations appear in bosonic quantum field theories [see refs (1)-(3) and (23)].

As it can be checked from figs. 2 and 4 the identification (2.24) is consistent with the graphical correspondences. The generators \( T(\theta) \) are operators acting on two vector spaces \( \mathcal{A} \) and \( \mathcal{V}' \). This is an important feature of YBZF algebras: their generators act in a pair of vector spaces and not in a single one.

Let us now discuss the internal symmetries of the YBZF algebras\([37]\). They are associated to transformations in the space \( \mathcal{A} \)

\[ T(\theta) \rightarrow g T(\theta) \] (2.26)

that leave eq. (2.11) invariant. That is a group of matrices \( g \) with the property

\[ \left[ g \otimes g \ , \ R(\theta) \right] = 0 \ , \ \forall g \in \mathcal{G} \] (2.27)

Inspection of the known \( R \)-matrices shows the connection between the \( \theta \)-dependence of \( R(\theta) \) and the symmetry group \( \mathcal{G} \) shown in table I. A family of commuting transfer matrices \( \tau_g(\theta) = T_g(\mathcal{A} \otimes g T(\theta)) \) follows from eq. (2.11) and (2.26) for each \( g \)

\[ \left[ \tau_g(\theta) \ , \ \tau_g(\theta') \right] = 0 \] (2.28)

Without loss of generality, we normalize \( g \) such that \( \det g = 1 \). The usual transfer matrix \( \tau(\theta) \) follows in the particular case \( g = 1 \).
The transfer matrices \( \tau^{[N]}(\theta) \) correspond to a vertex model with periodic boundary conditions in the horizontal direction. The transfer matrices \( \tau^{[N]}(\theta) \) describe boundary conditions where the operators at site \((N+1)\) and at site 1 are related by the group transformation \( g \). Other boundary conditions like free ends are compatible withintegrability[51].

For the "adjoint" representation (2.24) of the YBZF algebra horizontal and vertical spaces coincide. One finds from eqs. (2.5), (2.24) and (2.27)

\[
T_{ab}(\theta) = \sum_{cd} (g^{-1})_{ac} \left[ g^{-1} T_{cd}(\theta) g \right] g_{db} \tag{2.29}
\]

where

\[
G = g \otimes g \otimes \ldots \ldots \otimes g
\]

\((N\ \text{factors})\)

Taking trace in the space \( \mathcal{A} \) yields

\[
\left[ \tau^{(\gamma)}(\theta), G \right] = 0 \tag{2.30}
\]

and

\[
\left[ \tau^{(\gamma)}(\theta), G \right] = 0 \tag{2.31}
\]

More generally[37]

\[
\left[ \tau^{(\gamma)}(\theta), G \right] = 0
\]

if \( c \) belong to the center of \( \mathcal{G} \).

The YBZF algebra admits a local invariance whenever the \( R \) matrix is a function of the difference \( \theta - \theta' \). If \( t(\theta) \) fulfills the YBZF algebra (2.9) so does \( t(\theta - \alpha) \) (with fixed \( \alpha \))

\[
R(\theta - \alpha) \left[ t(\theta - \alpha) \circ t(\theta - \alpha) \right] = \left[ t(\theta - \alpha) \circ t(\theta - \alpha) \right] R(\theta - \alpha) \tag{2.33}
\]

The invariances (2.26) and (2.33) of the YBZF algebras allow the write the more general YBZF generator as[55]

\[
\tau^{(\gamma)}(\theta, \xi, \overline{\xi}) = \sum_{\alpha_1, \ldots, \alpha_N} \left[ \gamma, t(\theta, \xi, \overline{\xi}) \right]_{\alpha_1, \ldots, \alpha_N, \alpha_{N+1}} \otimes \cdots \otimes \left[ \gamma, t(\theta, \xi, \overline{\xi}) \right]_{\alpha_{N+1}, \ldots, \alpha_N} \tag{2.34}
\]

where \( \gamma = (\alpha_1, \ldots, \alpha_N) \) and \( \gamma = (\theta_1, \ldots, \theta_N) \).

This is the reproduction property of YBZF algebras: whenever \( t(\theta) \) obeys the YBZF algebra (2.9) with some \( R \)-matrix \( R(\theta - \theta') \) enjoying a symmetry group \( \mathcal{G} \) (eq.(2.27)), then \( T_{ab}^{(N)} \) defined by (2.34) fulfills the same YBZF algebra

\[
R(\theta - \theta') \left[ \tau^{(\gamma)}(\theta, \xi, \overline{\xi}) \circ \tau^{(\gamma)}(\theta', \xi', \overline{\xi}') \right] = \left[ \tau^{(\gamma)}(\theta, \xi, \overline{\xi}) \circ \tau^{(\gamma)}(\theta', \xi', \overline{\xi}') \right] R(\theta - \theta') \tag{2.35}
\]

Commuting transfer matrices are generated as usual.
\[ Z^{(N)}(\Theta, \xi, \zeta) = \sum_{a,s} T_{ab}^{(N)}(\Theta, \xi, \zeta) \]

\[ [ Z^{(N)}(\Theta, \xi, \zeta), Z^{(N)}(\Theta', \xi', \zeta') ] = 0 \quad (2.36) \]

One constructs in this way inhomogeneous integrable vertex models. More generally, one can shift the spectral parameter of a vertex with coordinates \((s, r)\) in a two-dimensional lattice as \([37, 38]\)

\[ \Theta \longrightarrow \Theta - \alpha_s - \beta_r \]

keeping the YBZF algebra invariant and hence the commutativity of transfer matrices.

In eq. (2.34) we multiply YBZF generators \(g_{ab}(\Theta - \alpha_i)\) from left to right to build a new YBZF generator \(T_{ab}(\Theta)^{(N)}\). One can also multiply the generators from right to left obtaining a new generator\([55]\)

\[ T_{ab}(\Theta, \xi, \zeta) = \sum_{\alpha, s} \left[ g_{ab}(\Theta - \alpha_i) \right]_{\alpha, s} \left[ g_{rs}(\Theta - \alpha_i) \right]_{s, \zeta} \Theta \]

\[ \ldots \Theta \left[ g_{\alpha, \zeta}(\Theta - \alpha_i) \right]_{\alpha, \zeta} \ldots \quad (2.37) \]

also fulfilling the YBZF algebra (2.35).

The inhomogeneous YBZF generator (2.34) also admits a graphical representation depicted in fig. 9. That is vertical and horizontal lines form angles \(\Theta - \alpha_i\) that vary site to site \((1 \leq i \leq N)\).

The generators \(T_{ab}^{(N)}(\Theta, \xi, \zeta)\) and \(T_{ab}^{(N)}(\Theta)\) (eqs. (2.34) and (2.5), respectively) can be considered as local gauge transformed of each other. A shift \(\alpha_i\) of \(\Theta\) and the action of \(g_i \in \mathfrak{g}\) are symmetries of the YBZF algebras. Since both operators are related by symmetry transformations depending upon the site, we can call them local gauge transformations\([56]\).

The symmetry transformations \(g \in \mathfrak{g}\) defined by eq. (2.26)-(2.27) can flow from horizontal to vertical spaces. For YBZF generators in the adjoint representation we find from eqs. (2.24) and (2.27)

\[ g_{ac} t_{cd}(\Theta) (g^{-1})_{db} = g^{-1} t_{ac}(\Theta) g \quad (2.38) \]

where the matrices \(g^{-1}\) and \(g\) in the r.h.s. act on the vertical space. Eq. (2.38) generalizes for YBZF generators \(T_{ab}\) acting on a generic vertical space \(\mathcal{V}\) as

\[ g_{ac} T_{cd}(\Theta) (g^{-1})_{db} = G(g)^{-1} T_{ac}(\Theta) G(g) \quad (2.39) \]

where \(G(g)\) is the representation of \(g\) acting on the space \(\mathcal{V}\). Using eqs. (2.38)-(2.39) for \(\alpha_j = 0 \ (1 \leq j \leq N)\) we can relate the local gauge
transformed YBZF generator (2.34) with the non-transformed one of eq. (2.5) by a horizontal symmetry transformation (2.26) plus similarity transformations in the vertical spaces. We find

$$ T_{\alpha \beta} (\theta, \theta') = \prod_{i=1}^{N} T_{\alpha \beta} (\theta) \prod_{i=1}^{N} T_{\alpha \beta} (\theta') g_{\alpha \beta} $$

(2.40)

Here $h^{-1}$ and $h$ act on the $i$-th site vertical space with

$$ h_i = \prod_{j=1}^{i} g_{j} $$

$$ G = \prod_{i=1}^{N} g_{\alpha \beta} $$

(2.41)

Therefore local gauge transformations (2.2) are equivalent to similarity transformations in the quantum space (vertical) plus a twist $G \prod_{i=1}^{N} g_{\alpha \beta}$ in the boundary conditions.

For an infinitesimal transformation $g_{\alpha \beta} \rightarrow g_{\alpha \beta} + \varepsilon \partial_{\alpha \beta}$, $G = \lambda + \varepsilon \in S$,

$\varepsilon \ll \lambda$,

and eq. (2.39) reduces to

$$ \left( [ \lambda, T(\theta) ]_{\alpha \beta} \right)_{a} + \left[ \frac{\partial}{\partial \theta}_{a} T_{\alpha \beta} (\theta) \right]_{b} = 0 $$

(2.42)

where $[\lambda, T(\theta)]_{ab}$ stands for the commutator in the auxiliary space

$$ [\lambda, T(\theta)]_{ab} = \partial_{ac} T_{cb} - T_{ac} \delta_{cb} $$

and $[\theta, T_{\alpha \beta}]_{\gamma \nu}$ for the commutator in the quantum space

$$ \left[ S, T_{\alpha \beta} (\theta) \right]_{\gamma \nu} = S T_{\alpha \beta} - T_{\alpha \beta} S $$

In particular the diagonal generators (Cartan subalgebra) of $\mathfrak{g}$ commute with the operators $T_{\alpha \beta} (\theta)$ (no sum over $a$)

$$ \left[ S, T_{\alpha \beta} (\theta) \right] = 0 $$

This generalizes eq. (2.31)-(2.32).

We have considered up to now integrable vertex models which are classical systems in two-dimensional statistical mechanics. These models are naturally related to quantum theories in one-dimension. One can consider the space $\mathfrak{g}$ as a quantum space of states for each site of a given horizontal line. The matrix $\tau_{a} [N(\theta)]_{ab}$ will be now a quantum operator in the total space $V$ of quantum states. These matrices are in general very complicated quantum operators that couple all sites in the line. Simpler quantum operators follow by noticing that for $\theta = 0$

$$ \partial_{a} \left[ V^{N} (\theta) \right] = C^{N} \partial_{a} $$

(2.43)

where eqs. (2.5), (2.21) and (2.27) were used and $\partial$ is the unit shift operator on the horizontal direction. So, $\partial$ is an operator with a direct physical meaning. Moreover, it can be shown (see Appendix and ref. [39]) that the operators
couple \((K+1)\) nearest neighbours on the horizontal line. Usually \(C_1\) can be identified with a quantum hamiltonian. The commutativity property of the transfer matrices (2.12) implies that

\[
\left\{ C_K, C_L \right\} = 0, \quad \forall K, L
\]  

(2.45)

So, we have an infinite number of commuting and conserved magnitudes. One can then conclude that one has an integrable theory.

The simplest example of the correspondance (2.39) between 2-D vertex models and 1-D quantum hamiltonian is the one between the 8-vertex or the 6-vertex models and the XYZ or XXZ magnetic chains, respectively[24] (see (A.9) ).

Expansion around points other than \(\Theta = 0\) of more general transfer matrices \(\tau[\Theta]\) has been considered in ref.[37]. This leads to non-local integrable hamiltonians.

The equations (2.43)-(2.44) are no more true when inhomogeneous models are considered since translational invariance is lost. However, it is still possible to obtain non-local integrable hamiltonians with inhomogeneous couplings by expanding \(\tau[N][\Theta, (\alpha_\delta)]\) around the points where \(R(\Theta)\) is singular\(^{[37]}\).

We shall now consider the theory of two-dimensional relativistic S-matrices. This may appear at first sight completely unrelated to vertex models but we shall see that this is not the case.

In two space-time dimensions it is very convenient to parametrize the energy and momentum of a particle of mass \(m\) in terms of the rapidity \(\Theta\)

\[
E = m \cosh \Theta
\]

\[
P = m \sinh \Theta
\]

(2.46)

The use of the same symbol \(\Theta\) for the rapidity and the spectral parameter is not fortuitous.

Let us call \(\omega\) the set of internal quantum numbers that characterizes the quantum state of the particle besides the rapidity \(\Theta\). So the relativistic invariant amplitude for a two body elastic scattering writes

\[
\sum_{\alpha, \delta} \left( \Theta_1 - \Theta_2 \right)
\]

(2.47)

where \(\alpha, \delta\) (\(\omega, \delta\)) are the quantum numbers of the initial (final) particles in the collision. \(\Theta_1 (\Theta_2)\) is the initial (final) rapidity and \(S\) depends only on the difference since it is a relativistic invariant. This amplitude is depicted in fig. 10.
Integrable two-dimensional S-matrices are characterized by the following postulates [22] 

I) There is no particle production. The number of particles of each type in the initial and final states coincide. The set of initial and final particle momenta coincide (particles can exchange their momenta during the collisions).

II) The N-particle S-matrix is expressed as a product of N(N-1)/2 two-particle S-matrices as if the processes of N-particle scattering were reduced to a sequence of pair collisions.

Integrable quantum field theories in two dimensions provide explicit realizations of postulates I and II (see sec. I).

In those theories the basic object is the two-body scattering matrix, since all amplitudes write as appropriate products of two body amplitudes.

There are a priori two inequivalent ways of writing the three-body scattering amplitude in terms of two-body amplitudes. (See fig. 7a and 7b). In fig. 7a particle 2 interacts first with particle 3, then 2 with 1 and finally 3 with 1. In fig. 7b the temporal order changes: first 1 with 3, then 1 with 2 and finally 2 with 3. This is analogous to eq. (2.15) for the permutation of operators (θi) (i = 1,2,3). As it was the case for eq. (2.15), one must here obtain the same three body S-matrix from 7a or 7b.

This leads to the following constraint on the 2-body S-matrices:

\[ S_{a_1 a_3} (θ_1 - θ_2) S_{c a_2} (θ_1 - θ_3) S_{d b_2} (θ_2 - θ_3) = \]

\[ S_{a_1 a_2} (θ_2 - θ_3) S_{c a_3} (θ_1 - θ_3) S_{d b_1} (θ_1 - θ_2) \]  

(2.4a)

This is the YBE equation (2.19) if we set

\[ R_{c d} (θ) = S_{d c} (θ) \]  

(2.4b)

So, we have a one-to-one formal correspondence between integrable 2-D vertex models and integrable 2-D S-matrices. This mapping can be visualized if we identify the particle trajectories (fig. 5 and 7) with the links in the lattice. Figs. 2 and 10 coincide if we take into account eqs. (2.24) and (2.45).

We said above formal correspondence between vertex weights and S-matrices since their physical properties are not identical. Vertex weights are usually real positive since they express probabilities. (However, a vertex model where some weights are negative or complex
may be also interesting). S-matrix elements can be complex but they must be meromorphic functions of \(\Theta\) due to the general principles of scattering theory [22]. Moreover space-time symmetries impose the following requirements[40]:

\[ S(\Theta) = S(\Theta)^T \quad \text{or} \quad S_{\alpha \lambda}(\Theta) = S^{\lambda \alpha}_{\lambda}(\Theta) \] 

(2.50)

Time-reversal invariance : 

\[ S(\Theta) = P S(\Theta) P \quad \text{or} \quad S^{\alpha \lambda}_{\beta \mu}(\Theta) = S^{	ext{b} \lambda}_{\text{c} \mu}(\Theta) \] 

(2.51)

Parity inversion invariance : 

In addition unless there are sources or sinks of particles \(S\) must be unitary

\[ S(\Theta) S(\Theta)^\dagger = 1 \] 

(2.52)

Moreover, it obeys real analyticity

\[ S(\Theta^*) = S(-\Theta) \] 

(2.53)

These properties [(2.50)-(2.53)] are related to those of vertex models. For example eqs. (2.50), (2.52) and (2.53) imply eq. (2.22) although the converse is not true. Eq. (2.50) was found before as eq. (2.23).

In particle theory one has in addition the crossing invariance. It says that the amplitudes of the process

\[ a + b \rightarrow c + d \]

and

\[ a + \tilde{d} \rightarrow c + \tilde{b} \] 

(2.54)

(where \(\tilde{b}, \tilde{d}\) means antiparticles of \(b\) and \(d\)) are related by appropriate analytic continuations in \(\Theta\). Crossing yields using eqs. (2.46)-(2.47)

\[ S^c d (\Theta) = S^c b (i\pi - \Theta) \] 

(2.55)

in a real basis of particle states and with a special normalization of \(\Theta\). In general crossing symmetry requires

\[ S(\Theta)^T = (1 \otimes W) S(-\Theta - \gamma) (1 \otimes W^{-1}) \] 

(2.56)

where \(\gamma\) means transpose in the first horizontal space, \(W\) is a constant matrix and \(\gamma\) a parameter that depends on the model.

The P and T symmetries (2.50)-(2.51) have a precise counterpart in the vertex language. T-invariance of the S-matrix implies invariance of the vertex weights under simultaneous up-down and left-right exchange. P-invariance means invariance of the weights under reflection over a line at \(+45^\circ\) from the horizontal axis. Crossing implies that a left-right
exchange in the vertices is equivalent to make \( \theta \rightarrow \eta \rightarrow \theta \) on the spectral parameter. Lack of any of these invariances can be interpreted as the presence of an external field in the vertex language.

The correspondence between an integrable vertex model and a factorizable S-matrix can be pushed even further as one sees from refs (52). That is, eqs. (2.48) define \( S(\theta) \) up to a c-number normalization \( p(\theta) \) that can be fixed by requiring unitarity [eq. (2.52)] and analyticity [eq. (2.53)]. It happens that these S-matrix requirements leads to a normalization that makes the free energy equal to zero for the corresponding vertex model in the thermodynamic limit. In other words they define a normalization \( p(\theta) \) where \( Z = 1 \) at \( N \rightarrow \infty \). Therefore, if one starts from a given normalization of the weights this factor \( p(\theta) \) is just the partition function per site, or [52]

\[
f = -\log p(\theta)
\]

(2.57)

It may be noticed that the partition function [eq. (2.8)] in the S-matrix language will be the trace of the S-matrix describing the scattering of N particles, all with rapidity \( \theta \), by M particles at rest.

Moreover the S-matrix interpretation of a YBZF algebra is not restricted to adjoint representations, that is R-matrices. Generally speaking we can associate a S-matrix amplitude to any YBZF generator \( t_{ab}(\theta) \) acting in a vertical space \( \mathcal{V} \) as

\[
S_{\alpha \gamma}^{\beta \delta}(\theta) = [t_{\alpha \gamma}(\theta)]_{\epsilon \delta} = \begin{array}{c}
\begin{array}{c}
\alpha \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta \\
\delta
\end{array}
\end{array}
\]

(2.58)

The spaces \( \mathcal{A} \) and \( \mathcal{V} \) describe the respective internal states of the two colliding particles. This S-matrix is factorizable thanks to eq. (2.9). In addition there must exist an amplitude

\[
S_{\alpha \gamma}^{\beta \delta}(\theta) = \begin{array}{c}
\begin{array}{c}
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\end{array} = R_{\alpha \beta}^{\gamma \delta}(\theta)
\]

when both particles belong to \( \mathcal{V} \). Once \( [t_{ab}(\theta)]_{\epsilon \delta} \) is known this last amplitude can be obtained by solving the linear equation

\[
R_{\alpha \beta}^{\gamma \delta}(\theta_1, \theta_2) \left[ t_{\alpha \gamma}(\theta_1) \right]_{\epsilon \delta} \left[ t_{\beta \delta}(\theta_2) \right]_{\epsilon \delta} = \left[ t_{\alpha \beta}(\theta_1) \right]_{\epsilon \delta} \left[ t_{\alpha \delta}(\theta_2) \right]_{\epsilon \delta} R_{\alpha \beta}^{\gamma \delta}(\theta_1, \theta_2)
\]

(2.59)

In summary a YBZF algebra consists in a set of operators \( T(I,J)(\theta) \) acting on couples of vector spaces \( (V^I, V^J) \). \( V^I \) stands for the auxiliary (quantum) space for this case. As in fig. 2

\[
\left[ T_{\alpha \beta}(\theta) \right]_{\alpha \gamma} = \theta
\]
1 \leq a, b \leq \dim V^l, 1 \leq \alpha, \beta \leq \dim V^j.

(2.60)

This set of operators is such that the following relation holds for any choice \{I, J, K\} of the vector spaces:

\[ T^{(K, J)}(\theta - \theta') T^{(I, J)}(\theta) T^{(I, J)}(\theta') = T^{(J, I)}(\theta') T^{(K, J)}(\theta) T^{(K, J)}(\theta - \theta') \]

(2.64)

where \( T(I, J)(\theta) \) is defined by eq.(2.60). The R-matrices correspond to the "diagonal" generator \( T(I, I)(\theta) \). Notice that there exists usually a fundamental R-matrix associated to the lowest dimensional \( V^l \).

(2.61)

As usual different types of lines are associated to different spaces:

\[ \quad = V^l, \quad \quad = V^j, \quad \quad = V^K \]

(2.62)

Eq.(2.61) writes in matrix form

\[
\left[ T_{a, \alpha} (\theta - \theta') \right]_{\beta, \beta'} \left[ T_{\beta, \beta'} (\theta) \right]_{\alpha', \alpha'} \left[ T_{\alpha', \alpha} (\theta') \right]_{\beta', \beta} = \]

(2.63)

\[
\left[ T_{a, \gamma} (\theta') \right]_{\alpha, \alpha'} \left[ T_{\beta, \beta} (\theta) \right]_{\gamma, \gamma'} \left[ T_{\gamma', \gamma} (\theta - \theta') \right]_{\alpha', \alpha} = \]

(2.65)

where latin and greek indices correspond to the spaces \( \mathcal{A} \) and \( \mathcal{V} \), respectively. We shall show that \( T(\theta)^{-1} \) exists by writing down its explicit expression.

Let us now discuss two important automorphisms of YBZF algebras possessing parity invariance [eq.(2.51)] : antipode and inversion [84]. One starts to define the inverse operator \( T(\theta)^{-1} \).

This inverse concerns the operatorial action of \( T(\theta) \) on spaces \( \mathcal{V} \) and \( \mathcal{A} \). That is \( T(\theta)^{-1} \) is such that

\[
T_{a, c} (\theta)_{\alpha, \gamma} T_{c, b} (\theta)_{\gamma, \beta} = T_{a, c} (\theta)_{\alpha, \gamma} T_{c, b} (\theta)_{\gamma, \beta} = S_{\alpha, \gamma} S_{\beta, \gamma} \]

(2.66)

There is a S-matrix associated to any local YBZF generator by eq.(2.58).
Assuming for $S$ unitarity [eq.(2.52)] and real analyticity [eq.(2.53)] yields for real $\theta$

$$S_{cY}^{a\xi}(-\theta)S_{cY}^{b\beta}(\theta) = S_{a\lambda}^{\xi}S_{\lambda Y}^{\beta}$$  \hspace{1cm} (2.66)

Comparing eqs. (2.54),(2.65) and (2.66) gives us explicitly the inverse generator

$$\left[ t_{-ab}^{-1}(\theta) \right]_{\alpha\lambda} = \left[ t_{ba}(-\theta) \right]_{\gamma\xi}$$ \hspace{1cm} (2.67)

The inverse of a general YBZF generator as given by eq. (2.34) follows using eq.(2.67) for each local factor. We find after a simple calculation

$$T_{-ab}^{-1}(\theta,\xi,\kappa) = \sum_{\alpha_1,\ldots,\alpha_N} \left[ t(-\theta+\alpha_i) \right]_{a\lambda}^{-1} t_{-ab}^{-1}(\theta) = \sum_{\alpha_1,\ldots,\alpha_N} \left[ t(-\theta+\alpha_i) \right]_{a\lambda}^{-1} t_{-ba}(\theta)$$ \hspace{1cm} (2.68)

This is just a YBZF generator constructed by reverse order as in eq.(2.37). More precisely

$$T_{-ab}^{-1}(\theta,\xi) = T_{-ba}(\theta,\xi)$$ \hspace{1cm} (2.69)

The antipode generator is defined by

$$t_{ab}(\theta) \rightarrow T_{ab}(\theta)$$

and

$$T_{\alpha\beta}^A(\theta) = T^{-1}_{\alpha\beta}(\theta)^T$$ \hspace{1cm} (2.70)

where $t$ means transpose in $\mathbb{A}$. That is

$$\left[ T(\theta) t \right]_{ab} = \left[ T_{ba}(-\theta) \right]_{ab}$$ \hspace{1cm} (2.71)

The antipode and the inversion

$$T_{-ab}(\theta) = T_{ba}(-\theta)$$ \hspace{1cm} (2.72)

are automorphisms of the YBZF algebra. It is easy to check from eq. (2.11) and (2.65) that

$$R(\theta,\theta') \left[ T_{\alpha\beta}(\theta) \otimes T_{\gamma\delta}(\theta') \right] = \left[ T_{\alpha\beta}(\theta') \otimes T_{\gamma\delta}(\theta) \right] R(\theta,\theta')$$

$$R(\theta,\theta') \left[ T_{\alpha\beta}(\theta) \otimes T_{\gamma\delta}(\theta') \right] = \left[ T_{\gamma\delta}(\theta') \otimes T_{\alpha\beta}(\theta) \right] R(\theta,\theta')$$

where the parity invariance [eq.(2.45)] is also used.

We conclude this section by stressing that the reproduction formulae (2.33) and (2.37) are the main properties of a YBZF algebra. Actually it should be possible to show that they actually imply eq. (2.11).

It should be noticed that eq.(2.33) for $N = 2$ together with the existence of the antipode (2.70) provide a Hopf algebra structure [85].
is a coproduct. \( \hat{T}(2) \) defines a second coproduct. This coproducts are non commutative so we have non-cocommutative Hopf algebras.

The YBZF algebras are not, strictly speaking, algebras in the usual sense since the sum of two generators \( T(\theta) \) fulfilling (2.9) does not verify this equation in general.

III. THE SIX VERTEX MODEL AND ITS DESCENDANTS.

The six vertex model corresponds to the trigonometric and hyperbolic solutions of the YBE (2.20) for \( q = 2 \) that is

\[
R(\theta) = \begin{pmatrix}
  a(\theta, \gamma) & 0 & 0 & 0 \\
  0 & c(\theta, \gamma) & b(\theta, \gamma) & 0 \\
  0 & b(\theta, \gamma) & c(\theta, \gamma) & 0 \\
  0 & 0 & 0 & a(\theta, \gamma)
\end{pmatrix}
\]  

(3.1)

We have here three different regimes.

I) \( a(\theta, \gamma) = \text{sh} (\gamma - \theta), b(\theta) = \text{sh} \theta, c(\gamma) = \text{sh} \gamma, \gamma > \theta > 0, \gamma > 0 \) in the antiferroelectric regime.

II) \( a(\theta, \gamma) = \sin (\gamma - \theta), b(\theta) = \sin \theta, c(\gamma) = \sin \gamma, 0 < \gamma < \pi, \gamma > \theta > 0 \) in the trigonometric regime. This regime is critical (gapless).

III) \( a(\theta, \gamma) = \text{sh} (\theta + \gamma), b(\theta) = \text{sh} \theta, c(\gamma) = \text{sh} \gamma, \theta > 0, \gamma > 0 \) in the ferroelectric regime.

The parameter \( \gamma \) describes the anisotropy of the model. The character of regimes I, II and III will be clear from the ground state and excitations obtained below.

This model enjoys the following symmetry group \( \mathcal{G} \) (in the sense of eq. (2.26))

\[
\mathcal{G} = \{ e^{i \omega \sigma_3}, 0 \leq \omega < 2\pi; \sigma \} \]

(3.2)

That is \( \mathcal{G} = \mathbb{U}(1) \oplus \mathbb{Z}_2 \). When \( \gamma = 0 \) this group enlarges to \( \text{SU}(2) \). This point corresponds to a Kosterlitz-Thouless type transition as we will see below from the explicit solution [eq. (6.26)]

It is called six vertex model, since the non-zero elements of the \( R \)-matrix, eq. (3.1) define six allowed configurations. The integrable eight-vertex model will not be considered here[2]. The state of a bond in the six-vertex (and eight-vertex) models is usually characterized by the sense of an arrow. This corresponds here to the values 1 or 2 of the vertical and horizontal indices. In fig. 11 the allowed configurations and their respective statistical weights are depicted.

It must be recalled that the trigonometric regime of the six-vertex
model (II) describes the critical (zero gap) limit of the eight-vertex model \cite{2}. As it will be clear from the solution one describes a critical line when \( \gamma \) varies from 0 to \( \pi \). As a S-matrix eq. (3.1) for regime II describes the scattering of a particle and its antiparticle with a conserved U(1) charge \cite{22}. The crossing symmetry (2.55) writes here

\[
\left[ \mathcal{R}(\theta) \right]^{\mathcal{T}_1} = (1 \otimes \sigma) \mathcal{R}(-\theta - \gamma) (1 \otimes \sigma)
\]

where

\[
\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

The YBZF generators read here (for one site)

\[
t_{11}(\phi) = \begin{pmatrix} a(\phi, \gamma) & 0 \\ 0 & a(\phi, \gamma) \end{pmatrix}, \quad t_{22}(\phi) = \begin{pmatrix} b(\phi, \gamma) & 0 \\ 0 & b(\phi, \gamma) \end{pmatrix}
\]

\[
t_{12} = c(\gamma, \phi) \sigma_-, \quad t_{21}(\phi) = c(\gamma, \phi) \sigma_+
\]

(3.3)

The YBZF generator \( T_{ab}(\theta) \) follows from eq. (2.34) where one inserts the \( t_{ab} \) given by eq. (3.3). One can then set

\[
T^{[N]}(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix}
\]

(3.4)

The YBZF algebra defined by the R-matrix (3.1) yields some number of bilinear algebraic relations between the \( T^{[N]}(\theta) \). Let us just write down the more useful ones for the subsequent derivations

\[
A(\phi)B(\phi') = q(\phi - \phi') B(\phi') A(\phi) - \tilde{t}_1(\phi' - \phi) B(\phi) A(\phi')
\]

\[
D(\phi)B(\phi') = q(\phi - \phi') B(\phi') D(\phi) - h(\phi - \phi') B(\phi) D(\phi')
\]

(3.5)

\[
A(\phi)B(\phi') = \frac{a(\phi, \gamma)}{b(\phi, \gamma)} B(\phi') A(\phi)
\]

and

\[
h(\theta) = \frac{c(\theta, \gamma)}{b(\theta, \gamma)}
\]

Let us now proceed to construct the exact eigenvectors and eigenvalues of

\[
\zeta^{[N]}(\theta) = T^{[N]}(\theta) = A(\theta) + D(\theta)
\]

(3.7)

using the algebraic Bethe Ansatz \cite{24}. We shall assume \( N \) to be even. One notices that the ferromagnetic state

\[
| \Omega \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

is an eigenvector of \( A(\theta) \) and \( D(\theta) \)

\[
A(\theta) | \Omega \rangle = \alpha(\theta, \gamma) | \Omega \rangle
\]

\[
D(\theta) | \Omega \rangle = \beta(\theta, \gamma) | \Omega \rangle
\]

(3.8)

In addition

\[
| \Sigma \rangle = 0
\]

(3.10)

whereas \( B(\theta)|\Omega\rangle \) is non-zero and not proportional to \( |\Omega\rangle \). The algebraic Bethe ansatz consist in looking for eigenvectors of \( \zeta(\theta) \) with the form

\[
\Psi(\theta_1, \ldots, \theta_r) = B(\theta_1) B(\theta_2) \cdots B(\theta_r) | \Omega \rangle
\]

(3.11)

Here, the complex number \( \theta_1, \ldots, \theta_r \) will be determined by requiring that

\( \psi(\theta_1, \ldots, \theta_r) \) is an eigenvector of \( \zeta(\theta) \).
In order to do that one applies $A(\theta) + D(\theta)$ to the r.h.s. of eq. (3.11) and pushes $A + D$ through the $B(\theta_j)$ with the help of eqs. (3.5). After using eqs. (3.5) $r$ times, $A$ and $D$ reach $|\Omega\rangle$ where their action is known from eqs. (3.9). These operations produced a lot of terms. Let us first write down explicitly those generated by the first term in eqs. (3.5):

$$
\hat{A}(\theta)\Psi(\theta_1, \ldots, \theta_r) = \prod_{\gamma=1}^{r} q(\gamma - \theta) a(\theta, \gamma)^N b(\theta_j) b(\theta_j) |\Omega\rangle + \text{unwanted terms} = 
$$

$$
= A(\theta)\Psi(\theta_1, \ldots, \theta_r) + \text{unwanted terms}
$$

and an analogous formula for $D(\theta)\Psi$.

The remaining terms are called "unwanted" since they are not proportional to $\Psi$ and hence they must finally cancel in order to get an eigenvector of $\sigma(\theta)$.

Now, let us concentrate in terms containing the vector

$$
B(\theta)B(\theta_2) \ldots B(\theta_r) |\Omega\rangle
$$

They originate when the second term in eq. (3.5) is used to express $A(\theta_j)B(\theta_j)$ and the first term for the rest when $A(\theta_j)$ is pushed through $B(\theta_j)$ $(2 \leq j \leq r)$. Hence, one finds

$$
-\hat{A}(\theta_1 - \theta) B(\theta) A(\theta_1) B(\theta_2) \ldots B(\theta_r) |\Omega\rangle = 
$$

$$
\sum_{k=1}^{r} \lambda^+(\theta, \theta_j) \Psi_k(\theta, \theta) \Psi(\theta_1, \ldots, \theta_r) = 
$$

$$
\Psi(\theta_1, \ldots, \theta_r) = \Lambda^+(\theta) \Psi(\theta_1, \ldots, \theta_r) + \sum_{k=1}^{r} \lambda^+(\theta, \theta_j) \Psi_k(\theta, \theta) \Psi(\theta_1, \ldots, \theta_r)
$$

where

$$
\Psi_k(\theta, \theta) = \prod_{\gamma \neq k} B(\theta) \prod_{\gamma \neq \theta} B(\theta_j) |\Omega\rangle
$$

and

$$
\lambda^+(\theta, \theta_j) = \hat{A}(\theta - \theta_j) a(\theta, \gamma)^N \prod_{\gamma \neq k} B(\theta_j) B(\theta_k) |\Omega\rangle
$$

One analogously finds

$$
D(\theta)\Psi(\theta_1, \ldots, \theta_r) = \Lambda^-(\theta) \Psi(\theta_1, \ldots, \theta_r) + \sum_{k=1}^{r} \lambda^-(\theta, \theta_j) \Psi_k(\theta, \theta) \Psi(\theta_1, \ldots, \theta_r)
$$

where

$$
\lambda^-(\theta, \theta_j) = \hat{D}(\theta - \theta) B(\theta) a(\theta, \gamma)^N \prod_{\gamma \neq k} B(\theta_j) B(\theta_k) |\Omega\rangle
$$
Now, in order to get an eigenvector of \( \tau(\Theta) \) we must require
\[
\Lambda_+^k(\Theta, \varphi) + \Lambda_-^k(\Theta, \varphi) = 0
\]
for regime II. The r.h.s. of eq. (3.21) would seem to have poles at \( \Theta = i(\lambda_j + \gamma)/2 \). However, the corresponding residues identically vanish due to eqs. (3.20). Actually one can use this property as a short-cut to derive the BAE when the construction of the explicit eigenvalues is more involved.

It is convenient to take logarithms of eqs. (3.20). One finds
\[
N \phi(\lambda, \gamma/2) = \sum_{\varepsilon = 1}^{r} \phi(\lambda - \lambda_{\varepsilon}, \gamma) + 2 \pi i \lambda_{\varepsilon}, \quad 1 \leq \varepsilon \leq r,
\]
where
\[
\phi(\lambda, \varphi) = i \log \frac{\sin \left( \lambda + i \varphi \right)}{\sin \left( \lambda - i \varphi \right)}
\]
for regime II
\[
\phi(\lambda, \varphi) = i \log \frac{\sinh \left( \lambda + i \varphi \right)}{\sinh \left( \lambda - i \varphi \right)}
\]
for regime I and III.

and \( l_j \in \mathbb{Z} \pm 1/2 \). The numbers \( l_1, \ldots, l_r \) characterize the eigenstate. The cut of the logarithm in eq. (3.23) is taken much that \( \phi(x, \lambda) \) is a continuous function for real \( x \in \mathbb{R} \). \( \phi(x, \lambda) \) is a monotonically increasing function and we choose \( \phi(0, \alpha) = \pi \). For large \( N \) and \( |\Theta| < \gamma/2 \) the first term in eq. (3.21) dominates. Therefore, one can set
\[
S_N(\Theta, \gamma) = -\frac{1}{N} \log \lambda(\Theta) = \frac{i}{N} \sum_{\varepsilon = 1}^{r} \frac{\phi(\lambda_{\varepsilon} + \Theta, \gamma/2)}{\phi(\lambda_{\varepsilon}, \gamma/2)} + o(e^{-N})
\]
with $c_i > 0$ (Here we have normalized the weight $a(\theta, \gamma)$ to unit).

When $\gamma = \pi/2$ in regime II eqs. (3.22) decouple from each other. In this case the model reduces to free fermions$^2$.

Let us analyze in the different regimes which is the ground state. That is the eigenvector of $\gamma(\theta)$ with maximum modulus eigenvalue. It follows from eq. (3.21) that $\Lambda_\pm$ dominates for large $N$ and fixed $r$ when $0 < \theta < \gamma/2$. It is then enough to compute $\lambda_\pm = a(\theta, \gamma)^\pm \Lambda_\pm (\theta, \gamma)$. Let us consider one pseudoparticle over $|\Omega\rangle$, that is $r = 1$. Eqs. (3.20)-(3.21) are easily solved with the result

$$\lambda_+ = \frac{\sin \theta - e^{\pm i \gamma} \sin (\theta + \gamma)}{\sin (\gamma - \theta) + e^{\pm i \gamma} \sin \theta} \quad \text{regime II}$$

Here $q = 2\pi i N (1 \leq i \leq N, \text{stands for the momentum of the pseudoparticle}). A simple calculation shows that

$$|\Lambda_\pm|^2 > 1 \quad \text{for } \theta > 0 \quad \text{regime I}$$

$$|\Lambda_\pm|^2 > 1 \quad \text{for } \theta > 0 \text{ and } \gamma' < q < 2\pi \quad \text{regime II}$$

$$|\Lambda_\pm|^2 < 1 \quad \text{for } \theta > 0 \text{ and all } q \quad \text{regime III}$$

Since $\Lambda_\pm$ decreases in regime III by adding pseudoparticles, $|\Omega\rangle$ is the ground state$^{32}$. This is indeed a ferroelectric regime and eq. (3.11) describes here states with $r \text{ spin waves interacting non-trivially. In regimes I and II we have the opposite behavior and the ground state follows by filling } |\Omega\rangle \text{ with pseudoparticles. The most regular filling is obtained for } r = N/2 \text{ and }$$

$$I_{j+1} - I_j = 1$$

as follows analyzing eq. (3.22) (see sec. VI for more details). Moreover, excitations around this antiferroelectric state decrease $|\Lambda_\pm|$ as it is shown below [eq. (3.53)]. Analogous conclusions for the states follows from the spectrum of the XXZ Hamiltonian [eq. (A.9)] (See ref. (50) for a rigorous discussion). For excited states the sequence $I_j$ exhibits jumps for some values of $j$

$$I_{j+1} - I_j = \lambda + \sum_{\lambda = 1}^{N_h} \delta_{\lambda j}$$

(3.25)

The values of $\lambda$ associated with these missing half-integers are called holes and denoted $\theta_h$.

In the QFT associated to vertex models, the vacuum (ground state) corresponds precisely to the antiferroelectric ground state. Let us concentrate on this state and excitations around it from now on. The operators $\mathcal{B}(\theta_i)$ play here the rôle of creation operators of excitations over
the bare vacuum $|\Omega\rangle$. That is pseudo-particles or "bare" particles. The
antiferroelectric ground state is the analog of the filled Dirac sea for free
fermions. However, the pseudoparticles are here not free, they interact
through two-body interactions. The functions $\phi(\lambda_1 - \lambda_j, \gamma)$ describe the
two-body phase-shift associated to such interactions.

As it is clear, one can easily solve eqs. (3.20) analytically for
small $r$ and $N$. For large $N$ the number of roots is very large but they
become closer and closer in the real axis so one can define a continuous
density in the $N - \infty$ limit

$$J_\infty (\lambda) = \lim_{N \to \infty} \frac{N}{N (\lambda_{i+1} - \lambda_i)}$$

(3.26)

Once this function is calculated the different physical magnitudes can be
computed by quadratures. That is

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{r} f(\lambda_j) = \int d\lambda J_\infty (\lambda) f(\lambda)$$

(3.27)

For example the free energy reads from eq. (3.24)

$$f(\theta, \lambda) = \lim_{N \to \infty} f_{N}(\theta, \lambda) = + \int d\lambda J_{\infty} (\lambda) \phi(\lambda + i\theta, \frac{\lambda}{2})$$

(3.28)

It is useful to introduce the function\cite{29}

$$Z_N (\lambda) \equiv \frac{1}{2\pi} \left[ \phi(\lambda, \frac{\lambda}{2}) - \sum_{j=1}^{N} \phi(\lambda - \lambda_j, \gamma) \right]$$

(3.29)

This function is continuous and monotonically increasing for real $\lambda$. At the
real roots of the BA eqs. (3.20)

$$Z_N (\lambda_i) = \frac{i}{N}$$

(3.30)

At the hole positions $\theta_h$

$$Z_N (\theta_h) = \frac{i \theta_h + 1}{N}$$

(3.31)

For large $N$, neighboring BA roots are very close and we have

$$\frac{d Z_N}{d\lambda} = \frac{Z_N (\lambda_{i+1}) - Z_N (\lambda_i)}{\lambda_{i+1} - \lambda_i} = \frac{1 + \sum_{h}^{N_h} \delta (\lambda - \theta_h)}{N (\lambda_{i+1} - \lambda_i)}$$

(3.32)

where we used eq. (3.25). Now in the $N - \infty$ limit using eq. (3.26)

$$\sigma_\infty (\lambda) \equiv \frac{d Z_N}{d\lambda} = J_\infty (\lambda) + \frac{1}{N} \sum_{h=1}^{N_h} \delta (\lambda - \theta_h)$$

(3.33)

Also, using eq. (3.26) and (3.28)

$$Z_N (\lambda) = \frac{1}{2\pi} \left[ \phi(\lambda, \frac{\lambda}{2}) - \int d\gamma \phi(\lambda - \gamma, \gamma) - \frac{1}{N} \sum_{h}^{N_h} \phi(\lambda - \theta_h, \gamma) \right]$$

(3.34)

Now, combining this with eq. (3.32) yields a linear integral equation for

$$\sigma_\infty (\lambda)$$

$$\sigma_\infty (\lambda) = \frac{1}{2\pi} \phi(\lambda, \frac{\lambda}{2}) - \int \frac{d\gamma}{2\pi} \sigma_\infty (\gamma) \phi(\lambda - \gamma, \gamma) +$$
\[ \frac{1}{2\pi N} \sum_{k=1}^{N_0} \phi(\lambda - \Theta_k, \nu) - \frac{1}{2\pi N} \sum_{k=1}^{N_0} [\phi(\lambda - \Theta_k, \nu) + \phi(\lambda - \Theta_k, \nu)] \] (3.34)

We denoted in eq. (3.32) and (3.33) by \( \xi, \zeta \) the complex roots (\( \text{Im} \xi, \zeta > 0 \)). They always appear in conjugate pairs. In the limiting case \( \gamma \to 0 \) (regime I or II) one has

\[ \phi(\lambda, \nu) = \frac{\lambda + i \nu}{\lambda - i \nu}, \quad \nu = 0 \quad (3.35) \]

The linear integral equations (3.34) can be easily solved by Fourier integrals (or Fourier series for regime I). In order to do that one needs the following Fourier representations of \( \phi(\lambda, \alpha) \):

\[ \phi(\lambda, \alpha) = \pi + \sum_{m \neq 0} \frac{2\sin \lambda - 2\sin \alpha}{\lambda - \alpha}, \quad \text{regime } I \text{ and III} \] (3.36)

\[ \phi(\lambda, \nu) = \pi + \sum_{k=1}^{\infty} \frac{d \sin(\nu k)}{d \nu} - 1, \quad \text{regime } II \] (3.37)

\[ \phi(\lambda, \nu) = \pi + \sum_{k=1}^{\infty} \frac{d \sin(\nu k)}{d \nu} - 1, \quad \text{regime } II \] (3.38)

The solution of eq. (3.34) reads

\[ \sigma_\infty(\lambda) = \sigma_\infty(\lambda) + \frac{1}{N} \left[ \sigma_h(\lambda) + \sigma_\zeta(\lambda) \right] \] (3.39)

\( \sigma_\infty(\lambda) \) corresponds to the ground state. One finds

\[ \sigma_\infty(\lambda) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \frac{e^{2i\pi \lambda}}{\cos \gamma, m, \nu} = \frac{K(k)}{\pi} \int_{-\pi}^{\pi} \frac{dz}{\cos(\nu z)}, \quad \text{regime I} \] (3.40)

Here \( K(k) / K(k) = \gamma / \pi \).

\[ \sigma_\infty(\lambda) = (1 - \cos \frac{\pi \lambda}{\nu})^{-1} \] (3.41)

\[ \sigma_\infty(\lambda) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \frac{e^{2i\pi \lambda}}{\cos \gamma, m, \nu} = \frac{4}{2\pi \cos \frac{\pi \lambda}{\nu}}, \quad \text{regime III} \] (3.42)

\( \sigma_h(\lambda) \) in eq. (3.39) stands for the hole contribution to the density of real roots. One finds \( \sigma_h(\lambda) = -\frac{1}{\pi} \sum_{k=1}^{N_0} \rho(\lambda - \Theta_k) \), where

\[ p(\lambda) = \frac{1}{2} + 2 \sum_{m, l} \frac{\cos 2\pi \lambda}{e^{2\pi \nu} + 1}, \quad \text{regime I} \] (3.43)

\[ p(\lambda) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(\gamma k)}{\cos(\gamma k/2)} \] (3.44)

\[ p(\lambda) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos(\gamma k)}{\cos(\gamma k/2)} \] (3.45)

The complex root contribution \( \sigma_c(\lambda) \) can be found in refs. (20), (41).

It must be noticed that \( p_h(\lambda) \), the hole contribution to \( p(\lambda) \) is minus the resolvent kernel \( R(\lambda) \) of the integral equation (3.34), defined by
\[ R(\lambda - \rho) + \int \frac{dr}{2\pi} \phi'(\lambda - \rho) R(\sigma - \rho) = \delta(\lambda - \rho) \] (3.46)

That is,
\[ f_h(\lambda) = \frac{1}{\pi} \left[ \delta(\lambda - \lambda_h) + \frac{i}{\pi} \phi(\lambda - \Theta_h) \right] = - \sum_{h=1}^{N_h} R(\lambda - \Theta_h) . \]

Therefore
\[ \hat{z}(\lambda) = - \int_{-\infty}^{\infty} \frac{dr}{2\pi} f_h(\lambda - \rho) \phi'(\rho, \frac{\lambda}{2}) + \frac{K}{2\pi} \] (3.47)

where \( K = \) a \( \lambda \)-independent constant. Eqs. (3.47) holds in regime II.

The densities (3.40)-(3.41) allow an easy calculation of the eigenvalues of \( \tau(\Theta) \) in the \( N \to \infty \) limit using eq. (3.27). One finds for example for the free energy per site
\[ f(\Theta, \gamma) = \Theta + \sum_{m \neq 0} \frac{e^{-m(2\Theta)}}{m \cosh(m(\gamma))}, \quad \text{regime I} \] (3.48)

\[ f(\Theta) = \int_{0}^{\gamma} d\gamma \frac{e^{-x}}{x} \frac{\cosh(2\Theta)}{\cosh x}, \quad \gamma = 0 \] (3.49)

\[ f(\Theta, \gamma) = \int_{0}^{\gamma} d\gamma \frac{\cosh(2\Theta) \cosh [x(\gamma - \gamma)]}{\cosh x} \frac{\cosh(x(\gamma))}{\cosh x}, \quad \text{regime II} \] (3.50)

It must be noticed that eq. (3.44) also gives \( \log S^{\theta_0}(\gamma) \theta(\gamma) \) where \( S^{\theta_0}(\gamma) \) is the soliton-soliton S-matrix in the sine-Gordon model as a function of the physical rapidity \( \gamma \).

The excited states eigenvalues of \( \tau(\Theta) \) have the following structure for large \( N \) due to eq. (3.39)
\[ \Lambda^{\text{exc}}(\Theta) = \lambda_{\Theta}(\Theta) e^{-i\gamma(\Theta)} \left[ \lambda + o(e^{-c_N}) \right] \] (3.51)

where \( \lambda_{\Theta}(\Theta) \) in the ground state eigenvalue and \( g(\Theta) \) is of order \( N^0 \) for \( N \gg 1 \). A look to eqs. (3.28) and (3.47) shows that the hole eigenvalues are given by
\[ \gamma(\Theta) = 2\pi \sum_{h=1}^{N_h} 2 \hat{z}(\Theta_h + \Theta) = \sum_{h} \gamma(\Theta, \Theta_h) \] (3.52)

Using now eq. (3.37) yields in regime II
\[ \gamma(\Theta, \Theta_h) = \arctan \left[ \frac{\pi}{\beta(\Theta_h + \Theta)} \right], \quad \text{regime II} \] (3.53)

This is clearly a gapless regime since \( g(\Theta, \infty) = 0 \). Moreover \( |e^{-g(\Theta)}| < 1 \) for \( 0 < \Theta < \gamma \). This shows that our identification of the ground state is correct since any deviation from it decreases the eigenvalue of \( \tau(\Theta) \).

Interesting complex roots appear for \( \pi/2 < \gamma < \pi \) in regime II.

They appear in strings of length \( n \), where \( n \) may be \( \leq [\pi/(\pi - \gamma)] \cdot 1 \) and \( [x] \) stands for integer part of \( x \).
\[ \lambda_r = \sigma + i\gamma/2 - \lambda(r + \gamma/2)(\pi - \gamma), \quad 0 \leq r \leq \sqrt{n - \gamma/2} \] (3.54)
\[ \lambda_s = \sigma + \frac{i\pi}{2} - iS(\pi - Y), \quad 0 \leq S \leq \frac{\lambda - 1}{2} \tag{3.55} \]

where \( \sigma \) is the common real part of the roots. The associate eigenvalue of \( \tau(\theta) \) can be shown to be \[^{[41]} \]

\[ g_n(\theta, \sigma) = 2 \arctan \left( \frac{\frac{1}{2} + i\frac{\pi}{2}}{\frac{\pi}{2} + i\theta} \right) \frac{\sin \frac{\pi m_n}{2} (\pi - Y)}{\sin \frac{\pi m_n}{2} (\pi - Y)} \tag{3.56} \]

Let us derive for future reference the asymptotic behaviour of \( g(\theta, \Theta_h) \) and \( g_n(\theta, \sigma) \) for \( \theta \to -\infty \). Eqs. \( (3.53) \) and \( (3.56) \) yield in this limit

\[ g(\theta, \Theta_h) = 2 e^{\frac{\pi}{2} i \Theta_h (\theta + \Theta_h)} + O(e^{-\frac{2\pi}{N} \theta}) \tag{3.57} \]

\[ g_n(\theta, \sigma) = -\pi + i \frac{\pi}{2} e^{\frac{\pi}{2} i \theta} \sin \frac{\pi m_n}{2} (\pi - Y) + O(e^{-\frac{3\pi}{2} i \theta}) \]

The coefficients in this formulae give the mass spectrum of the QFT provided by the light-cone transfer matrix approach : the massive Thirring model \[^{[23,42]} \] (see sec. V).

The momentum can be defined in terms of the logarithm of \( \tau[N](0) \) since this is the unit shift operator [see eq. (2.43)]

\[ P_N = i \log \left[ e^{-\lambda N} \tau[N](0) \right] \tag{3.58} \]

We get from eq. (3.21) for its eigenvalues

\[ P_N = \sum_{j=1}^{c} \phi(\lambda_j, Y_j) \tag{3.59} \]

where we choose \( c \) such that \( P_N \) vanishes for the reference state \( |\Omega> \). We find using eqs. (3.22) and (3.59) and the fact that \( \phi(\lambda, Y) \) is an odd function of \( \lambda \)

\[ P_N = \frac{2\pi}{N} \sum_{j=1}^{c} I_j \tag{3.60} \]

This formula allows to compute \( P_N \) directly from the half-integers \( I_j \) characterizing the state. It shows that the \( P_N \) of excited states differs in multiples of \( 2\pi/N \) of that of the ground state which can be set equal to zero by appropriately choosing \( I_1 \).

For large \( N \) the momentum of a hole excitation at \( \Theta_h \) writes

\[ P(\Theta_h) = 2 \arctan \left( \frac{\pi}{2} \Theta_h \right) \tag{3.61} \]

where we used eqs. \( (3.27), (3.59), (3.51) \) and \( (3.53) \).

The hole states eigenvalues of \( \tau(\theta) \) write regime I

\[ g(\theta, \Theta_h) = 2 \pi 2^{\frac{\pi}{2}} \left( \Theta_h + \frac{\pi}{2} \theta \right) = -i \log \left\{ s_n \left[ \frac{2k}{\pi} (\Theta_h + \theta) \right] \right\} \tag{3.62} \]

\[ = -ic\log \left\{ \frac{2k}{\pi} (\Theta_h + \theta) \right\} + \frac{\pi}{4} \Theta_h + \frac{\pi}{2} \sum_{j=1}^{c} P(\lambda) \delta \lambda \]

In this regime we have a non-zero gap given by the end-point excitations \( \Theta_h \pm \pi/2 \).
\[ g(\theta, \pm \frac{\pi}{2}) = \pi + i \log \frac{\sin \left( \frac{2k\theta}{\pi}, k' \right)}{1 - k' \sin \left( \frac{2k\theta}{\pi}, k' \right)} \]  

(3.63)

This gap vanishes for \( \gamma \to 0^+ \) as

\[ g(\theta, \pm \frac{\pi}{2}) - \pi = \frac{1}{2} \sin \theta \cos \theta + O(\theta) \]  

(3.64)

Therefore the six-vertex model is critical (gapless) for regime II and massive for regime I.

Eq. (2.40) gives for the six-vertex model symmetry (3.2) (rotations around \( z \))

\[ [A(\theta), S_z] = [D(\theta), S_z] = 0 \]  

(3.65)

\[ [S_z, B(\theta)] = -B(\theta), [S_z, C(\theta)] = C(\theta) \]

where \( S_z = \frac{1}{2} \sum_{\xi=1}^{\sigma} \sigma_z \) acts in the vertical space. Therefore \( B[\theta] [C[\theta]] \) lowers (raises) the \( z \)-component of the spin in one unit.

In particular we find that the state (3.11) is an eigenvector of \( S_z \)

\[ S_z \Psi (\theta_1, \ldots, \theta_r) = \left( \frac{N}{2} - r \right) \Psi (\theta_1, \ldots, \theta_r) \]  

(3.66)

\[ \mathbb{Z}_q \otimes (\lambda)^{q-1} \]  

(4.3)

**IV. Multi-State Integrable Models.**

In the previous section a general introduction to vertex model and YBZF algebras was given. General vertex models where each link can be in many different states were introduced there.

We present now an integrable model of this type. The R-matrix reads [25]

\[ R_{ij}^{ab}(\theta) = \frac{\sin \gamma}{\sin (\pi \gamma \theta)} \sin (\delta \theta) \cos (\delta \theta) + \frac{\sin \theta}{\sin (\pi \gamma \theta)} \sin (\delta \theta) \cos (\delta \theta) \]  

(4.1)

Here \( 1 \leq i,j,a,b \leq q \) where \( q \) can take any value \( \geq 2 \). This is a regular R-matrix, since

\[ R_{ij}^{ab}(\theta) = \sin \delta_{jk} \]  

(4.2)

The YBZF algebra defined by the R-matrix (4.1) is invariant under the group

\[ \mathbb{Z}_q \otimes (\lambda)^{q-1} \]  

(4.3)
Here $D$ is the one-dimensional dilation group and the cyclic group $Z_q$ is generated by the powers of the matrix $\delta_a$ defined as

$$\delta_{a+1} = \delta_a \delta_b$$

where

$$\delta_{a+1} = \delta_{a+1} = \delta_a \delta_b$$

(one has $\delta_0 = 1$). The matrices of $D^{q-1}$ read

$$D_{ab} (\gamma) = \delta_{ab} e^{\gamma} \quad \text{where} \quad \frac{\gamma}{2\pi} z_a = 1$$

$R$ enjoys in addition the following symmetry property

$$R_{ac} (\theta) = R_{bc} (\theta)$$

(4.6)

An integrable vertex model can be associated to the $R$-matrix (4.1) through eq. (2.24) and fig. 2. That is a model with statistical weights

$$w(c d | a b) = \left[ t_{ab} (\theta) \right]_{cd} = R_{bd}^{ac} (\theta)$$

(4.7)

There are $q(2q-1)$ non-vanishing weights. Eq. (4.6) tells that the model is invariant under a $180^\circ$ rotation of the whole lattice. For $q = 2$ one recovers the six-vertex model.

In this model the links can be in $q$ different states and the statistical weights are given in figure 18. In regime III, for $\gamma > 1$, $2\theta+\gamma$ fixed, the model has a long-range generalised ferroelectric order and the dominant configurations are formed mostly by vertices of type $c_1$ (for $\theta > 0$) and some $c_{b-1}$. There are $q$ different predominant patterns following one from each other by shifting by one the state of all links. This generalises the six vertex ($q = 2$) situation [2]. Regimes I and III map into each other through $\gamma \rightarrow - \gamma + 2\pi$.

We construct in this section the exact solution of this $q(2q-1)$ vertex model. That is the eigenvectors and eigenvalues of the transfer matrix $\tau(N)(\theta, \alpha \gamma)$ associated to the weights (4.7) through eqs. (2.5), (2.7) and (2.34)[43]. The case $q = 2$ (six-vertex model) is considered in sec. III. For $q > 2$ a more sophisticated construction is needed for the eigenvectors of $\tau(N)$. They are built as a set of nested Bethe ansatz. This type of construction is needed when non-abelian internal structures are present both in vertex models and field theory [10,13,16,17,21].

It is easy to find eigenstates of $\tau(N)(\theta, \alpha \gamma)$ with ferromagnetic character for the model (4.1)-(4.7). For example

$$\left| \chi \right> = \bigotimes_{s=1}^N \left| \chi \right>^{(s)}$$

(4.8)
where the $q$-component vectors $|i\rangle$ have all components zero except the $i^{th}$ that equals one.

The state (4.8) has special properties under the action of the monodromy operator $T_{ab}^{[N]}(\theta, \varphi)$. One finds

$$T_{ab}^{[N]}(\theta, \varphi) |1\rangle = |1\rangle \quad (4.9)$$

$$T_{k\ell}^{[N]}(\theta, \varphi) |1\rangle = \prod_{j=1}^{N} \frac{1}{g(\theta - \varphi_j)} |1\rangle \quad (4.10)$$

$$2 \leq k < \ell$$

$$T_{k\ell}^{[N]}(\theta, \varphi) |\ell\rangle = 0 \quad \text{for} \ 2 \leq k \leq \ell \ \text{or} \ \ell \leq k \geq 2 \quad (4.11)$$

and

$$T_{k\ell}^{[N]}(\theta, \varphi) |\ell\rangle \neq 0$$

Here

$$g(\theta) = \frac{s_k(\gamma \tau \theta)}{s_k(\theta)}$$

The operators $B_i(\theta, \varphi) = T_{i1}^{[N]}(\theta, \varphi)$ give new states when applied to the reference state $|1\rangle$ and they will be called creation operators.

The basic idea of the algebraic Bethe ansatz is to obtain all the physical states by acting with the $B_i$ on the reference state many times. One can get in this way states of antiferromagnetic character from $|1\rangle$.

The properties of the reference state $|1\rangle$ suggest to decompose $T_{ab}^{[N]}$ and $R$ in blocks of the following form [43]

$$T_{ab}^{[N]}(\theta, \varphi) = \begin{pmatrix} A(\theta, \varphi) & B_{ij}(\theta, \varphi) \\ C_{ij}(\theta, \varphi) & D_{ij}(\theta, \varphi) \end{pmatrix} \quad (4.12)$$

Here $A(\theta) = T_{11}^{[N]}(\theta)$, $C_{ij}(\theta, \varphi) = T_{ij}^{[N]}(\theta, \varphi)$ and $D_{ij}(\theta, \varphi) = T_{ij}^{[N]}(\theta, \varphi)$, $2 \leq i, j \leq q$. The $R$-matrix reads

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_{ij} c_-(\theta) & \delta_{ij} \frac{1}{g(\theta)} & 0 \\ 0 & \delta_{ij} \frac{1}{g(\theta)} & \delta_{ij} c_+(\theta) & 0 \\ 0 & 0 & 0 & R_{ij}^{(2)} e(\theta) \end{pmatrix} \quad (4.13)$$

Here

$$c_{\pm}(\theta) = \frac{s_k(\gamma \tau \theta)}{s_k(\theta)} e^{\pm \theta}$$

and

$$2 \leq i, j, k, \ell \leq q$$
The bilinear algebra for the operators $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ follows by inserting eqns. (4.12) and (4.13) in eqn. (2.33). One finds \[ A(\theta') A(\theta) = A(\theta') A(\theta) \]

\[ B(\theta') B(\theta) = \frac{1}{\theta(\theta-\theta')} A(\theta') B(\theta) + C_+ (\theta-\theta') B(\theta') A(\theta) \]

\[ [B(\theta) \otimes B(\theta)] R^{(2)}(\theta-\theta) = B(\theta') B(\theta) \]

\[ - c_-(\theta-\theta') B(\theta) A(\theta') + \frac{1}{\theta(\theta-\theta')} D(\theta) B(\theta) \]

\[ = [B(\theta') \otimes D(\theta)] R^{(2)}(\theta-\theta) \]

where $B(\theta) = (B_2(\theta), \ldots, B_1(\theta))$.

It is useful for later derivations to write these equations as

\[
A(\theta) B(\theta') = \frac{1}{\theta(\theta-\theta')} [B(\theta') A(\theta) - A(\theta') B(\theta)] R^{(2)}(\theta-\theta')
\]

\[
B(\theta) \otimes B(\theta') = R^{(2)}(\theta-\theta') [B(\theta') \otimes B(\theta)] =
\]

\[
[ B(\theta') \otimes B(\theta)] R^{(2)}(\theta-\theta')
\]

\[
D(\theta) \otimes D(\theta') = \frac{1}{\theta(\theta-\theta')} [B(\theta') \otimes D(\theta)] R^{(2)}(\theta-\theta')
\]

\[
- c_-(\theta-\theta') B(\theta) D(\theta')
\]

(4.19)

Here $\lambda_+ (\theta) = \frac{\sqrt{\gamma}}{\sqrt{\lambda_+ \theta}} \epsilon^{+ \theta}$.

The tensor product notations read for eqn. (4.19)

\[
D_{\lambda\mu}(\theta) B_{\epsilon}(\theta') = q(\theta-\theta') R^{(2)}(\theta-\theta')
\]

\[
- \lambda_-(\theta-\theta') B_{\gamma}(\theta) D_{\lambda\mu}(\theta')
\]

(4.20)

using the fact that

\[
R^{(2)}(\theta) \gamma (\theta) = \delta_{\lambda\mu} \delta_{\mu\alpha}
\]

(4.21)

Eqn. (4.19) can also be written as

\[
D(\theta) \otimes B(\theta') = \frac{1}{\theta(\theta-\theta')} [B(\theta') \otimes D(\theta)] R^{(2)}(\theta-\theta')
\]

\[
- c_-(\theta-\theta') [B(\theta) \otimes D(\theta')] R^{(2)}(\theta-\theta')
\]

(4.22)

In eqns. (4.14)-(4.22) and in what follows the dependence of the operators $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ in $\{\theta\}$ is implicit. We seek now for an eigenstate of $\tau^{[N]}(\theta, \theta')$ with the following structure [43].

\[
\Psi (\lambda_1, \ldots, \lambda_r) = \sum_{i_1 \ldots i_r = 2} \prod_{i_1 \ldots i_r} \chi_{i_1 \ldots i_r}
\]

\[
\Psi (\lambda_1, \ldots, \lambda_r)
\]

(4.23)
\[ X^+ B(\mu_1) \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \|1\rangle \] (4.23)

So, \( X \) is a vector in the tensor product of \( r \) restricted horizontal spaces of dimension \( q-1 \). The numbers \( \mu_1, \ldots, \mu_r \) and \( X_{1 \ldots r} \) will be determined by the eigenvalue equation

\[ \zeta^{(N)}(\theta, \xi) \psi(\lambda) = \Lambda(\theta, \xi) \psi(\lambda) \] (4.24)

Notice that \( \psi \) is assumed to be independent of \( \theta \). This is reasonable since eq. (2.36) indicates that the family \( \zeta^{(N)}(\theta, \xi) \) may have common eigenvectors \( \psi \) for all \( \theta \). We have here

\[ \zeta^{(N)}(\theta, \xi) = A(\theta) + \xi r^{(2)} D(\theta) \] (4.25)

where

\[ \xi r^{(2)} D(\theta) = \sum_{\alpha = 2}^{1} D_{\alpha - \alpha}(\theta) \]

That is \( r^{(2)} \) is the trace in the restricted horizontal space (q-1 dimensional). Following the general strategy of the algebraic Bethe ansatz we apply either \( A(\theta) \) or \( D_{ab}(\theta) \) on \( \psi \). Then we push \( A(\theta) \) or \( D_{ab}(\theta) \) through the \( B_{ij}(\mu_1) \) to the right using the commutation relations (4.17)-(4.22).

When \( A(\theta) \) or \( D_{ab}(\theta) \) reaches \( \|1\rangle \) they reproduce it as eqs. (4.9) and (4.10) state. Since eqs. (4.17) and (4.19) possess two terms in the r.h.s. this procedure generates a lot of terms (2\( r \) in principle). We classify them in two types of terms: wanted and unwanted terms. Wanted terms are those containing the product

\[ B(\mu_1) \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \] (4.26)

unchanged. Unwanted terms are those where some \( B(\mu_j) \) in eq. (4.26) is replaced by \( B(\theta) \). We shall collect the unwanted terms and require them to have a vanishing sum. The wanted terms will be required to be proportional to \( \psi \) providing in this way an eigenvector and its eigenvalue.

From \( A(\theta) \psi(\mu_1, \ldots, \mu_r) \) one gets a wanted term by using repeatedly the first term of the r.h.s. of eq. (4.17). It reads

\[ \prod_{j=1}^{r} \zeta(\mu_j - \theta) \psi(\mu_1, \ldots, \mu_r) \] (4.27)

An unwanted term where \( B(\theta) \) replaces \( B(\mu_j) \) follows from the second term of the r.h.s. of eq. (4.17) in the commutation \( A(\theta) \) \( B(\mu_1) \) and the first term in the subsequent commutations \( A(\theta) B(\mu_j) \) \( \cdots \) \( r \). It reads
The computation of the rest of the unwanted terms is greatly simplified by the use of the properties of the product (4.26) under the actions of cyclic permutations of its factors \( B(\mu_j) \). One finds using repeatedly eq. (4.18)

\[
B_{i_1}(\mu_1) \cdots B_{i_r}(\mu_r) = R_{i_1 i_2}^{(2)} \, \cdots \, R_{i_r i_1}^{(r)} (\mu_r - \mu_1) R_{a_1 a_2}^{(r)} (\mu_r - \mu_1)
\]

or in a more compact notation

\[
B(\mu_r) \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) =
\]

\[
= B(\mu_r) \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \otimes B(\mu_r) \sum_{\alpha = 2}^{q} T_{[\alpha]^{(2)}} (\theta, \zeta) (\mu_r, \mu_r)
\]

(4.29)

where

\[
C_{[\alpha]}^{(r)} (\theta, \zeta) = \sum_{\alpha = 2}^{q} T_{[\alpha]^{(2)}} (\theta, \zeta) (\mu_r, \mu_r)
\]

(4.30)

and \( T_{[\alpha]^{(2)}}^{[\mu]} \) is given by eq. (4.7) and (2.34) with \( N = r \) and the indices \( a_i \)

running from 2 to \( q \). In the derivation of eq. (4.29) we also used eq. (4.21).

So, the cyclic permutation \( B(\mu_r) \rightarrow B(\mu_1) \) followed by the multiplication by the matrix

\[
M \equiv C_{[\alpha]}^{(r)} (\mu_r, \zeta)
\]

(4.31)

leaves the product \( B(\mu_r) \otimes \cdots \otimes B(\mu_r) \) invariant. We can analogously compute the unwanted term where \( B(\theta) \) replaces \( B(\mu_2) \) just applying this symmetry operation to eq. (4.28). This yields

\[
- \lambda_+ (\mu_r - \Theta) \sum_{\alpha = 2}^{q} T_{[\alpha]^{(2)}}^{[\mu]} (\mu_r - \mu_1) (\mu_r \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \otimes \cdots \otimes B(\mu_r) \otimes B(\mu_r) \otimes B(\mu_r) \otimes B(\mu_r) \otimes B(\mu_r) \otimes B(\mu_r)
\]

(4.32)

This completes the calculation of \( A(\theta) \psi \). Let us now compute the action of \( \text{tr}^{(2)} \, \phi(\theta, \zeta) \) on \( \psi \). As before, wanted and unwanted terms appear. The wanted term follows by using repeatedly the first term in the r.h.s. of
eq. (4.20) or eq. (4.22) when \( \text{tr} \, D^{(2)}(\theta) \) is commuted through the \( B(\mu, j) \) \((1 \leq j \leq r)\). It reads

\[
\sum_{r} \left[ C_{c_2}^{(r)} (\theta, \omega) \right] \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \sigma_j)} \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \omega_j)} \rho \left( \phi (\theta - \sigma_j) \right) \rho \left( \phi (\theta_1 - \omega_j) \right)
\]

The unwanted terms where \( B(\mu_1) \) is replaced by \( B(\theta) \) follow from the second term in the r.h.s. of eqs. (4.20) or (4.22) when commuting \( \text{tr}^{(2)} D^{(2)}(\theta) \) through \( B(\mu_1) \) and using the first one in the subsequent commutations

\[
\sum_{r} \left[ C_{c_2}^{(r)} (\mu, \omega) \right] \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \sigma_j)} \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \omega_j)} \rho \left( \phi (\theta - \sigma_j) \right) \rho \left( \phi (\theta_1 - \omega_j) \right)
\]

The sum of unwanted terms reads from eqs. (4.32) and (4.35)

\[
- \sum_{k=1}^{r} \left[ C_{c_2}^{(r)} (\theta, \omega) \right] \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \sigma_j)} \prod_{j=1}^{r} \frac{1}{\gamma_j (\theta - \omega_j)} \rho \left( \phi (\theta - \sigma_j) \right) \rho \left( \phi (\theta_1 - \omega_j) \right)
\]

Now, the eigenvalue condition (4.24) imposes that the sum of unwanted terms must vanish and that the sum of wanted terms must be proportional to \( \psi \). This yields the eigenvalue conditions

\[
C^{(r)}_{c_2} (\theta, \omega) \propto \Lambda_{c_2}^{(r)} (\theta, \omega)
\]
Eqs. (4.38) means that $X$ is an eigenvector of the transfer matrix $\tau^{[r]}(\theta, \mu^{(r)})$. So, we reduce the eigenvalue problem for the matrix $\tau^{[r]}(\theta, \mu^{(r)})$ acting on the tensor product of $N$ vertical spaces of dimension $q$ to the eigenvalue problem of the matrix $\tau^{(r)}(\theta, \mu^{(r)})$ acting on the tensor product of $r$ vertical spaces of dimension $q-1$. The change from $N$ to $r$ is inessential, the crucial simplification is the shift from $q$ to $q-1$. If $q = 2$ eq. (4.39) solves the problem since $\Lambda^{(2)}(\theta) = 1$ in this particular case. For $q > 2$ we can repeat the construction since $T_{[r]}(\theta, \mu^{(r)})$ and $R^{(2)}(\theta)$ have the same block structure than $T_{[r]}(\theta, \mu^{(r)})$ and $R^{(2)}(\theta)$ (eqs. (4.12) and (4.13)).

We set

$$
\tau^{[r]}(\theta, \mu^{(r)}) = \begin{pmatrix}
\Lambda^{(2)}(\theta, \mu^{(r)}) & B^{(2)}(\theta, \mu^{(r)}) \\
C^{(2)}(\theta, \mu^{(r)}) & \Lambda^{(2)}(\theta, \mu^{(r)})
\end{pmatrix},
$$

where $\mu^{(1)} = \mu^{(1)}(\theta)$ for $1 \leq \ell \leq p_1 = r$. We write for $X^{(1)} = X$ an ansatz analogous to eq. (4.23)

$$
X^{(r)} = \bigotimes_{k=1}^{p_k} B^{(1)}(\mu^{(r)}, \zeta^{(r)}) \otimes \ldots \otimes B^{(p_k)}(\mu^{(r)}, \zeta^{(r)}) \Big| \phi^{(r)} \Big>,
$$

(4.40)

Here $(\zeta^{(r)}) = \bigotimes_{k=1}^{p_k} |\zeta^{(r)}\rangle$ and $|\zeta^{(r)}\rangle$ is a $(q-1)$ component vector where the first component equals one and the rest vanishes.

As it is the whole argumentation from eq. (4.24) to eq. (4.40) can be repeated as many times as necessary to reduce the dimension of the vertical spaces up to one. Then the eigenvalue problem of $\tau^{[r]}(\theta)$ is solved in the sense that it reduces to the following set of algebraic equations [43]

$$
\Lambda^{(r)}(\theta, \zeta^{(r)}; \mu^{(r)}) = \prod_{j=1}^{p_r} \mathcal{g}(\mu^{(r)}_j - \theta) \Bigg[ \prod_{j=1}^{p_{r-1}} \mathcal{g}(\theta - \mu^{(r)}_j) \bigwedge \Lambda^{(r-1)}(\theta, \zeta^{(r-1)}) \bigwedge \Lambda^{(r-1)}(\theta, \zeta^{(r-1)}) \Bigg] (4.41)
$$

$$
\Lambda^{(r)}(\theta, \zeta^{(r)}) = \Lambda^{(r)}(\theta) = 1 \quad (4.42)
$$

$$
\Lambda^{(r)}(\theta, \zeta^{(r)}) = \Lambda^{(r)}(\theta, \zeta^{(r)}) = 1 \quad (4.43)
$$

Here $\mu^{(r)}_j = \mu^{(r)}_j(\theta)$ for $1 \leq j \leq p_r$, $\zeta^{(r)}(\theta, \zeta^{(r)}) = \Lambda^{(r)}(\theta, \zeta^{(r)}) \Lambda^{(r)}(\theta, \zeta^{(r)})$. As it was the case for $\Lambda^{(r)}(\theta)$ of the six vertex model [eq. (3.21)] the functions $\Lambda^{(r)}(\theta, \zeta^{(r)})$ are not singular at the points $\theta = \mu^{(r)}_j(\theta)$ for $1 \leq j \leq p_r$.
as it must be since the finite dimensional matrix \( c^{[N]}(\theta) \) is an analytic function of \( \theta \). In fact one can use this property of \( \wedge N \sim \mu^{k-1} \sim \nu \) to derive eqs. (4.43) from eqs. (4.41).

The generators of the continuous symmetry \( D^{q-1} \) (see eq. (4.3)) of the R-matrix naturally commute with the transfer matrix \( c^{[N]}(\theta, \alpha) \). We can take as generators \( S_k = E_k - E_{k+1} \)

where
\[
E_k = \sum_{s=1}^{N} c^{(s)}_{\lambda}(\theta, \alpha) \quad 1 \leq k \leq q \quad (4.44)
\]

They satisfy
\[
[ S_k, c^{[N]}(\theta, \alpha) ] = 0 \quad (4.45)
\]

For \( q = 2 \) this reduces to the third component of the spin.

Eq. (2.41) gives for the commutators of \( E_k \) and the creation operators \( B_i(\theta) \)
\[
[ E_{i}, B_{i}(\theta) ] = - B_{i}(\theta) \quad 2 \leq i \leq q \quad (4.46)
\]

\[
[ E_{k}, B_{i}(\theta) ] = S_{ik} B_{i}(\theta)
\]

Applying \( E_1 \) to our state (4.23) gives
\[
E_1 \Psi(\alpha) = (N - \kappa_1) \Psi(\alpha)
\]

where we used eq. (4.46) and \( E_k \parallel 4 \rangle = N \parallel 4 \rangle \).

Let us now apply \( E_2 \) to \( \Psi(\alpha) \). We find with the help of eq. (4.46)
\[
E_2 \Psi(\alpha) = \left[ E_{2}^{(1)} \ X^{(1)} \right] \left[ B_{2}^{(1)}(\theta) B_{3}(\alpha) \cdots B_{q}(\alpha) \right] \parallel 4 \rangle
\]

where \( E_{2}^{(1)} \) is the restriction of the operator \( E_2 \) to a \((q-1)\) dimensional vertical space where the first component of the original vertical space has been deleted. By analogy with eq. (4.46), we find
\[
[ E_2, B_{j}^{(1)}(\theta) ] = - B_{j}^{(1)}(\theta), \quad 3 \leq j \leq q
\]

Therefore
\[
E_{2}^{(1)} \ X^{(1)} = (\kappa_1 - \kappa_{2}) \ X^{(1)}
\]

and
\[
E_2 \Psi(\alpha) = (\kappa_1 - \kappa_{2}) \Psi(\alpha)
\]

This procedure works for all the generators \( E_k \) giving the eigenvalues
\[
E_k = \kappa_{k-1} - \kappa_k \quad 1 \leq k \leq q \quad (4.47)
\]

and
\[
S_k = \kappa_{k-1} + \kappa_{k+1} - 2 \kappa_k
\]
Closed expressions for the eigenvalues $\Lambda_{\kappa}^{(k)}$ can be only derived in the thermodynamic limit $N = \infty$ where eqs. (4.43) become easily solvable. It is convenient to set

$$
\Lambda_{\kappa}^{(k)} = \lambda_{\kappa}^{(k)} - k \gamma / 2 ; \quad 1 \leq \kappa \leq q-1 , \quad 1 \leq k \leq p_{\kappa} ,
$$

and take the logarithm of eqs. (4.43). One finds

$$
\sum_{\ell = 1}^{p_{x+1}} \phi_{\ell} \left( \lambda_{\ell}^{(k)} - \lambda_{\ell}^{(k+1)} , \frac{\gamma}{2} \right) = \sum_{\ell = 1}^{p_{x}} \phi_{\ell} \left( \lambda_{\ell}^{(k)} - \lambda_{\ell}^{(k)} , \gamma \right) + \sum_{\ell = 1}^{p_{x-1}} \phi_{\ell} \left( \lambda_{\ell}^{(k)} - \lambda_{\ell}^{(k+1)} , \frac{\gamma}{2} \right) = 2 \pi \Gamma_{\ell}^{(k)} j
$$

(4.49)

Here (cfr. eq.(3.23))

$$
\phi_{\ell} (\gamma , \omega) = i \log \frac{\sin (\gamma + i \omega)}{\sin (\gamma - i \omega)}
$$

(4.50)

The $\Gamma_{\ell}^{(k)}$ are half-odd integers. We recall that $\lambda_{\ell}^{(0)} = -i \phi ; \quad 1 \leq \ell \leq p_{x} = N$ in eq. (4.49) and $p_{x+1} = 0$.

Up to now we have studied hyperbolic regimes (I and III) for this $q(2q-1)$ vertex model. Besides these regimes, a trigonometric one (II) follows by letting $\gamma \rightarrow i \gamma$ and $\theta \rightarrow i \theta$ in eq. (4.1). That is

$$
\mathcal{R}_{\kappa \ell}^{(k)} (\theta) = \frac{\sin \gamma}{\sin (\gamma - \theta)} \delta_{\kappa \ell} \delta_{bd} \delta_{\kappa \ell} + \frac{\sin \theta}{\sin (\gamma - \theta)} \delta_{bc} \delta_{\kappa \ell}
$$

(4.51)

In this regime II the weights are complex. When $q = 2$ one can get rid of the complex phases by a gauge transformation and one obtains the real weights (3.1) for regime II. This does not seem to be the case here neither in the elliptic $Z_{q} \otimes Z_{q}$ vertex model. The solution of the model in the regime (4.51) is nevertheless very interesting. A third regime of ferroelectric character is also present (see p.62).

The eigenvector construction (4.23)-(4.43) holds in all regimes I-III. One must simply use the appropriate form of $\phi (\gamma , \omega)$ for each regime.

$$
\phi (\gamma , \omega) = i \log \frac{\sin (\gamma + i \omega)}{\sin (\gamma - i \omega)}
$$

(4.52)

for regime II (cfr. eq.(3.23)).

Let us now describe the solution of the nested Bethe Ansatz equations (4.49) (NBAE) in the $N = \infty$ limit for the antiferroelectric cases I and II. The roots tend to have a continuous distribution (as in sec. III) with densities

$$
\phi_{\ell} (\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N \left( \lambda_{\ell+1}^{(2)} - \lambda_{\ell}^{(2)} \right)}
$$

(4.53)

The half-odd integers $l_{j}^{(k)}$ display in monotonous sequences for the ground state. In general we can have jumps in the sequences.
\[ I^{(k)}_{j_k+1} - I^{(k)}_{j_k} = k + \sum_{j = 1}^{N_h^{(k)}} \delta_{j_k} \delta_{n^{(k)}} \]  

(4.54)

Here we assume \( N_h^{(k)} \) holes at the \( k^{th} \) level of the NBA \((1 \leq j_k^{(k)} \leq N_h^{(k)})\). This generalizes eq.(3.25). Now we have \( q-1 \) different branches of exciton states. In addition, there are \( q \) degenerate states for the maximum eigenvalue of \( \tilde{\tau}^{(N)}(\theta) \) in the \( N = \infty \) limit [41]. They are connected by the \( Z_q \) symmetry and their monotonous sequences differ by integers

\[ I^{(k)}_{j,\tilde{q}} = I^{(k)}_{j,\tilde{q}} + \tilde{q}, \quad 1 \leq \tilde{q} \leq q-1 \]  

\[ I^{(k)}_{j+\tilde{q},\tilde{q}} - I^{(k)}_{j,\tilde{q}} = 1 \]  

(4.55)

In the same way as eq.(3.22) lead to the integral equation (3.34) for \( N = \infty \), the system of equations (4.49) yield the following system of integral equations when \( N \to \infty \):

\[ \sigma_0(\lambda) - \frac{1}{\tilde{N}} \sum_{\xi = 1}^{\tilde{N}} \int_{\lambda - \Delta}^\lambda K^{(e)}_{\xi} (\lambda - \mu) \sigma_0(\mu) \, d\mu = \frac{\delta_{01}}{2\pi} \Delta \sigma_0(\lambda) \]  

(4.56)

\[ + \frac{1}{\tilde{N}} \sum_{\xi = 1}^{\tilde{N}} \left\{ \sum_{j = 1}^{N_h^{(e)}} K^{(e)}_{\xi} (\lambda - \mu_j) + \delta_{\xi} \sigma^{(e)}_0 (\lambda - \mu_0) \right\} \]

Here \( \tilde{N} = \pi / 2 \) or \( + \infty \) in the regimes I and II, respectively. We set

\[ \sigma^{(e)}_0 (\lambda) = \sigma^{(e)}_0 (\lambda) + \frac{1}{\tilde{N}} \sum_{\xi = 1}^{\tilde{N}} \sigma_0(\lambda - \mu_0) \]  

(4.57)

The \( (\xi, \epsilon, \tilde{\xi}, \tilde{\epsilon}) \) stand for the complex roots of the NBAE. They always appear in complex conjugated pairs. The kernel \( K^{(e)}_{\xi}(\lambda) \) in eq. (4.56) reads

\[ 2\pi K^{(e)}_{\xi}(\lambda) = \phi^{(e)}(\lambda, \frac{\pi}{\tilde{N}}) \left( \delta_{\xi,\tilde{\xi}} + \delta_{\xi,\tilde{\xi}} \right) \]  

(4.58)

Eqs.(4.56) are easily solved by Fourier expansions. The solution reads

\[ \sigma_0(\lambda) = \sigma_0^{(e)}(\lambda) + \frac{1}{\tilde{N}} \left[ \sigma_0^{(e)}(\lambda) + \sigma_0^{(e)}(\lambda) \right] \]  

(4.59)

As for the six-vertex case [eq.(3.39)] \( \sigma_0^{(e)}(\lambda) \) and \( \sigma_0^{(e)}(\lambda) \) stand for the holes and complex roots contributions, respectively. Setting

\[ \sigma^{(e)}_0(\lambda) = \sum_{\mu \in \mathbb{Z}} e^{2i\pi \lambda} \frac{\sigma^{(e)}_0(\lambda)}{2\pi} \]  

\text{regime I}

(4.60)
\[ \sigma_e(\lambda) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\lambda} \hat{\sigma}_e(k) \text{ regime II} \]

We find for the ground state (vacuum)
\[
\hat{\sigma}^V_e(m) = 1 \frac{\delta_k [\delta_m(m-e) \chi_m(\gamma+\frac{\pi}{2})]}{\Delta_k [\chi_m(\gamma+\frac{\pi}{2})]} \text{ regime I} \]
\[ \hat{\sigma}_e^V(\varphi) = 2 \left(1 - \frac{\varphi}{\tau} \right) \]  
(4.61)
\[
\hat{\sigma}_e^V(k) = \frac{\delta_k [(\gamma-k+1)/2]}{\Delta_k [\chi_m(\gamma+\frac{\pi}{2})]} \text{ regime II} \]  
(4.62)

This gives
\[ \lim_{N \to \infty} \frac{p^V_e}{N} = \int_{-A}^{A} d\lambda \sigma_e^V(\lambda) = 1 - \frac{\varphi}{\tau} \text{ regimes I and II} \]  
(4.63)

Therefore the spins \( s_k \) (\( 1 \leq k \leq q - 1 \)) vanish in the ground state.

Let us now compute the transfer matrix eigenvalue \( \Lambda_N(\theta) \). For \( |\theta| < \gamma/2 \), the first term dominates in eq.(4.41) for \( k = 1 \) (recall that \( \Lambda(\theta) = \Lambda_{[1]} \)) and we find for the free energy
\[ f(\theta, \gamma, 1) = -\lim_{N \to \infty} \frac{1}{N} \log \mathcal{N}(\theta) = -i \int_{-A}^{A} d\lambda \sigma^V_e(\lambda) \phi(\lambda + i\theta, \gamma/2) \]
(4.64)

This integral gives in the regimes I and II using eqs. (4.60) - (4.62)
\[ \mathcal{F}(\theta, \gamma, 1) = 2\theta(1 - \frac{\gamma}{\tau}) + 2 \sum_{m \neq 0} \frac{e^{-\gamma}}{m} \frac{\Delta_k [\chi_m(\gamma-1)]}{\Delta_k [\chi_m(\gamma+\frac{\pi}{2})]} \]  
(4.65)
\[ \mathcal{F}(\theta, \gamma, 1) = 2 \int_{0}^{\gamma} dx \frac{\Delta_k (2x\theta)}{\Delta_k (\pi x)} \frac{\Delta_k [\chi_m(\gamma-x)]}{\Delta_k [\chi_m(\gamma+\frac{\pi}{2})]} \]  
(4.66)

Notice that \( f(\theta, \gamma, 2) \) coincides with the six-vertex free energy [eqs.(3.48) - (3.50)].

The solution of eq.(4.56) can be appropriately written in general in terms of the resolvent \( R = (1 - K)^{-1} \). That is
\[ R_{\xi\lambda}(\theta) = \sum_{\gamma=1}^{q-1} \int_{-A}^{A} d\gamma K_{\lambda\gamma}(\theta, \gamma) R_{\gamma\lambda}(\gamma) = \delta_{\xi\lambda} \delta(\lambda) \]  
(4.67)

One finds upon Fourier transformation,
where we used eqs. (4.56) and (4.67). It must be noticed that $\sigma \xi^h(\lambda)$ is a continuous function of $\lambda$ [without $\delta$-like singularities]. The vacuum density of roots $\sigma \xi^v(\lambda)$ can be also expressed in terms of the resolvent as

$$\sigma^v_\lambda(\lambda) = \sum_{\ell = 1}^{N-1} \int_{-1}^{1} \frac{d\mu}{2\pi} \phi'(\mu, \frac{\pi}{2}) R_{\xi^v}(\lambda - \mu),$$  \hspace{1cm} (4.72)$$

where eq. (4.59) and (4.67) were used.

Let us finally compute the transfer matrix eigenvalues for a state with a hole at $\Theta_h$ in the $l^{th}$ branch. As before [eq. (3.51)] this eigenvalue behaves for large $N$ as

$$\lambda N \xi_{\Theta_h}(\Theta) = \lambda_0(\Theta) e^{\pi \ell} \left[-i \phi(\Theta, \Theta_h)\right] + o(e^{-\pi \ell}),$$  \hspace{1cm} (4.73)$$

where $\lambda_0 = \exp(-N\ell(\Theta, \psi, \omega))$ and $g_\ell$ follows from eqs. (4.64), (4.59) and (4.71)

$$g_\ell(\Theta, \Theta_h) = \sum_{\ell = 1}^{N-1} \int_{-1}^{1} \phi(\lambda, \Theta, \frac{\pi}{2}) d\lambda R_{\xi^v}(\lambda - \Theta),$$  \hspace{1cm} (4.74)$$

A look to eqs. (4.72) and (4.74) shows that

$$g_\ell(\Theta, \Theta_h) = \delta(\Theta - \Theta_h) + \frac{K_\Theta(x)}{2\pi},$$  \hspace{1cm} (4.75)$$

in the trigonometric and hyperbolic regimes respectively. Using eq. (4.58), (3.36), (3.37), (4.67)-(4.69) yield as explicit solutions:

$$\hat{R}_{\xi^v}(m) = \frac{\exp(i m \pi) \eta_0(\frac{m \pi}{N})}{\eta(\frac{m \pi}{N})},$$  \hspace{1cm} (I)$$

$$\hat{R}_{\xi^v}(x) = \frac{\exp(2 \pi i x) \eta_0(\frac{\pi}{N})}{\eta(\frac{\pi}{N})},$$  \hspace{1cm} (II)$$

Now, it is easy to express the hole contributions to the roots density $\sigma(\lambda)$ as

$$\sigma^h(\lambda) = \sum_{\ell = 1}^{N-1} \left\{ s_{\ell^1} \delta(\lambda - \Theta_h^{(1)}) - R_{\xi^v}(\lambda - \Theta_h^{(1)}) \right\}$$  \hspace{1cm} (4.71)$$
It must be noticed that the physical excitations eigenvalues have a
structure very similar to the "bare" ones in the gapless regime. Bare
excitations are those over the reference state $|\Omega\rangle$ [cf. eq. (4.41)]. Only a
finite renormalization of the variables in the function $\phi(z, \alpha)$ takes place.
Relations of the type (3.52) and (4.75) between eigenvalues of $\tau(\theta)$
associated to holes and the function $Z_\theta(\lambda)$ are rather general in models
solvable by Bethe Ansatz.

Since the R-matrix (4.1) fulfills the regularity condition (2.20)
[see eq. (4.2)] one gets a quantum Hamiltonian for a chain of SU(q) spins
from the logarithmic derivative of $\tau(\theta)^N$ at $\theta = 0$. We set [43] (cf.
Appendix)

$$
\tilde{H}(\eta) = -\frac{\hbar Y}{2} \log \left[ \frac{\tau(\theta, 2)}{\tau(0, 2)} \right] \bigg|_{\theta = 0} - N c h Y \left( \eta + \frac{iy}{\hbar} \right) \quad (4.80)
$$

This yields

$$
\tilde{H}(\eta) = -\sum_{s=1}^{N} \left\{ \sum_{r \neq s}^{N} e^{(s)} e^{(s+1)} + \cos \theta \sum_{r=1}^{N} e_{rr} e^{(s+1)} +
+ n Y \sum_{r, s \neq 1}^{N} \frac{\hbar^2}{2} \left( e^{(s)} \right) e^{(s+1)} \right\} \quad (4.81)
$$

For $q = 2$ we recover the XXZ Heisenberg model. For real $Y$, $H_q$
describes a ferromagnetic interaction. For $3m \gamma = \pi$, $H_q$
descibes an

where

$$
Z_\theta(\lambda) = \int_{-\infty}^{\lambda} \frac{d\tau}{2\pi i} e^{-\frac{i}{2\pi} \lambda \tau} \quad (4.76)
$$

and

$$
K(\theta, \lambda) = -\frac{1}{2} \frac{\lambda^2 - \frac{\pi}{2}}{\lambda - \frac{\pi}{2}},
$$

in the trigonometric regime ($L$).

Moreover, in this regime $Z_\theta(\lambda)$ expresses in terms of elementary functions

$$
Z_\theta(\lambda) = \frac{1}{2\pi} \left| 1 + 2 \sum_{m=1}^{\infty} \frac{\cos(2\pi m \lambda)}{\sin(\pi n \theta)} \right| \quad (4.77)
$$

In the hyperbolic regime ($L$) we find [41]

$$
\sigma_\theta(\lambda) = \frac{1}{\pi} \left| 1 + \frac{\pi}{2} \sum_{m=1}^{\infty} \frac{\cos(2\pi m \lambda)}{\sin(\pi n \theta)} \right| \quad (4.78)
$$

This function and its primitive $Z_\theta(\lambda)$ can be expressed in terms of elliptic functions:

$$
Z_\theta(\lambda) = \frac{1}{2\pi i} \log \left( \frac{\delta_\gamma (\lambda - \frac{\gamma}{\pi})}{\delta_\gamma (\frac{\lambda}{\pi} - \frac{\gamma}{\pi})} \right) +
+ \left( \frac{3}{2} - \frac{\gamma}{\pi} \right) (\pi + 2 \lambda) \quad (4.79)
$$

where $\theta_q(z/\tau)$ can be found in ref. [54].

We see from eqs. (4.77) and (4.79) that the trigonometric regime is
gapless [ $g(Q, -\infty) = 0$ ] whereas a non-zero gap exists in the hyperbolic
regime for $Y > 0$. This was already the case in the six-vertex model (see sec. 3).
V. THE LIGHT-CONE LATTICE APPROACH.

This approach starts by discretizing the two-dimensional Minkowski space-time in light-cone coordinates \( x_± = x \pm t \). Space time is thus approximated by a diagonal lattice. This discretization scheme turns to be an useful regularization method for integrable quantum field theories since they become naturally connected with integrable vertex models in their scaling limit[23].

The sites in the light-cone lattice (fig. 15) are considered as world events. Each site (event) is joined by light-like links to its four nearest neighbours along \( x_+ \) and \( x_- \). There diagonal links are possible world lines for the propagation upwards in time of "bare" massless particles. Particles on right-oriented (R) and left-oriented (L) links are called respectively right and left-movers.

One then associates microscopic amplitudes to each site (world event) where two oppositely oriented world lines cross. These amplitudes describe the different processes that can take place, and must verify general invariance properties like unitarity.

Let us start for the simplest case where each link describes only two different configurations. We assume that these two cases correspond to the presence or absence of a bare fermion without internal degrees of freedom. In general, there can be 16 different amplitudes per site antiferromagnetic interaction. This corresponds to the antiferroelectric regime I. In the regime II \( H_q \) becomes non-hermitian. In the limiting cases \( \gamma = 0 \) and \( \gamma = \pm 1 \) we recover the hamiltonian of ref. [26].

It is instructive to write \( H_q \) in terms of spin operators. One finds for \( q = 3 \)

\[
H(3) = -\frac{1}{2} \sum_{\gamma=1}^{3} \left\{ \sum_{x=1}^{2} S_x^{(\gamma)} S_x^{(\gamma+1)} + \frac{1}{2} \sum_{x',x'} T_{xx'}^{(\gamma)} T_{xx'}^{(\gamma+1)} \right\} + \\
+ (\Delta \gamma - 1) \left[ S_3^{(\gamma)} S_3^{(\gamma+1)} + \frac{3}{4} T_{SS}^{(\gamma)} T_{SS}^{(\gamma+1)} \right] + \\
+ \frac{\Delta \gamma}{2} \left[ S_3^{(\gamma)} T_{SS}^{(\gamma+1)} - T_{SS}^{(\gamma)} S_3^{(\gamma+1)} \right]
\]

(4.82)

where \( S_\alpha \) are the spin-one matrices :

\[
S_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & 1 \\ 1 & 0 & i \\ 0 & 1 & 0 \end{pmatrix}, \quad S_\times = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}
\]

\[
S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

and

\[
T_{XX'} = S_X S_{X'} + S_{X'} S_X - \frac{\gamma}{3} S_{XX'}
\]

This can be considered as an integrable anisotropic interaction with bilinear plus biquadratic exchange[44].
corresponding to the 16 configurations (occupied/empty) of the four links joining there. Only $U(1)$ invariant microscopic amplitudes will be considered here such that the number of particles is conserved at each site. $U(1)$ transformations act on the link states by

$$
|0\rangle \rightarrow e^{\Delta} |0\rangle \\
|1\rangle \rightarrow e^{-\Delta} |1\rangle 
$$

(5.1)

where $|0\rangle$ is (empty) and $|1\rangle$ is (occupied). Therefore, there are only six non-zero amplitudes as depicted in fig. 16. The correspondence with the general (non-symmetric) six-vertex model is evident.

Of course, space-time translational invariance implies that the amplitudes are the same in all sites of the lattice. It is natural (and causes no loss of generality) to set the nothing-to-nothing amplitude to be 1. Unitarity then requires

$$
\Omega \Sigma^+ = 1 , \quad \Sigma = \begin{pmatrix} \omega_3 & \omega_5 \\ \omega_6 & \omega_4 \end{pmatrix} , \quad |\omega_i|^2 = 1
$$

(5.2)

While $\omega_3$ and $\omega_4$ are naturally interpreted as amplitudes for free propagation (being therefore related to kinetic energies in the continuum limit), $\omega_5$ and $\omega_6$ play the role of mass terms since they couple right and left movers.

Symmetry under parity transformation holds if

$$
\omega_3 = \omega_4 = b , \quad \omega_5 = \omega_6 = c
$$

(5.3)

This corresponds now to an integrable six vertex model. Unitarity now reads

$$
|b|^2 + |c|^2 = 1 , \quad b \overline{c} + \overline{b} c = 0
$$

(5.4)

One can organize these microscopic amplitudes at a site into a 4x4 unitarity "bare" S-matrix

$$
R^{\alpha'}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}
$$

(5.5)

where $\omega = \omega_2$ and $\alpha, \beta, \alpha', \beta'$ take the values 0 or 1 for empty or occupied links like in eq. (5.1).

The amplitude for a global process, from a given state at $t = t_0$ to another given state at a later time, is obtained by summing over the amplitudes of all allowed vertex configurations compatible with initial and final conditions and with boundary conditions. Each of these is given by a product of microscopic amplitudes $\omega_i$. It clearly corresponds to the sum over all possible paths of an arbitrary, but constant in time number of particles. At any instant, a particle can move to the left or to the right at the speed of light. We are thus dealing with a discretization of Feynman path integral for fermions.
It is convenient to parametrize $b$ and $c$ following the constraints (5.4) as

$$b = b(\theta, \gamma) = \frac{\alpha^2 \theta}{\sin (\theta + \gamma)}, \quad c(\theta, \gamma) = \frac{\sin \gamma}{\sin (\theta + \gamma)}$$

$$0 < \theta < \omega, \quad 0 < \gamma < \pi$$

(5.6)

This makes (5.5) identical the six-vertex model $R$ matrix (3.1) up to an overall factors $\sin (\theta + \gamma)$ and a redefinition of $\theta \rightarrow i \theta$ when $\omega = 1$. Actually $\omega = 1$ corresponds to a six-vertex model in an external field.

Let us now describe the operator formalism for the light-cone approach[23]. The unit evolution operators in the light-cone direction (R or L) are given by simply juxtaposition of the microscopic $S$-matrices (5.5) at the same horizontal level. That is

$$U_R(\theta) = \begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12}
\end{array}$$

(5.7)

$$U_L(\theta) = \begin{array}{cccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12}
\end{array}$$

(5.8)

Here $N$ is assumed to be even and $\alpha_{j+N} = \alpha_j$. Notice that there are no summations in eq. (5.7)-(5.8). One can now define the two light-cone lattice evolution generators as

$$H \pm P = \frac{2 \sqrt{\alpha}}{\alpha} \log U_R(\theta)$$

(5.9)

where $H$ and $P$ stand for lattice hamiltonian and momentum and $\alpha$ is the lattice spacing.

Eq. (5.9) is extremely suggestive since it provides a lattice version of field-theoretic $H$ and $P$ in terms of lattice vertex transfer matrices $U_R$ and $U_L$. The natural question is now to find the eigenvectors of them. It will be shown now that this is possible using the techniques of sec. III and IV (and their generalizations) provided $R(\theta)$ verifies the YB algebra (2.20)[23].

Let us consider the row-to-row transfer matrix $\tau_N(\theta, \gamma)$ of eq. (2.33) [with $\gamma_1 = \ldots = \gamma_N$] with the particular choice of inhomogeneities (recall $N$ = even)

$$\tau_{k} = (-1)^{k+1} \theta$$

$$\tau_{N} = (\theta, -\theta, \ldots, (-1)^{k+1} \theta, \ldots, \theta, -\theta)$$

(5.10)

It then follow from eqs. (2.33) and (2.35) using eq. (2.20) that

$$\tau(\theta, \theta) = U_L(\theta)$$

(5.11)

$$\tau(-\theta, \theta) = U_R(\theta)^{\dagger}$$

(5.12)

Let us check (5.12). Setting (5.10) in (2.23)-(2.35) yields

$$\tau(\theta, \theta) = \sum_{\beta_1 \ldots \beta_N} \delta_{\beta_1 \beta_N} \delta_{\beta_2 \beta_2} R(\theta)$$

$$= \sum_{\beta_1 \ldots \beta_N} \delta_{\beta_1 \beta_N} \delta_{\beta_2 \beta_2} \ldots \delta_{\beta_{N-1} \beta_{N-1}} \delta_{\beta_N \beta_1}$$

$$= 1$$
after using eq. (2.20) repeatedly.

The key relations (5.9), (5.11)-(5.12) connect the lattice $H$ and $P$ with the row-to-row transfer matrices whose eigenvectors and eigenvalues can be constructed by the algebraic Bethe Ansatz developed in sec. III and IV. The light-cone or diagonal-to-diagonal transfer matrices resulted to be particular cases of the inhomogeneous row-to-row transfer matrices. The commutativity properly (2.38) gives in addition

\[
\begin{align*}
\left[ \mathcal{Z}(\lambda, \Theta), \mathcal{U}_L(\Theta) \right] &= 0, \quad \forall \lambda \in \mathbb{C} \\
\left[ \mathcal{Z}(\lambda, \Theta), \mathcal{U}_R^\dagger(\Theta) \right] &= 0 \\
\left[ \mathcal{U}_L(\Theta), \mathcal{U}_R(\Theta) \right] &= 0
\end{align*}
\tag{5.14}
\]

One can consider the infinite sequence of commuting operators ($1 \leq K < \infty$)

\[
C_K = \alpha_1^K \lambda \frac{\partial}{\partial \lambda} \mathcal{Z}(\lambda, \Theta) \bigg|_{\lambda = \Theta}
\tag{5.15}
\]

They all commute with $\mathcal{U}_L(\nu)$, $\mathcal{U}_R^\dagger(\nu)$ and with each other.

Let us now consider the continuum limit ($\alpha \to 0$) of the lattice models through eq. (5.9). The ground state of $\mathcal{Z}(\lambda, \Theta)$ corresponds just to the physical vacuum (filled Dirac sea) of the QFT defined by $H$ and $P$. The particle states follow from the lowest excitations. Since a factor $a^{-1}$ appears in $H \pm P$ [see eq. (5.9)] only gapless models yield finite energy states in the scaling limit. Moreover, in order to compute the energy and momentum in the scaling limit it is enough to know the eigenvalues of $\mathcal{Z}(\pm \Theta, \tilde{\Theta})$ close to the bottom of the spectrum. The low-lying excitations are associated to holes and complex solutions with large (real) rapidity. Moreover, their eigenvalues normalized to the vacuum ones [as in eq. (3.71)] are independent of the inhomogeneities.

Let us start by the fermion model (with $\omega = \tau$) associated to the six-vertex models (eqs. (5.1)-(5.6)). The excitation spectrum is given in sec. III in terms of the function $g(\Theta)$ (eqs. (3.51)-(3.53)). Combining eq. (3.53) with (5.9) yields

\[
\mathcal{E} \pm \mathcal{P} = \pm \frac{g(\pm \nu)}{\alpha}, \quad \forall \nu \in \mathbb{R}
\tag{5.16}
\]

Let us start by a hole excitation. One finds for large $\nu$ from eq. (3.51) and (3.57)

\[
g(\pm \nu) = \pm 2 e^{\pm \frac{\pi \nu}{\sqrt{\nu}}} e^{-\frac{\pi \nu}{\sqrt{\nu}}} + \alpha(e^{-\frac{2 \pi \nu}{\sqrt{\nu}}})
\tag{5.17}
\]

after discarding an irrelevant $\pi$ (it does not contribute to the eigenvalue of $\mathcal{Z}(\Theta)$ since the holes appear always by pairs). One finds a relativistic spectrum provided $\nu \to \infty$ when $\alpha \to 0$ keeping fixed the renormalized mass.
\[ m = \frac{\eta}{a} e^{-\frac{\pi \nu}{Y}} \]  

(5.18)

The dispersion law results
\[ \epsilon = m \cos \frac{\pi \Theta}{Y} \]
\[ p = m \sin \frac{\pi \Theta}{Y} \]  

(5.19)

So, \( \pi \Theta / Y \) is the physical rapidity of the particle [see eq. (2.5)]. Besides these holes that are identified with the fermions of the massive Thirring model\([23,42]\) one finds the string solutions (3.45)-(3.48). They provide relativistic particles in the same scaling limit (5.18) with masses
\[ \eta \gamma = 2m \sin \left[ \frac{\pi n}{2Y} \right], \quad 1 \leq n \leq \left[ \frac{\pi}{{\pi - Y}} \right] - 1 \]  

(5.20)

as follows from eqs. (5.16) and (3.51). This set of particles are fermion-antifermion bound states. They relate semiclassically to the breather of sine-Gordon as the fermions (or holes) (5.19) correspond to sine-Gordon solitons.

The preceding exposition of the light-cone lattice method applies to all gapless vertex models. In ref. (31) the models with rational R-matrices associated to simple Lie algebras are analysed. The model of section IV is also considered in its gapless regime.

Within this light-cone approach it is possible to construct explicitly the canonical bare fields on the lattice and to show that in the scaling limit (5.18) the massive Thirring model emerges\([23]\).

One introduces lattices fermion fields \( \psi_{R,n} \) and \( \psi_{L,n} \). They are associated to the links stemming upwards from each site at a fixed time (see fig. 17). They satisfy usual anticommutations rules

\[ \{ \psi_{A,m}, \psi_{B,n} \} = 0 \quad , \quad A, B = R, L \]
\[ \{ \psi_{A,m}, \psi_{B,n}^+ \} = s_{m-n} s_{m-n}^{-} , \quad 1 \leq m, n \leq \mathcal{N} \]  

(5.21)

\( \psi_{R,n} \) and \( \psi_{L,n} \) can be assembled in a two-component spinors. In this representation \( \gamma_5 \) is obviously diagonal since chiral rotations act locally on the \( \psi \)'s.

\[ \psi_{R,n} \rightarrow e^{i\xi} \psi_{R,n} \quad \psi_{L,n} \rightarrow e^{-i\xi} \psi_{L,n} \]

These lattice fermions are quite different from Kogut-Susskind fermions. In our case the species doubling is avoided thanks to the non-locality (on the lattice) of the hamiltonian (5.9). To simplify the notation we write

\[ \psi_{R,m} = \psi_{2m} \quad , \quad \psi_{L,n} = \psi_{2m+1} , \quad 1 \leq m \leq \mathcal{N} \]  

(5.22)

so that eq. (5.21) is replaced by
\{ \psi_m, \psi_m^* \} = 0 \quad \{ \psi_m, \psi_m^t \} = \delta_{mm} \quad (5.23)

Consider now the bilocal, unitary, even operators

\[ R_{nm} = 1 + b \ K_{nm} + (c-1) \ K_{nm}^2 + (\omega - 1) \ (\psi_m^t \psi_m + \psi_m \psi_m^t) \quad (5.24) \]

where

\[ K_{nm} = \psi_n^t \psi_m + \psi_m^t \psi_n \quad (5.25) \]

and \( b, c \) and \( \omega \) are given by eq. (5.2) and (5.5). The matrix elements \( R_{nm} \) in the bare representation \( \| \chi \rangle \) read

\[ \langle \chi | R_{nm} | \chi' \rangle = R_{nm}^\chi \prod_{n} \xi_{n}^\chi \left( A_{n}^{-1} A_{n}^{-1} \right) \quad (5.26) \]

where as usual

\[ |0, \ldots, 1_{n_{1}}, \ldots, 1_{n_{K}}, \ldots, 1_{n_{T}} \rangle = \psi_{n_{1}}^{t} \psi_{n_{2}}^{t} \ldots \psi_{n_{T}}^{t} |0 \rangle \]

with \( n_{1} < n_{2} < \ldots < n_{H} \).

The second quantized representation (5.24)-(5.26) for \( R \) allows to write the light-cone transfer matrix \( U_{R} (\nu) \) in second-quantized language using eq. (5.7)-(5.8). Now it is easy matter to derive the lattice equations of motion for the fermion operators \( \psi_{n} \) and \( \psi_{n}^{+} \). One finds \[ U_{R} \psi_{2n-1} \ U_{R}^{+} = U_{L} \psi_{2n} \ U_{L}^{+} = R_{2n-1,2n} \psi_{2n-1} \ psi_{2n} \]

\[ U_{R} \psi_{2n-1} \ U_{R}^{+} = U_{L} \psi_{2n+1} \ U_{L}^{+} = R_{2n-1,2n} \psi_{2n+1} \ psi_{2n} \]

(5.27)

This equation holds for any form of the 4 \times 4 \( R \) matrix. Inserting now in (5.27) the explicit form (5.24)-(5.25) yields

\[ U_{R} \psi_{2n-1} \ U_{R}^{+} = U_{L} \psi_{2n} \ U_{L}^{+} = \bar{b} \ psi_{2n} + \bar{c} \ psi_{2n-1} \]

\[ + (\frac{c}{\omega} - \bar{c}) \psi_{2n}^{+} \psi_{2n+1} \psi_{2n-1} \psi_{2n} - (\frac{b}{\omega} + \bar{b}) \psi_{2n-1}^{+} \psi_{2n+1} \]

\[ U_{L} \psi_{2n-1} \ U_{L}^{+} = U_{R} \psi_{2n+1} \ U_{R}^{+} = \bar{b} \ psi_{2n+1} + \bar{c} \ psi_{2n} \]

\[ + (\frac{c}{\omega} - \bar{c}) \psi_{2n-1}^{+} \psi_{2n+1} \psi_{2n} - (\frac{b}{\omega} + \bar{b}) \psi_{2n-1}^{+} \psi_{2n+1} \]

(5.28)

These second quantized field equations are perfectly defined on the lattice. The bare continuum limit \( a \to \infty \) is rather subtle. The detailed proof of ref. (23) shows that it leads to the continuum MTM provided the lattice parameters \( b \) and \( c \) scale as

\[ b = e^{i \nu} \left[ 1 + O(a^{-1}) \right], \quad c = -i e^{i \nu} \frac{m_{0} a}{2} \left[ 1 + O(a^{-1}) \right] \]

(5.29)

\[ a \to \infty \]

Notice that this bare limit is different from the renormalized one.
[eq. (5.18)]. Where \( \mu \) and \( m_o \) are fixed parameters that characterize the bare scaling limit. The lattice fermion operators leads to the continuum ones \( \psi_R(x), \psi_L(x) \) in the following way

\[
\psi_{R,n} = \sqrt{a} \psi_R(x + \frac{n}{a}) \quad \psi_{L,n} = \sqrt{a} \psi_L(x - \frac{n}{a}) \tag{5.30}
\]

\( x = na \) and \( 0 < \xi < 1/2 \) is a fixed number whose precise value is irrelevant in the limit. Then the continuum hamiltonian and momentum follow from

\[
\frac{1}{2} (H + \not{p}) = \lim_{a \rightarrow 0} \frac{-i}{a} \left[ e^{-i\not{Q}} \Omega_R - 1 \right] \tag{5.31}
\]

where \( Q \) is the bare (U(1)) charge

\[
Q = \int_0^L dx \left( \psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L \right) \tag{5.32}
\]

where \( L = Na \).

Notice that \( \lim_{a \rightarrow 0} \Omega_R = \lim_{a \rightarrow 0} \Omega_L = e^{iQ} \neq 1 \). After some calculations\(^{[23]}\) it can be shown that

\[
P = -i \int_0^L dx \psi^\dagger \partial_x \psi
\]

and

\[
H = \int_0^L dx \left[ -i \psi^\dagger (\gamma \cdot \partial_x + i m_o \gamma^0) \psi + \frac{\beta}{2} (\overline{\psi} \gamma^\dagger \gamma \psi)^2 \right] \tag{5.33}
\]

where \( \psi = (\psi_R, \psi_L) \) and \( \beta = -2 \coth(\mu/m_o), \omega = e^{2i\gamma_o}, \gamma_x = -c_0 \gamma^0, \gamma^0 = \gamma^0, \gamma^\dagger = \sigma^\dagger \gamma \).

That is we find the massive Thirring model (MTM) in the scaling limit (5.29). Eqs. (5.30)-(5.33) give the bare operators and eq.(5.29) defines the bare scaling limit. This is different to the renormalized scaling limit giving the physical sector of the Fock space [eq.(5.18)]. Both the particle spectrum and the physical S-matrices follow rigorously in the renormalized limit computed by this light-cone approach. The bare limit (5.30)-(5.33) tells us which model one is actually solving. We use the word "rigorous" since we solve in this approach a lattice model exactly, then we take the infinite volume limit and finally the \( a \rightarrow 0 \) (scaling) limit. In other words, here one solves (exactly) a model with both UV and volume cutoffs and then lets the cutoffs to infinity in a precise way. This is clearly much better than the coordinate Bethe-Ansatz (CBA) where the UV cutoff is introduced after the obtention of the solution. For the MTM and the chiral Gross-Neveu model the results of the CBA coincide with the light-cone approach for on-shell magnitudes. Hence the CBA works well in these cases. This is not the case for the multilavor Chiral fermion model treated in ref.(69) by CBA. As it is shown in ref.(23) the results of ref.(69) are not correct.

Starting from richer vertex models than the six-vertex a large set of QFTs arises \(^{[23]}\). Let us first summarize the integrable vertex models classification in terms of simple Lie Algebras.

A deep connection exists between integrable theories and simple Lie algebras \(^{[64,65]}\). It is possible to associate an integrable vertex model to each representation of a simple Lie algebra. These rational models are invariant under the corresponding Lie group \( G \), since this \( R \) matrix obeys
\[ R(\theta), \varphi \theta \gamma] = 0, \quad \forall \gamma \in G \tag{5.34} \]

Moreover, the structure of their BAE looks like the one of their respective Dynkin diagram. It must be noticed that a proof that these BAE lead to the eigenvectors and eigenvalues of the transfer matrix has been explicited only for a subset of models: those associated to \( U(N) \) (see sec.IV), \( Sp(2N) \) [27c], and \( SO(2N) \) [26] and some others. However, these statements are extremely likely to hold for all semisimple Lie algebras. Moreover, the whole structure of the BAE deforms in a very simple and suggestive way for the trigonometric/hyperbolic models where the symmetry contracts to the Cartan subalgebra of \( G \).

Let us describe the BAE for the trigonometric models. The derivation of these equations (for a subset of Lie algebras) are in sec.IV and refs. [27, 28]. The eigenvalues of the transfer matrix can be written as a sum of terms. The dominant one in the infinite volume limit (\( N \to \infty \)) is

\[ \lambda_{\alpha}(\theta, \lambda^{(i)}(\theta)) = \prod_{\alpha = 1}^{N} \prod_{k = 1}^{r} \left[ \lambda^{(i)}(\theta - \alpha_\alpha) + \lambda^{(i)}(\theta - \alpha_\alpha) + \lambda^{(i)}(\theta - \alpha_\alpha) + \lambda^{(i)}(\theta - \alpha_\alpha) \right] \tag{5.35} \]

for \( \theta \) in the vicinity of \( \theta = 0, |\theta| < \theta_0 \). Here \( \theta_1, \theta_2, \ldots, \theta_N \) are given numbers describing inhomogeneities of the lattice as discussed in sec.II [2, 37]. The \( \omega_\alpha \) are fundamental weights and \( \alpha_k \) are the simple roots of \( G \) whose rank is \( r \). \( (\alpha, \beta) \) stands for the usual inner product in

The normalization of the simple roots can be absorbed as a multiplicative factor on the \( \lambda^{(i)}(\theta) \).

Taking the logarithm of eq.(5.36) leads, in the homogeneous case \( (\theta_0 = 0) \) to

\[ \lambda_{\lambda}(\theta) = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) \right] = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) - \lambda^{(c)(i)}(\theta) \right] = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) - \lambda^{(c)(i)}(\theta) \right] = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) - \lambda^{(c)(i)}(\theta) \right] = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) - \lambda^{(c)(i)}(\theta) \right] = \prod_{i = 1}^{r} \left[ \lambda^{(c)(i)}(\theta) - \lambda^{(c)(i)}(\theta) \right] \tag{5.37} \]
\[ N \phi \left( \lambda_{i}, \gamma < \omega_{i}, \omega_{c} > \right) = \sum_{k=1}^{r} \sum_{j=1}^{p_{k}} \phi \left( \lambda_{j} - \gamma_{k} \right) \neq \gamma < \nu_{i}, \nu_{c} > \right) \] + \left( 5.38 \right)

\[ + 2 \pi I_{j}^{(c)} \quad 1 \leq j \leq p_{i}, \quad 1 \leq i \leq r \]

where the \( I_{j}^{(c)} \) are half-odd integers and \( \phi \left( z, \alpha \right) \) is given by eq.(3.36).

Actually eqs.(5.38) also hold in the rational and hyperbolic cases using the appropriate expression for \( \phi \left( z, \alpha \right) \) [eq.(3.35) or (3.37)] respectively. In the thermodynamic limit eq. (5.38) yields for any G a system of linear integral equations for the root densities analogous to eq. (4.56) for the trigonometric A_{q-1} model:

\[ \sigma_{g}^{(c)} \left( \lambda \right) = \sum_{\varepsilon = 1}^{r} \int_{-A}^{A} d\nu \left( K_{g}^{(c)} \left( \lambda - \nu \right) \sigma_{e} / r \right) = \frac{2}{2\pi} \phi' \left( \lambda, \gamma < \omega_{e}, \omega_{c} > \right) \] (5.39)

\[ - \frac{4}{N} \sum_{\varepsilon = 1}^{r} \left\{ \sum_{j=1}^{p_{j}} K_{g}^{(c)} \left( \lambda - \Theta_{g}^{(c)} \right) \right\} - \left[ K_{g}^{(c)} \left( \lambda - \delta_{g}^{(c)} \right) + K_{g}^{(c)} \left( \lambda - \delta_{g}^{(c)} \right) \right] \]

where

\[ K_{g}^{(c)} \left( \lambda \right) = \frac{2}{2\pi} \phi' \left( \lambda, \gamma < \omega_{e}, \omega_{c} > \right) \] (5.40)

or in fourier space [eq.(4.60)]

\[ \hat{K}_{g}^{(c)} \left( \lambda \right) = - \sigma_{g}^{(c)} \left[ < \nu_{g}, < \omega_{c} > \right] e^{- \left[ < \nu_{g}, < \omega_{c} > \right] \lambda} \] (5.41)

\[ \left( \text{rational case} \right) \]

\[ \hat{K}_{g}^{(c)} \left( \lambda \right) = - \sigma_{g}^{(c)} \left[ < \nu_{g}, < \omega_{c} > \right] e^{- \left[ < \nu_{g}, < \omega_{c} > \right] \lambda} \]

\[ \left( \text{trigonometric case} \right) \]

\[ \hat{K}_{g}^{(c)} \left( \lambda \right) = 2 \sigma_{g}^{(c)} \left[ < \nu_{g}, < \omega_{c} > \right] e^{- \left[ < \nu_{g}, < \omega_{c} > \right] \lambda} \]

\[ \left( \text{hyperbolic case} \right) \]

where \( \sigma_{g}^{(c)} \left( \gamma < \omega_{e}, \omega_{c} > \right) \neq 0 \).

The resolvent of the integral equation (5.39) follows as the inverse of the \( r \times r \) matrix

\[ \hat{R}_{g}^{(c)} \left( \lambda \right) = \left[ \hat{1} - \hat{K} \left( \lambda \right) \right]^{-1} \]

(5.44)

This is not a formidable inversion problem since it is a sparse matrix [67] whose characteristic diagram is precisely the Dynkin diagram of G. Explicit formulae for \( R_{j} \left( x \right) \) can be derived for each Lie algebra G.

We find for A_{q-1} the result in eq.(4.70). For D_{n} see refs. [28] and [17]. For E_{6}, E_{7} and E_{8}, \( R_{j} \) can be calculated explicitly by hand. For non-simply laced Lie algebras, the ground state is formed by complex roots and hence this treatment needs to be generalized. This is also the case for
non-fundamental representations of $G$. That is, the models obtained by fusion.

Eqs. (4.71)-(4.82) valid for Aq-1, easily generalize for any simple-laced Lie algebra. In order to study the scaling limit we need the excitation eigenvalues $g_k(\theta, \theta_k)$. They write [cfr. eq.(4.74)]

$$g_k(\theta, \theta_k) = \sum_{k=-n}^{n} \int d \lambda \hat{R}_k (\lambda - \theta_k) \hat{\varphi}(\lambda + \theta_k)$$

(5.45)

In the trigonometric (gapless) regime this can be recasted as [31]

$$g_k(\theta, \theta_k) = \sum_{k=-n}^{n} \int \frac{dx}{2\pi} \frac{\lambda_k [\varepsilon(\frac{\pi}{2} - \omega_k \lambda)] e^{i\xi (\theta_{\kappa} + \sigma)} [(1 - \hat{k}(\kappa)]^{-1}}{\lambda_k (\frac{\pi}{2})}$$

(5.46)

where we used eqs.(3.37), (4.72) and (5.44). As before [eqs.(5.17)-(5.19)] only the large $i\theta$ behavior is relevant for the scaling limit. Eq.(5.46) tells us that this behavior is determined by the zeros of $\det [1 - \hat{K}(x)]$ closer to the real axis. These values are clearly $\xi$ independent. One finds from eq.(5.46) by the residue method [31]

$$g_k(\theta, \theta_k) = \frac{m_\xi}{\theta} \exp \left[ \frac{\mu_\xi}{\theta} (\theta_k + i \theta) \right] \left( 1 + O \left( e^{-i \theta_\kappa} \right) \right)$$

(5.47)

where $\xi > 0$. The parameters $\kappa$ and $m_\xi$ are given in Table II. $\kappa$ is just $2\pi$ times the length squared of the shortest simple root in the normalization where [63]

$$B_\xi (\xi, \xi) = -1$$

and $B(x, y)$ is the Killing form.

Light-cone evolution operators can be defined through eqs.(5.7)-(5.9) for any R-matrix. Let us see that a relativistic dispersion law arises from any excitation spectrum as given by eq.(5.47). Let us call $E_\xi(\sigma)$ and $p_\xi(\sigma)$ the eigenvalues of $H$ and $P$, respectively. Eqs. (5.9) and (5.47) yield

$$E_\xi(\sigma) = \frac{e^{-2\mu_\xi \sigma}}{\theta} e^{-2\mu_\xi \sigma} + O \left( e^{-2\mu_\xi \sigma} \right)$$

(5.48)

$$p_\xi(\sigma) = \frac{e^{-2\mu_\xi \sigma}}{\theta} e^{-2\mu_\xi \sigma} + O \left( e^{-2\mu_\xi \sigma} \right)$$

(5.49)

It is then natural to define the scaling limit according to

$$a \to 0, \quad i\sigma \to \infty, \quad \mu_\xi = \frac{i \chi \theta}{\pi}$$

is the renormalised or physical mass scale and the particle mass spectrum of these integrable QFT's is given by
\[ M_\xi = \mu m_\xi \quad (5.50) \]

We recognize in eq.(5.48) \( \kappa \theta \) as the physical particle rapidity.

This is a very general way of constructing integrable QFTs. The operators \( H \) and \( P \) given by eq. (5.9) are well defined on the lattice as well as all the higher conserved charges. In the continuum limit \( \theta \to 0 \), they provide the energy and momentum of a relativistic invariant QFT, as long as the spectrum of the initial vertex model is gapless. This is usually the case for rational or trigonometric weights. In addition to the particle spectrum, the S-matrix is exactly calculable from the BAE by standard methods[1,68].

As it was the case for the MTM, the evolution operators \( U_R \) and \( U_L \) are much simpler than \( H \) and \( P \) on the lattice. This was exploited before [eqs.(5.27)-(5.28)] to obtain the lattice field equations for the fermionic fields of the MTM regularized by the lattice. An analogous local construction would be very interesting to obtain in the general case of a Lie algebra \( G \). We present here a lattice construction for the current operators for all rational models discussed before. The \( H \) and \( P \) are always given by eqs. (5.7) and (5.9). The renormalized scaling limit (5.49) yields the mass spectrum (5.50) [see Table II]. Now the \( R \)-matrix for all rational models has the asymptotic behavior

\[ R(\theta) = P \left[ 1 + \frac{\kappa + \lambda}{i \theta} + O \left( \frac{1}{\theta^2} \right) \right] \quad (5.51) \]

where \( \lambda \) is a numerical constant, the exchange operator \( P \) was defined in eq. (2.17) and

\[ T^{x \alpha}_m = \underbrace{1 \otimes \cdots \otimes}_{m \text{ site}} T^{x \alpha}_1 \otimes \cdots \otimes 1 \quad (5.53) \]

We then introduce the lattice operator

\[ \prod_{\alpha=1}^{d_{\text{im},G}} T^{x \alpha}_1 \otimes T^{x \alpha}_2 \otimes \cdots \otimes T^{x \alpha}_{d_{\text{im},G}} \quad (5.52) \]

Using eqs.(5.7)-(5.8),(5.51) and (5.52) and the Lie Algebra commutators

\[ [ T^{x \alpha}, T^{y \beta} ] = i \int_{\mathbb{R}^2} T^{x \alpha} \otimes T^{y \beta} \]

we can show that the \( T^{x \alpha}_m \) obey the local equations of motion on the lattice

\[ U_R T^{x \alpha}_{z+m} U_R^{-1} U_L T^{x \alpha}_{z+m} U_L^{-1} = T^{x \alpha}_{z+m} + 2i \theta \int_T T^{x \beta}_{z+1} T^{x \gamma}_{z-1} \theta + o \left( \frac{1}{\theta^2} \right) \quad (5.54) \]

\[ U_R T^{x \alpha}_{z+m} U_R^{-1} U_L T^{x \alpha}_{z+m} U_L^{-1} = T^{x \alpha}_{z+m} - 2i \theta \int_T T^{x \beta}_{z+1} T^{x \gamma}_{z-1} \theta + o \left( \frac{1}{\theta^2} \right) \quad (5.55) \]

The bare scaling limit is now defined as \( \theta \to 0 \), \( x \to \infty \), \( x = na \) fixed. We get
\[ \partial_x \mathcal{J}^{x}(x) = 0 \]

\[ \partial_y \mathcal{J}^{y}(y) - \partial_z \mathcal{J}^{z}(z) + i \gamma^\rho \gamma^\lambda \left[ \mathcal{J}_\rho^{x}, \mathcal{J}_\lambda^{z} \right] = 0 \quad (5.55) \]

where

\[ \mathcal{J}^{x}(x) = \frac{1}{g \alpha} \mathcal{T}_{2a}^{x}, \quad \mathcal{J}^{z}(x) = \frac{1}{g \alpha} \mathcal{T}_{2a}^{z} \quad (5.56) \]

Therefore we have a lattice version of the G-algebra currents \( \mathcal{J}_\mu(x) \) associated to an exactly solvable discretization of the field theoretic models. These equations (5.55) characterize the currents in the non-abelian Thirring model associated to the group G. This theory, also called Chiral Gross-Neveu model, has as Lagrangian,

\[ \mathcal{L} = i \overline{\psi} \gamma^\rho \gamma^\lambda \left( \mathcal{J}^{x}, \mathcal{J}^{z} \right) \mathcal{T}^{\lambda} \mathcal{T}^{\rho} \mathcal{K}_{\lambda} \mathcal{K}_{\rho} \quad (5.57) \]

Here \( \psi \) transforms under the irreducible representation \( \rho \) of G, \( \mathcal{T}^{\lambda} \) are the G-generators in that representation and \( \mathcal{K}_{\lambda} \) is proportional to the inverse of the Killing form. Actually the H and P constructed from eqs. (5.7)-(5.9) with the R-matrix (5.54) describe the zero-chirality sector of the model (5.57) and we can identify

\[ \mathcal{J}^{x}(x) = \overline{\psi} \gamma^\rho \gamma^\lambda \psi \quad (5.58) \]

The field theoretic models discussed up to here correspond to finite dimensional \( \mathcal{H} \) and \( \mathcal{V} \). Namely, a finite dimensional vector space at each link in the light-cone lattice. This is clearly appropriate for fermion or parafermion fields. Since there exists infinite dimensional representations of YBZF algebras, also bosonic QFTs may be described in this framework. The infinite spin representation of the SU(2) invariant R-matrix (rational limit of the six-vertex model) relates to the SU(2) principal chiral model (PCM) as it is investigated in ref. [70]. For arbitrary spin \( s \), this R-matrix writes [71]

\[ R_{12}(\theta) = \frac{\Gamma(2s+1+i\theta) \Gamma(1+s+i\theta)}{\Gamma(2s+1-i\theta) \Gamma(1+s-i\theta)} \quad (5.59) \]

where the operator \( J \) is defined by

\[ J(J+1) = 2s(s+1) + 2 \tilde{S}_1 \cdot \tilde{S}_2 \quad (5.60) \]

\( \tilde{S}_1 \) and \( \tilde{S}_2 \) are spin S operators acting on the spaces \( \mathcal{H} \) and \( \mathcal{V} \) respectively \( (\tilde{S}_1)^2 = (\tilde{S}_2)^2 = S(S+1) \). The light-cone hamiltonian (5.7)-(5.9) provides particle states that yield all particle masses and S-matrix amplitudes for the PCM letting \( S = \infty \). However, this \( H \) is not the full hamiltonian of the PCM as it is proven in refs. (72) and (70). There is a very simple explanation for this, the physical particle states for this model transform under the group \( \text{SU}(2)_L \oplus \text{SU}(2)_R \) and from the present construction only left or right operators can be obtained. Therefore all states obtained in this way are left (or right) singlets. The detailed counting of states in ref. (70) is confirmed by the simple proof of ref. (72).
The lattice current construction, eqs. (5.53)-(5.56) also applies to the PCM. For large \( \theta \) the R-matrix (5.59) admits a semiclassical expansion of the type (5.51). Therefore, the whole construction holds. It must be noticed that we get only one conserved and curvatureless current: either the SU(2)_L or the SU(2)_R.

This whole construction generalizes to the SU(N) PCM. It also applies for Chiral fermion models and PCM with one anisotropy axis (trigonometric YBZF algebras) [73].

VI. FINITE-SIZE CORRECTIONS FROM THE BETHE-ANSATZ AND CONFORMAL INVARIANCE.

As we see in previous sections the Bethe Ansatz provides the exact eigenvalues and eigenvectors of an integrable model from the resolution of a system of coupled algebraic equations. Two typical and relevant examples are given by eqs. (3.20) (the six-vertex model) and eqs. (4.49) [the q(2q-1) vertex model]. As we have seen, the explicit solution of the BAE in the \( N \rightarrow \infty \) limit is straightforward. The density of roots follows from a linear integral equation [eq. (3.33) or eqs. (4.56)] explicitly solvable by Fourier transformation. However, the analytic resolution of the BAE for a finite number of sites is a formidable task as soon as \( N \) is not very small.

A systematic procedure for computing finite size corrections for integrable theories was proposed in ref. (29). This method as well as subsequent improvements will be reported in this section.

We treat here the finite size corrections in the six-vertex model both in the zero gap and non-zero gap regimes. The generalization to multistate models can be find in ref. (31).

Let us consider the generalized transfer matrix \( \tau_\alpha(\theta) \) [37] considered in sec. 2. For the six-vertex model we can take

\[
\tau_\alpha = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}
\]  

Then,

\[
\tau_\alpha(\theta) = e^{i\theta} A(\theta) + e^{-i\theta} D(\theta)
\]  

As discussed in sec. 2, \( \tau_\alpha(\theta) \) describes boundary conditions where the spins at the sites \( N+1 \) and 1 are related by a rotation \( g \). That is

\[
\sigma_{N+1}^\pm = e^{\pm i\theta} \sigma_1^\pm
\]

The BA construction of the eigenvectors from sec. 3 applies with minor changes to the BAE and eigenvalue expressions. We have now as BAE instead of eq. (3.22)[37]

\[
N \phi(\lambda; \gamma) = 2 \alpha + \sum_{\epsilon=1}^r \phi(\lambda_i - \lambda_r; \gamma) + 2 \pi i \epsilon, \quad 1 \leq \epsilon \leq r
\]
The respective eigenvalue of $\tau_\infty(\theta)$ expresses as

$$\Lambda_\pm(\theta) = e^{i\theta} \Lambda_+ (\theta) + e^{-i\theta} \Lambda_- (\theta)$$ \hspace{1cm} (6.5)

where $\Lambda_\pm(\theta)$ are given by eqs. (3.21). As we know the first term dominates in eq. (6.5) for $|\theta| < \pi/2$. The $N \to \infty$ limit is $\alpha$ independent. Therefore the results of sec. 3 hold also here. Let us consider the finite size corrections

$$L_N(\theta) \equiv -\frac{1}{N} \log \Lambda_\pm(\theta) + \frac{i}{N} \left[ \frac{1}{N} \log \Lambda_\pm(\theta) \right] \hspace{1cm} (6.6)$$

Obviously $L_\infty(\theta) = 0$. Using eqs. (3.21) and (3.33) we can recast $L_N(\theta)$ in a more convenient form to study the large (but finite) $N$ regime

$$L_N(\theta) = -\frac{i}{N} \sum_{\lambda} \frac{1}{\lambda} \phi(\lambda + i \theta, \frac{y}{2}) - \frac{i}{N} \sum_{\lambda} \phi(\lambda + i \theta, \frac{y}{2}) \Lambda(\lambda)$$

$$= -\frac{i}{N} \sum_{\lambda} \left[ \frac{1}{\lambda} \phi(\lambda + i \theta, \frac{y}{2}) \right] \sigma_\infty(\lambda) + \sigma_\infty(\lambda)$$ \hspace{1cm} (6.7)

$$+ \frac{1}{N} \int_{-\pi}^{\pi} d\lambda \phi(\lambda + i \theta, \frac{y}{2}) \left[ \frac{1}{N} \sum_{\lambda} \delta(\lambda - \lambda_k) - \sigma_\infty(\lambda) \right]$$

where $A = \pi/2$ for regime I and $A = +\infty$ for regime II, we have used eq. (3.33) and

$$\sigma_\infty(\lambda) = \frac{d}{d\lambda} \xi_N(\lambda)$$ \hspace{1cm} (6.8)

where

$$\xi_N(\lambda) = \frac{1}{2\pi} \left[ \phi(\lambda, \frac{y}{2}) - \frac{2\alpha}{N} \sum_{k=1}^{N} \phi(\lambda - \lambda_k, \frac{y}{2}) \right]$$ \hspace{1cm} (6.9)

The $\lambda_k$ [1 $\leq$ $k$ $\leq$ $N$] are here the real roots of eq. (6.4). The function $\xi_N(\lambda)$ fulfills

$$\xi_N(\lambda) = \frac{i}{N} \sum_{k=1}^{N} \phi(\lambda - \lambda_k, \frac{y}{2})$$ \hspace{1cm} (6.10)

as in sec. III eq. (3.30)-(3.31). Notice that the phase $\alpha$ drops in the $N \to \infty$ limit and hence $\sigma_\infty(\lambda)$ also obeys eq. (3.31). Let us now study the function $\sigma_N(\lambda) - \sigma_\infty(\lambda)$. Subtraction of eqs. (3.31) and the derivative of (6.9) yields

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) + \int_{-\pi}^{\pi} d\lambda \phi(\lambda + i \theta, \frac{y}{2}) \left[ \sigma_N(\lambda) - \sigma_\infty(\lambda) \right] = -\int_{-\pi}^{\pi} d\lambda \frac{d}{d\lambda} \phi(\lambda + i \theta, \frac{y}{2}) \xi_N(\lambda)$$ \hspace{1cm} (6.11)

where

$$\xi_N(\lambda) \equiv \frac{i}{N} \sum_{k=1}^{i} \delta(\lambda - \lambda_k) + \frac{1}{N} \sum_{k=1}^{N} \delta(\lambda - \lambda_k) - \sigma_\infty(\lambda)$$ \hspace{1cm} (6.12)

Eq. (6.11) is to be considered as a linear integral equation for $\sigma_N(\lambda) - \sigma_\infty(\lambda)$ with the r.h.s. as inhomogeneous term. Solving it with the help of the resolvent (3.46) yields

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) = -\int_{-\pi}^{\pi} d\lambda \phi(\lambda + i \theta, \frac{y}{2}) \xi_N(\lambda)$$ \hspace{1cm} (6.13)

Inserting eq. (6.1 ) in eq. (6.7) and using eq. (3.47) gives
\[ \mathcal{L}_N(\theta) \equiv \int_{-A}^{A} d\lambda \left[ 2\pi \frac{z_N(\lambda + i\theta)}{2} + K_Y(\lambda) \right] S_N(\lambda) - \frac{i\xi}{N} \tag{6.14} \]

The constant \( K_Y(\lambda) \) is given by
\[ K_Y(\lambda) = \frac{x}{\lambda} + \sum_{n=1}^{\lambda + \pi/2} \lambda_n \tag{6.15} \]
where \( p(\lambda) \) is given by eq. (3.43). The calculation of large but finite \( N \) effects involve the evaluation for large \( N \) of expressions like
\[ \mathcal{I}_N = \int_{-A}^{A} d\lambda \, f(\lambda) \, S_N(\lambda), \tag{6.16} \]
where \( f(\lambda) \) is explicitly known. Notice that \( I_\infty = 0 \).

It is convenient to change in eq. (6.16) to the integration variable \( z_N(\lambda) \) as defined by eq. (6.9). Using eqs. (6.8) and (6.10) yields
\[ \mathcal{I}_N = \int_{-\delta}^{\delta} d\zeta \, f(\zeta(\lambda)) \left\{ \frac{1}{N} \sum_{k=1}^{M+N_\delta} \delta(\zeta - 3\lambda) - 1 \right\}, \tag{6.17} \]
where
\[ \zeta = \int_{-A}^{A} d\lambda \, \sigma_N(\lambda) = \frac{M + N_\delta}{N} \]
\( \lambda_N(2) \) is the inverse function of the monotonous function \( z_N(\lambda) \). We choose

\[ \zeta = \frac{k + \sqrt{k^2 - \delta^2}}{N} \quad 1 \leq k \leq M + N_\delta \]

By Fourier expanding the periodic \( \delta(z) \) with period \( \delta \), one gets
\[ \frac{1}{N} \sum_{k=1}^{M+N_\delta} \delta(\zeta - 3\lambda) = \sum_{s=-w}^{w} (-1)^s e^{2\pi i s N_\delta} \]
Inserting this formula in eq. (6.17) gives
\[ \mathcal{I}_N = \sum_{s \in \mathbb{Z}} (-1)^s T_{N_\delta} \tag{6.18} \]
where
\[ T_m = \int_{-A}^{A} d\lambda \, f(\lambda) \, \sigma_N(\lambda) \, e^{2\pi i m \zeta} \frac{2\pi i m z_N(\lambda)}{2 \lambda} \tag{6.19} \]
Expressions (6.18) and (6.19) are exact for all values of \( N \). Now we can proceed to obtain their asymptotic behavior for large \( N \). The procedure is different depending if we are in the massive regime I or in the massless one (regime II). In the former case the roots \( \lambda \) lie in a finite interval \((-\pi/2, \pi/2)\) for all \( N \). When the gap vanishes the BAE roots are not anymore bounded. The root density (3.42) (valid for \( N = \infty \)) permits to estimate that the largest roots are at \( \lambda = \pm A \) with
\[
\lambda_{\pm} \simeq \frac{1}{N} \left[ \rho_{nN} + \beta_{\pm} \right] \quad N \to \infty
\] (6.20)

where \( \beta_{\pm} = O(1) \) for \( N \to \infty \). The dominant finite size corrections to physical quantities depend on the value of \( \beta_{\pm} \). Therefore, more information than that contained in the \( N = \infty \) densities is needed.

Let us start by the massive case \( (a = \pi/2) \). In eq. (5.18) we have a sum of \( T_n \) with argument \( n - N \) always much larger than one (in absolute value) since \( |s| > 1 \). Therefore, we can try to evaluate \( T_n \) from eq. (6.19) by stationary point methods since \( n \) only appears in the exponent. We need to find the points \( \lambda_0 \) where
\[
\sigma_{nN}(\lambda_{nN}) = \frac{d^2 \sigma_{nN}}{d\lambda^2}(\lambda_{nN}) = 0
\] (6.21)

Moreover, the dominant large \( N \) behavior follows by replacing \( T_n \) by \( T_{\infty} \) where
\[
T_{n_{\infty}}^{ad} = \int_{-A}^{A} f(\lambda) \sigma_{nN}(\lambda) e^{2\pi i (m + z \omega)(\lambda)} d\lambda
\] (6.22)

and
\[
\sigma_{nN}(\lambda_{nN}) = \frac{d^2 \sigma_{nN}}{d\lambda^2}(\lambda_{nN}) = 0
\] (6.23)

The solutions of eq. (6.23) are exactly calculable from eq. (3.40) with the result
\[
\lambda_0 = \frac{\pi}{2} + \frac{\xi}{2} \quad \text{mod} (\pi, \xi)
\] (6.24)

where \( \sqrt{k_1 = (1 - k')/k} \) is the modulus associated with the nome \( e^{-2\gamma} \). In particular \( [49] \)
\[
\sqrt{k_1} = 2 e^{-\gamma/2} \prod_{\lambda = 0}^{\infty} \left[ \frac{1 + e^{-2\gamma(2\xi + 2)} e^{-\pi i (\xi + 2\xi)}}{1 + e^{-2\gamma(2\xi + 2)}} \right]^{2}
\] (6.25)

We see that \( 0 < k_1 < 1 \) for \( \gamma > 0 \). The same is true for the second derivative at \( \lambda_0 \) since
\[
2\pi i \sigma_{nN}(\lambda_{nN}) = -\frac{\xi K^2}{2}
\]

The infinite product (6.25) is slowly convergent for small \( \gamma \). Applying the Poisson summation formula to \( \log k_1 \) yields
\[
\sqrt{k_1} = \prod_{m=0}^{\infty} \left[ \frac{1 + e^{-\pi i (m + \frac{1}{2})}}{1 + e^{-\pi i (m + \frac{1}{2})}} \right]^{2}
\]

Then for \( \gamma \to 0 \),
\[
\lambda_0 = 1 - 8 e^{-\pi^2/2\gamma} + O(e^{-\pi^2/2\gamma})
\] (6.26)
we find a typical Kosterlitz-Thouless behavior. Hence, $T_n$ is exponentially small for large $n$

$$T_n = O \left( \frac{1}{n^{3/2}} \right) \quad \text{for} \quad n \gg 1$$

(6.27)

and the series (6.18) is dominated by the terms with $s = \pm 1$. This result holds for any expression with the form (6.16). We find for the finite size corrections to the free energy with $\alpha = 0$ [eq. (6.14)], after some calculations\[29]

$$L_N(\theta) = -\frac{2\pi \lambda}{K \sqrt{2 \Delta^2 N'}} \left\{ \sum_{m=1}^{\infty} \frac{e^{-m^2 \gamma}}{m^2} \right\}$$

(6.28)

$$\exp \left[ \frac{2\pi \lambda}{K \sqrt{2 \Delta^2 N'}} \right] \left\{ \sum_{m=1}^{\infty} \frac{e^{-m^2 \gamma}}{m^2} \right\}$$

Further results are reported in refs. (29). In summary, finite size corrections appear as an asymptotic series in (positive) powers of $\Delta^2 N'$. When $\gamma$ (regime I) tends to $0^+$, $\Delta^2 \gamma \Rightarrow 0^+$ and this expansion ceases to be useful. For small $\gamma$ these asymptotic formulae hold for regime I provided

$$N \gg e^\pi \frac{\Delta^2}{2\gamma} \quad \text{or} \quad \Delta^2 N \gg e^\pi \frac{\Delta^2}{2\gamma}$$

(6.29)

This is related to the vanishing of the mass gap [eq. (3.64)] when $\gamma \rightarrow 0^+$.

Let us now turn to the gapless regime. In this case, one can try to use the stationary point method as before. It follows that (6.21) only has

infinite solutions (i.e. $\lambda_0 = \pm \infty$) for $\sigma_\infty(\gamma)$ given by eqs. (3.41)-(3.42). In other words the integrals in eq. (6.19) are dominated by their end-points of integration ($R = \infty$) for large $n$. It is possible to evaluate in this way the integral in eq. (6.19) with the result that the $T_n^\alpha$ have for large $n$ an asymptotic expansion in powers of $1/n$. In this way the finite size corrections to the free energy result for $\alpha = 0$:

$$L_N(\theta) = -\frac{2\pi \lambda}{N^2} \sum_{\lambda, \nu} \frac{e^{i\theta N}}{} + o \left( \frac{1}{N^2} \right)$$

(6.30)

However for the study of excited states and the $\alpha \neq 0$ case is more effective to analyse $1_N$ in the following way\[30]. Let $\pm \Delta^2 \gamma$ be the largest real roots and assume that there are no holes within the interval $(-\Delta^2, \Delta^2)$. We assume also no complex roots for the moment. In this way their energy will be as small as possible. The motivation to study the finite size corrections to such lower states comes from conformal invariance. Since the model is here gapless one expects to find conformal invariant behavior in this regime.

As before, we study an expression of the form of eq. (6.16) (now with $R = \infty$)

$$I_N = \int d\lambda \ f(\lambda) S_N(\lambda) = \int d\lambda \ \sum_{k=1}^{M+K} \frac{1}{N} \int f(\lambda_k) \sigma_N(\lambda_k) d\lambda$$

(6.31)

where eq. (6.17) was used. The sums in the r.h.s. of eq. (6.31) can be
approximated for \( N \gg 1 \) using Euler-Maclaurin type formulae:

\[
I_N = \frac{1}{2N} \left[ \frac{d}{dN} \left( \frac{\sigma(N)}{\sigma(N-1)} \right) \right]
\]

\[
+ \int_{N-1}^{N} \left[ \frac{d}{dN} \left( \frac{\sigma(N)/\sigma(N-1)}{\sigma(N-1)} \right) \right] dN = O\left( \frac{1}{N^2} \right)
\]

We shall apply this approximation both to eq. (6.13) determining \( \sigma_N(\nu) \) and to eq. (6.14) expressing \( L_N(\nu) \). Eq. (6.13) gives

\[
\sigma_N(\nu) - \sigma_\infty(\nu) = \int_{\nu}^{\nu+1} \frac{d\nu}{\pi} \left[ \frac{\nu}{\nu'} - \frac{\nu}{\nu''} \right] + O\left( \frac{1}{N^3} \right)
\]

The relevant information about the finite size corrections to the lowest states comes from the regions around \( \nu = \Lambda_+ \) and \( \nu = -\Lambda_- \) [eq. (6.20)]. It is then useful to define

\[
\chi(t) = \sigma_N \left( t + \Lambda \right)
\]

and the Fourier transforms

\[
\chi^\pm(\omega) = \int_{-\Lambda}^{\Lambda} e^{i\omega t} \chi(t) dt
\]

which are analytic functions of \( \omega \) in \( \pm \Im \omega > 0 \). The contributions from the region around \( \nu = -\Lambda_- \) are treated analogously and added at the end.

Fourier transforming eq. (6.33) yields a Riemann-Hilbert (RH) problem for \( \chi(\omega) \):

\[
\chi_-(\omega) + \hat{R}(\omega) \chi_+(\omega) = e^{-i\omega \Lambda} \hat{\sigma}(\omega) + \]

\[
+ \frac{1}{2N} \left[ -1 + \hat{R}(\omega) \right] - \frac{i\omega}{\sigma_{N}(\Lambda_+)} \frac{1 - \hat{R}(\omega)}{\sigma^\prime_{N}(\Lambda_+)}
\]

That is, we find an equation relating the functions \( \chi_+(\omega) \) and \( \chi_-(\omega) \) which are analytic for \( \Im \omega > 0 \) and \( \Im \omega < 0 \) respectively. Here \( \hat{R}(\omega) \) and \( \hat{\sigma}(\omega) \) are the Fourier transform of the resolvent (3.46) and the vacuum density of roots (3.42)

\[
\hat{R}(\omega) = \frac{\mathcal{A}(\omega \tau/2)}{2 \mathcal{A}(\pi - \tau/2) \mathcal{A}(\omega \tau/2)} \hat{\sigma}(\omega) = \frac{A}{2 \cos(A/2)}
\]

In eq. (6.36) we only consider terms coming from \( \sim \Lambda_+ \), we also neglect contributions or order \( e^{-2\pi\Lambda_+} \).

In order to solve the RH problem (6.36) one starts to factorize \( \hat{R}(\omega) \) as

\[
\hat{R}(\omega)^{-1} = G_+(\omega) G_-(\omega)
\]

where \( G_\pm(\omega) \) are analytic functions in \( \pm \Im \omega > 0 \). The explicit form of \( G_\pm(\omega) \) is known in terms of \( \Gamma \) functions [50] but we shall not need it here. It will be enough to notice that

\[
G_+(\omega) = G_-(\omega)
\]
Therefore
\[ G_+ (\omega)^2 = \hat{R} (\omega)^{-1} = 2 \left( 1 - \frac{x}{\pi} \right) \] (6.40)

In addition, we have an integral representation that follows from eq. (6.38) and Cauchy theorem
\[ \log G_\pm (z) = \frac{1}{2 \pi i} \int_{-\infty}^{+\infty} \frac{d \omega}{\omega - z} \log \hat{R} (\omega), \text{Im} z > 0 \] (6.41)

Expanding eq. (6.41) for large \( z \) yields
\[ G_\pm (\omega) = 1 - \frac{3}{\omega} + \frac{3}{2 \omega^2} + o \left( \frac{1}{\omega^3} \right) \quad \text{as} \quad \omega \to \infty \] (6.42)

where \( g \) is a numerical constant. As we shall see below it cancels in physical results. Now using eq. (6.38) the RH problem (6.36) can be written as
\[ G_-(\omega) X_-(\omega) - Q_-(\omega) + \frac{G_-(\omega)}{2 \pi i} \left[ 1 + \frac{i \omega}{6 N \sigma_\omega (\Lambda^+)} \right] = \]
\[ = -G_+(\omega) X_+(\omega) + \frac{G_+(\omega)}{2 \pi i} \left[ 1 + \frac{i \omega}{6 N \sigma_\omega (\Lambda^+)} \right] + Q_+(\omega) = -P_+(\omega) \] (6.43)

where the function \( Q_+(\omega) \) is holomorphic in \( \text{Im} \omega > 0 \) and fulfill
\[ Q_+(\omega) + Q_-(\omega) = e^{-i \omega \Lambda^+} G_-(\omega) \sigma^+(\omega) \] (6.44)

Eq. (6.43) tells us that \( p(\omega) \) is an entire function of \( \omega \). It yields in addition the solution of our RH problem as
\[ X_+(\omega) = G_+(\omega) \left[ P(\omega) + Q_+(\omega) \right] + \frac{1}{2 \pi i} \left[ 1 + \frac{i \omega}{6 N \sigma_\omega (\Lambda^+)} \right] \] (6.45)

\( \rho(\omega) \) is obtained by letting \( \omega \to \infty \) in eq. (6.43) and using \( X_+(\infty) = 0 \). We find
\[ P(\omega) = -\frac{1}{2 \pi i} \left[ 1 + \frac{i \omega}{6 N \sigma_\omega (\Lambda^+)} \right] \] (6.46)

In addition, eqs. (6.37) and eq. (6.44) give
\[ Q_+(\omega) = \frac{1}{i \sigma} e^{-i \pi A^+ / \sigma} G_+(\omega) \frac{i \sigma}{\omega + i \pi / \sigma} \] (6.47)

Contour integration and eq. (6.35) yield
\[ \sigma N (\Lambda^+) = \int_{-\infty}^{+\infty} \frac{d \omega}{\pi} \omega X_+(\omega) = -i \text{Im} \left[ \omega X_+(\omega) \right] \quad \text{as} \quad \omega \to \infty \] (6.48)

Combining eq. (6.43) and eq. (6.48), we find
\[ N \sigma_\omega (\Lambda^+) = -\frac{N}{\gamma} e^{-\frac{\pi A^+}{\sigma}} G_+(\omega) \frac{i \sigma}{\omega + i \pi / \sigma} \] (6.49)
Up to now, we have not specified the physical state. That is, we must choose the integers \( \ell_1 \) (and \( \ell_n \)) in the BA eqs. (3.22) or equivalently in eq. (6.14). In the present case this information enters in the RH solution through the value of

\[
\Xi_+(\alpha) = \sum_{\Lambda_+} \omega_\Lambda^N(\lambda) e^{i\lambda} = \frac{1}{\Lambda_+} \cdot \Xi_+(\infty) = 2 \frac{\omega_\Lambda^N(\infty)}{\Lambda_+}
\]  

(6.50)

and the analogous contribution from \( \lambda \sim -\Lambda_- \). We find from eq. (6.9)

\[
2 \omega(\infty) = 1 - \frac{c}{N} - \frac{(4 - \frac{c}{2})}{N} - \frac{\alpha^2}{N^2}
\]

\[
2 \omega(-\infty) = \frac{\gamma}{N} \left( \frac{4}{2} - \frac{c}{N} \right) - \frac{\alpha^2}{N^2}
\]

(6.51)

where we used [see eq. (3.23)] the formulae

\[
\phi(\infty, \alpha) = 2(\pi - \alpha), \quad \phi(-\infty, \alpha) = 2\alpha
\]

(6.52)

valid for \( 0 < \alpha < \pi/2 \).

The value of \( \omega(\Lambda_+) \) is related to the half-integer associated to the last positive root. For the ground state we have a monotonous sequence of \( N/2 \) half-integers running from \( 1/2 \) till \((N-1)/2\). Therefore

\[
2 \omega(\Lambda_+) = \frac{4}{2} - \frac{\lambda}{2N}
\]

(6.53)

\[
2 \omega(-\Lambda_-) = \frac{4}{2N} \quad \text{(no holes)}
\]

Now, if we put \( h_+ \) holes beyond \( \Lambda_+ \) and \( h_- \) before \( -\Lambda_- \), a shift in the sequence \( \ell_1 \) is produced and we find

\[
2 \omega(\Lambda_+) = 1 - \frac{c}{N} - \frac{\lambda}{2N} - \frac{h_+}{N}
\]

\[
2 \omega(-\Lambda_-) = \frac{4}{2N} + \frac{h_-}{N}
\]

(6.54)

The total number of real roots is given by

\[
r = 1 + N \left[ z_N(\Lambda_+) - z_N(-\Lambda_-) \right]
\]

Combining this with eq. (6.54) yields

\[
r = \left( \frac{N - h_+ - h_-}{2} \right)
\]

(6.54a)

Since we assumed \( N \) to be even, this shows that the total number of holes must also be even.

Then eqs. (6.50)-(6.51) and (6.54) yield

\[
\Xi_+(\alpha) = \frac{4}{2N} - \frac{\gamma}{N} \left( \frac{4}{2} - \frac{c}{N} \right) + \frac{h_+}{N} - \frac{\alpha^2}{N^2}
\]

(6.55)

where \( S = N/2 - r = (h_+ + h_-)/2 \) is the spin of the state (the eigenvalue of \( S_a = \frac{1}{2} \sum_{a=1}^{N} \sigma_3^a \)). From eq. (6.45), (6.46), (6.47) and (6.55) we finally get

\[
e^{-\frac{\pi A^+}{8}} G_1 \left( \frac{\pi^2}{8} \right) = \frac{\pi}{2N} \left[ 1 - \frac{\gamma}{6N^2(1/e)} \right] + \frac{\pi h_+ - \gamma - \alpha^2}{N^2 G_1(\alpha)}
\]

(6.56)
Let us apply this approximation scheme to the finite size corrections $L_N(\theta)$. We find from eqs. (6.14) and (6.32)

$$L_N(\theta) = 2\pi i \sum_{\lambda \lambda^*} \sigma_N(\lambda) \sigma_N(\lambda + i\theta) - \frac{\epsilon N}{\pi} \left[ 2\sigma_N(\lambda + i\theta) + \sigma_N(-\lambda - i\theta) \right]$$

$$- \frac{\epsilon N^2}{6} \left[ \sigma_N(-\lambda + i\theta) - \frac{\epsilon N}{\pi} \sigma_N(-\lambda - i\theta) \right] - \frac{\epsilon N^2}{2}.$$  

(6.57)

Now, we can approximate the integrals here as

$$2\pi i \sum_{\lambda \lambda^*} \sigma_N(\lambda) \sigma_N(\lambda + i\theta) = \frac{\epsilon N}{2} \sigma(\lambda),$$

$$- \frac{\epsilon N^2}{6} \left[ \sigma_N(-\lambda + i\theta) - \frac{\epsilon N}{2} \sigma_N(-\lambda - i\theta) \right] = \frac{\epsilon N^2}{6} \sigma(\lambda).$$  

(6.58)

where eqs. (3.53), (6.34) and (6.35) were used. From eqs. (6.45)-(6.47), (6.49), (6.56) and (6.58) we derive the final expressions for $L_N(\theta)$ at large $N$. Let us first write the result for the ground state ($h_+ = h_- = S = 0$)

$$L_{N}^{G,S}(\theta) = - \frac{\epsilon N}{2} \sin \left( \frac{\pi \theta}{\gamma} \right) \left[ 1 - \frac{6 \epsilon N}{\pi (\pi - \gamma)} \right] + O \left( \frac{1}{N} \right).$$  

(6.59)

The contributions from $\lambda < -\lambda_*$ contained in eq. (6.59) follow by a procedure analogous to eqs. (6.34)-(6.58).

We see that eq. (6.59) reproduces eq. (6.30) when $\alpha = 0$. Let us now compare with the conformal theory predictions. For periodic boundary conditions one expects a leading finite size correction equal to

$$- \frac{\pi c}{6 N^2}.$$

(6.60)

where $c$ is the central charge. However, one cannot blindly identify eqs. (6.59) and (6.60). For large distances one expects rotational invariance in the gapless regime. This invariance can be seen in the spectrum of low energy excitations is derived in sec. III. In the present context the hamiltonian can be identified with

$$\delta E = - \text{Re} \log \mathcal{C}(\theta)$$

(6.61)

whereas the momentum is given by eq. (3.58). The low-lying eigenvalues of $H$ and $\mathcal{P}$ follow from eqs. (3.53) and (3.51)

$$E \approx P \sin \frac{\pi \theta}{\gamma}$$

(6.62)

This shows that we must renormalize the energy by the "speed of sound", $\sin(\pi \theta/\gamma)$ in order to recover an ultra-relativistic dispersion law and hence rotational invariance for large distances (Large compared with the lattice spacing). After this renormalization

$$L_{N}^{G,S}(\theta) = \frac{1}{\sin \frac{\pi \theta}{\gamma}} L_{N}^{G,S}(\theta) = - \frac{\pi c}{6 N^2} \left[ 1 - \frac{6 \epsilon N}{\pi (\pi - \gamma)} \right].$$

(6.63)

In conclusion

$$c = 1 - \frac{6 \epsilon N}{\pi (\pi - \gamma)}$$

(6.64)
In particular we recover $c = 1$ for the six-vertex model ($\alpha = 0$).

Let us now consider the low lying excited states with $h^\pm$ holes near $\pm \infty$. We find after some computations from eqs. (6.55)-(6.58) and their analogous for the contributions around $\lambda = -\Lambda$,

\[
L_N^{\text{exc}}(\theta) = - \frac{\pi}{6N^2} \sin \frac{\pi}{\frac{g}{2N^2}} \left\{ e^{\frac{-\pi \theta}{\gamma}} \frac{\left[ h + \frac{\gamma}{g} - \frac{\alpha}{\pi} \right]^2}{1 - \frac{\gamma}{g}} \right\} + \text{higher orders} \quad (6.65)
\]

where we disregard multiples of $2\pi i$.

For the six-vertex model ($\alpha = 0$), eqs. (6.59) and (6.65) can be recast as

\[
L_N^{\text{exc}}(\theta) - L_N^{\text{G,S}} = \frac{2\pi}{N^2} \left[ (\Delta^+ + \Delta^-) \sin \frac{\pi \theta}{\gamma} + \xi (\Delta^+ - \Delta^-) \cos \frac{\pi \theta}{\gamma} \right] \quad (6.66)
\]

where

\[
\Delta^+ = \frac{1}{\gamma} \left( \frac{h + \frac{\gamma}{g}}{1 - \frac{\gamma}{g}} \right)^2, \quad \Delta^- = \frac{1}{\gamma} \left( \frac{h - \frac{\gamma}{g}}{1 - \frac{\gamma}{g}} \right)^2 \quad (6.67)
\]

Eq. (6.66) fits with conformal theory predictions[57]. It may be considered as a proof of the conformal invariance of the six-vertex model giving in addition the conformal weights $\Delta$ and $\Delta^+$ of the low-lying states of it and central charge equal to one. The same results hold for the Heisenberg XXZ chain[30b,d].

For $\alpha = \gamma$ the critical Potts model properties follow from eq. (6.64)-(6.65)[30c,f]. Also the conformal content of the RSOS model of ABF[58] are derived in an analogous way[59].

Eqs. (6.67) give the conformal dimensions of primary fields. Actually some complex roots are present in these states besides the holes near $\pm \infty$. Otherwise one finds secondary fields with conformal dimensions $\Delta = K, \Delta^+ = K$ where $K$ and $K^+$ are positive integers[30f].

Methods analogous to those exposed in this section allow to extract from the BAE the conformal properties of the XXZ model, the critical Ashkin-Teller and the critical q-state Potts chain with free boundaries[30a,60]. Also the algebra properties of the critical O(n) model has been derived in this framework[61].

Moreover, the present methods has been generalized to nested Bethe Ansatz system[31]. In this way the central charge and conformal dimensions of the model exposed in sec. IV were found as well as for vertex models associated to all simply laced Lie algebras $G$. We find...
that gives \( c = q - 1 \) for the model of sec. IV. That is, each stage of the nested Bethe Ansatz contributes in one unit to \( c \). The conformal weights turn to be

\[
\Delta = \frac{1}{8(1 - \frac{2}{\pi})} \sum_{\epsilon, \epsilon' = 1}^{r} \left( h_{\epsilon}^+ - \frac{\pi}{\tau} S_{\epsilon} \right) \left( h_{\epsilon'}^+ - \frac{\pi}{\tau} S_{\epsilon'} \right)
\]

and a similar formula for \( \bar{\Delta} \) with \( h_{\epsilon}^- \) instead of \( h_{\epsilon}^+ \). Here \( h_{\epsilon}^+ \) are the number of holes near \( z = \infty \) in the \( \ell \)th branch, \( S_{\epsilon} \) the \( \ell \)th spin of the state [cf. eq. (4.47)], \( M \) the Cartan matrix of the underlying Lie Algebra. For the simply laced Lie algebras

\[
M_{\epsilon e} = \delta_{\epsilon e} + \frac{\pi}{\tau} \left( \epsilon < \sigma_\epsilon, \omega_\epsilon \right)
\]

In particular for \( A_{q-1} \) [the model of sec. IV]

\[
M_{\epsilon e} = 2 \delta_{\epsilon e} - \delta_{\epsilon, e+1} - \delta_{\epsilon, e-1}
\]

Once more \( \Delta \) and \( \bar{\Delta} \) vary continuously with \( \gamma \). In particular when \( \gamma = \frac{\pi}{\tau} (m + 1) \), \( m = q + 1, q + 2, \) one recovers the conformal weights of theories possessing extended Virasoro invariance (W-algebras)\[^{62}\]. More precisely one must consider the RSOS version of these multistate vertex models\[^{63}\].

In this way the central charge takes the values

\[
c = (q - 1) \left[ \frac{q^2}{m (m + 1)} \right], \quad m \geq q + 1
\]

These integrable lattice models provide explicit realizations of the extended Virasoro algebra through their long-range behaviour. They may be a very useful framework to uncover the physical meaning of the extended conformal symmetries.

In the present review we only consider the dominant corrections for large \( N \). From the subdominant ones one identifies irrelevant operators of the models\[^{30b}\]. In addition one sees that these subdominant powers of \( N^{-1} \) coincide with the previously computed conformal dimensions plus positive integers. That is secondary conformal fields.

In the rational limit \( \gamma \to 0 \) besides powers corrections, logarithmic corrections emerge as one could expect. These logarithmic corrections has been also computed with the methods here exposed\[^{30b,e,60}\].

It must be remarked that all central charges and conformal weights are independent of the spectral parameter \( \Theta \). Moreover the phase
transitions in vertex models are associated to changes in \( \gamma \) or in the elliptic modulus \( k \). Therefore \( \theta \) plays the rôle of an irrelevant parameter in integrable statistical models.

**VII. QUANTUM GROUPS AND BRAID GROUPS.**

The large \( \theta \) or semi-classical expansion of the R-matrix \( R(\theta) \) reads quite generally

\[
R(\theta) = 1 + \frac{\theta}{2} R(0) + O\left( \frac{1}{\theta^2} \right)
\]  

(7.1)

where \( R(0) - \Pi/\theta \) obeys the so-called classical YB equation

\[
\{ r_{23}(\Theta, -\Theta), r_{12}(\Theta, -\Theta) \} +
\]

\[+ \{ r_{23}(\Theta, -\Theta), r_{13}(\Theta, -\Theta) \} + [ r_{23}(\Theta, -\Theta), r_{23}(\Theta, -\Theta) ] = 0
\]  

(7.2)

The word semi-classical here means \( \hbar \to 0 \) as usual, where \( \hbar \) is Planck's constant for the QFT associated to \( R(\theta) \). The classification of solutions of eq. (6.2) can be shown to be given by the classical Lie algebras\[45\]. On the contrary the solutions of the full Yang-Baxter equation (2.19) lead to something new. This new structures underlying the trigonometric/hyperbolic YBZF algebras are called "quantum groups". The name "quantum group"\[46\] follows from the fact that these structures are related to the Lie algebras and groups in the same way as quantum mechanics relates to classical mechanics. In other terms, the YBZF algebras would be related to "quantum" deformations of the classical Lie groups. Let us recall their general form (2.11)

\[
R(\theta, \theta') = \left[ T(\theta) \otimes T(\theta') \right] R(\theta, \theta') R(\theta, \theta')^{-1}
\]  

(7.3)

Let us review now how \( T(\theta) \) is seen to be the quantum version of a Lie group element in the context of the construction of ref. (3). An explicit construction of YBZF algebras is made in ref. (3) for the models characterized by the equations of motion

\[
\partial_x \hat{A}^+ (x) = 0
\]

\[
\partial_x \hat{A}_+ - \partial_x \hat{A}_- + [ \hat{A}_-, \hat{A}_+ ] = 0
\]  

(7.4)

where the vector current \( \hat{A}_\mu (x) \) takes values in a Lie algebra \( \mathfrak{g} \). These models are classically conformal invariant and integrable. Concrete examples of eqs. (7.4) are general sigma models (including principal chiral models, CP\(^{N-1} \), Sp\(^n \) models, etc.) the fermionic chiral models of ref. (14) the Gross-Neveu models and the WZW sigma model\[12-13\]. Eqs. (7.4) are the compatibility condition for the following linear system\[9\]

\[
\partial_x \varphi (x, \lambda) = L_\lambda \varphi (x, \lambda)
\]

\[
L_\lambda \varphi (x, \lambda) = \frac{\hat{A}^+ - \lambda \hat{E}_\lambda A^-}{\mu - \lambda^2}
\]  

(7.5)

Here \( \lambda \in \mathbb{C} \) is a spectral parameter. Imposing finite energy boundary conditions for the field theory

\[
\lim_{|x| \to \infty} \hat{A}^+ (x^0, x^+) = 0
\]  

(7.6)
implies that \(\varphi(x,\lambda)\) leads to a constant matrix when \(x^1 \to \pm \infty\). Therefore, we can define solutions \(T(x,\lambda)\) of eq. (7.5) such that

\[
T(x^1 = \pm \infty, \lambda) = 1
\]  
(7.7)

The scattering data for the linear system (7.5)

\[
T(\lambda) = T(x^1 = -\infty, \lambda)
\]  
(7.8)

are time independent: \(T(\lambda)\) is often called monodromy matrix. It provides an infinite number of integrals of motion for all models where equations of motion have the form (7.4). The solution \(T(x,\lambda)\) of eqs. (7.5)-(7.7) can be written as[47]

\[
T(x, \lambda) = P e^{x^1 \int \frac{dx^\mu}{\lambda - \lambda^\mu} L_\mu(x, \lambda)}
\]  
(7.9)

where \(P\) stands for ordered exponential. Since \(L_\mu(x,\lambda)\) is a flat connection vanishing at \(x^1 \to \pm \infty\) [eq. (7.6)] we can deform the integration path in the two-dimensional Minkowski space. Taking a path along the \(x^1\)-axis yields

\[
T(\lambda) = P e^{x^1 \int_{-\infty}^{+\infty} \frac{dx^1}{\lambda - \lambda^\mu} L_\mu(x, \lambda)}
\]  
(7.10)

Taking a contour in the light-cone directions \((x^1 \to \pm x^n)\) gives

\[
T(\lambda) = P e^{x^1 \int_{-\infty}^{+\infty} \frac{dx^1}{\lambda \pm 1} \tilde{A}_\pm(x)}
\]  
(7.11)

where we used

\[
L_\pm(x, \lambda) = \frac{1}{\lambda \pm 1} \tilde{A}_\pm(x)
\]  
(7.12)

Since \(A_\mu\) takes values in the Lie algebra \(\mathfrak{g}\) the classical monodromy matrix \(T(\lambda)\) is in the corresponding group \(G\).

At the quantum level not every theory in the class of eqs. (7.4) is integrable. For those theories which are quantum mechanically integrable the quantum operators \(\hat{T}_{ab}(\lambda)\) has been constructed in refs [3,48] (examples of such integrable field theories are sigma models on symmetric spaces \(G/H\) where \(H\) is simple, the fermionic models of ref. (14) and the Gross-Neveu model). It is also possible to expand the RHS of eq. (7.10) in powers of \(\lambda\) and renormalize term by term. This gives the same renormalized operator as the general construction (in ref. [3,48] a check for the first two orders in \(\lambda\) was done but there is no doubt about the general validity). Although the renormalization of \(\hat{T}(\lambda)\) as given by eq. (7.10) would seem a formidable problem, there exists a relatively simple solution for quantum integrable field theories. The final result can be summarized as follows. The matrix elements of \(\hat{T}_{ab}(\lambda)\) between physical particle states can be expressed in terms of two-body S-matrix elements of the same theory[3]. For one particle states \((\Theta, \alpha)\) one finds

\[
\langle \Theta', \alpha' | \hat{T}_{ab}(\lambda) | \Theta, \alpha \rangle = \delta(\Theta - \Theta') \int S_{ab}(\lambda, \lambda') (\Theta + \Theta')
\]  
(7.13)
Here $\alpha, \beta$ as well as $a, b$ label internal particle states, $\Theta$ and $\Theta'$ are the physical rapidities. The physical two-body collision associated to $S_{ab}(\Theta - \Theta')$ is depicted in fig. 3 and $\Theta = c \lambda$, where $c$ depends on the model.

A general matrix element of $\hat{T}_{ab}(\lambda)$ writes

$$\langle \theta^{\prime} | \hat{T}_{ab}(\lambda) | \theta \rangle = S_{ab} \prod_{i=1}^{N} \delta(\theta_{i} - \theta_{i}^{\prime}) \sum_{a_{1}, a_{2}, \ldots} S_{\alpha_{1}, \alpha_{2}, \ldots} (\Theta_{1} + \Theta_{2}) \ldots S_{\alpha_{N-1}, \alpha_{N}} (\Theta_{N} + \Theta_{1}).$$

(7.14)

Notice that eq. (7.14) has the same structure as the reproduction property of YBZF algebras (2.34). Therefore, for integrable QFT where $S_{ab}(\theta)$ obeys the factorization equation (2.19), $\hat{T}_{ab}(\lambda)$ fulfills a YB algebra as operators in the whole Fock space [3, 48].

$$R(\lambda, \rho) [\hat{T}(\lambda) \otimes \hat{T}(\rho)] = [\hat{T}(\rho) \otimes \hat{T}(\lambda)] R(\lambda, \rho)$$

(7.15)

where

$$R(\lambda, \rho) = P S (\Theta_{1} - \Theta_{2})$$

$\hat{T}(\lambda)$ follows by quantization from $T(\lambda)$ that takes values in the classical group $G$. Hence, the name "quantum group" for the operators $\hat{T}_{ab}(\lambda)$ is fully justified here.

In addition this construction provides an explicit link between representation of YBZF algebras and Kac-Moody algebras [47]. Notice that $\hat{A}_{+}(x)$ in eq. (7.11) obeys a loop algebra as a classical current and a

Kac-Moody algebra as a quantum field theoretic operator.

Let us now see how braid groups and quantum groups follow from YBZF algebras. More precisely from the $\Theta \to -\infty$ limit of a hyperbolic YBZF algebra. Since R-matrices (and more generally YBZF generators) can be interpreted as S-matrices (see sec. II) and $\Theta$ as the rapidity, the $\Theta \to -\infty$ limits correspond to extreme high-energy scattering where the $S$-matrix is usually diagonal (in the basis of particle states). That is, with an appropriate normalization

$$\lim_{\Theta \to -\infty} R(\Theta) = R_{\pm}$$

(7.16)

where

$$R_{\pm} = c_{\pm} \delta_{\lambda}^{k} \delta_{\lambda}^{k}$$

(7.17)

These R-matrix can be represented graphically as in fig. 12. The Kronecker delta in eq. (7.17) are associated to continuous lines in fig. 12, as usual. However eq. (7.17) will not be used in what follows. We will only use the fact that the $\Theta \to -\infty$ limit exists and is non-zero. Letting $\Theta_{1}$ and $\Theta_{3}$ equal to $-\infty$ with fixed $\Theta_{2}$ in eq. (2.19) yields

$$R_{23}^{-1} R_{12}^{-1} R_{13}^{-1} = R_{42}^{-1} R_{23} ^{-1} R_{12}^{-1}$$

(7.18)

where $R_{12} = R_{-1}$ and $R_{23} = 1 \circ R$. In addition eq. (2.22) tells us that $R_{+}$ and $R_{-}$ are inverses of each other

$$R_{+} R_{-} = R_{-} R_{+} = 1$$

(7.19)
(where an appropriate normalization has been chosen for $R$).

The matrices $R_+$ and $R_-$ give a representation of a braid group in the following way. Let us consider the operators $X_i(\theta)$ acting in the tensor product of $n$ auxiliary spaces $\mathbb{C}_l[74]

$$X_i(\theta) = 1 \otimes \cdots \otimes R_i(\theta) \otimes \cdots \otimes 1_{i+1} \cdots 1_{n} \quad (7.20)$$

That is

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} X_1(\theta) & \cdots & X_n(\theta) \end{pmatrix} = R_i^{(a_i \rightarrow a_{i+1})}(\theta) \prod_{k=1}^{n} R_k^{(b_k \rightarrow b_{k+1})} \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \quad (7.21)$$

They fulfill the relations

$$\left[ X_i(\theta), X_j(\theta') \right] = 0 \quad \text{if} \quad |i-j| \geq 2, \quad \forall i, j$$

$$X_i(\theta) X_{i+1}(\theta+\theta') X_i(\theta') = X_{i+1}(\theta') X_i(\theta+\theta') X_i(\theta')$$

$$X_i(\theta) X_i(-\theta) = 1 \quad (7.22)$$

that follow from eqs.(2.19) and (7.20)-(7.21). These operators are clearly of "light-cone" type. They are closely related to the light-cone evolution operators discussed in sec. V. We find

$$U_+ = X_1 X_3 \cdots X_{n-1}$$

$$U_- = X_2 X_4 \cdots X_n$$

where[23] (see eqs.(5.7)-(5.8))

$$U_T = U_L V = U_R V^\dagger, \quad U_- = VU_L = V^+U_K$$

Here $V (V^\dagger)$ is the shift operator affecting one-half translation to the right (to the left)[23]. In the $\theta \to \infty$ limit we get

$$b_i = \lim_{\theta \to \infty} X_i(\theta), \quad b_i^{-1} = \lim_{\theta \to \infty} X_i(\theta)$$

These $b_i (1 \leq i \leq n)$ precisely obey the relations of the $n$-braid group generators[75]

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad b_i b_j = b_j b_i, \quad |i-j| \geq 2 \quad (7.24)$$

Let us briefly recall the notion of a braid group. Braids are formed when $n$ points in a straight line are connected by $n$ lines with other points on a paraller line as shown in fig.13. When the lines connecting the points have no intersections, the braid is called trivial. A general $n$-braid is obtained from the trivial one applying successively the operations $b_i$ and/or the inverses $b_i^{-1} (1 \leq i \leq n-1)$. The operations $b_i$ and $b_i^{-1}$ are depicted in fig.14. Then each topologically equivalent class of braids is identified with an element in $B_n$. Eq.(7.24) shows that the $\theta \to \infty$ limit of
hyperbolic $R$-matrices provide a representation of $B_n$. This connection between YBZF algebras and braid groups revealed recently very fruitful to obtain knot invariants and link polynomials [76, 77].

The exchange of points in the $n$-point conformal blocks forming the conformal invariant correlation functions yields a representation of a braid group (7.24) [78]. We want to remark that the $R$-matrix associated to such braid groups defines a lattice statistical model whose critical behavior is described precisely by the conformal theory yielding this braid group.

Let us now briefly discuss the quantum group associated to the six-vertex YBZF algebra. It is convenient for this purpose to write the six-vertex S-matrix in the antiferroelectric regime (other regimes can be treated analogously) as

$$S(\theta) = \sin \left( \theta + \frac{\pi}{2} \left[ 1 + \sigma_3 \otimes \sigma_3 \right] \right) + \sin \left( \sigma_+ \otimes \sigma_+ + \sigma_- \otimes \sigma_- \right) \sin \lambda \left( \lambda + \frac{\pi}{2} \right)$$

(7.25)

where we used eq.(3.1). We define for brevity of subsequent notations

$$\mu \equiv \theta - \lambda, \quad \lambda \equiv \theta + \frac{\pi}{2}$$

Then,

$$S(\lambda) = \sin \left( \lambda + \frac{\pi}{2} \left[ 1 + \sigma_3 \otimes \sigma_3 \right] \right) + \sin \left( \sigma_+ \otimes \sigma_+ + \sigma_- \otimes \sigma_- \right) \sin \lambda \left( \lambda + \frac{\pi}{2} \right)$$

(7.26)

This provides the YBZF generator for a vertical space $\mathcal{V} = C^2$ isomorphic to the horizontal one $\mathcal{A} = C^2$ . Eq. (7.26) suggests for the YBZF generator acting on a general vertical space $\mathcal{V} = C^{2k+1}$ the following ansatz [79]

$$S^{(4, \gamma)}(\lambda) = \sin \left( \lambda + \mu \sigma_3 \otimes J_3 \right) + \left( \sigma_+ \otimes J_+ + \sigma_- \otimes J_- \right) \sin \gamma \left( \lambda + \frac{1}{2} \right)$$

(7.27)

where the operators $J_\pm$ and $J_3$ act on $\mathcal{V}$ are to be found. As a YBZF generator $S^{(4, \gamma)}(\theta)$ writes [see eq.(2.58)]

$$t_{\alpha \beta}(\lambda) = \begin{pmatrix} \sin (\lambda + \mu J_3) & J_+ \sin \mu \\ J_- \sin \mu & \sin (\lambda - \mu J_3) \end{pmatrix}$$

(7.28)

where $1 \leq a, b \leq 2$. Inserting eq.(7.28) in the YBZF algebra definition (2.9) with R-matrix (3.1) yields as sufficient conditions [79]

$$[ J_3, J_\pm ] = \pm J_\pm$$

(7.29)

$$[ J_+, J_- ] = \frac{\sin (2 \mu J_3)}{2 \sin \mu} \frac{\sin (2 \mu J_3)}{2 \sin \mu}$$

(7.30)

That is (7.27)-(7.28) obey the six-vertex YBZF algebra for all $\theta$, provided $J_\pm$ and $J_3$ fulfill eq.(7.29)-(7.30). When $s = 1/2$ $(\mathcal{V} = C^2)$, eq. (7.29) is obviously satisfied by $J_3 = \sigma_3 / 2$. When $s > 1/2$, eqs. (7.29)-(7.30) define a deformation of the angular momentum algebra since in the isotropic limit $(\gamma \to 0)$ one recovers the usual $SU(2)$ commutators.
It must be noticed that this deformed structure is $\theta$-independent and only depends on $\mu = \pi - \theta$.

It is possible to relate the operators $J_\pm, J_3$ with the usual spin operators $S_\pm, S_3$ obeying
\[ [S_+, S_-] = 2S_3, \quad [S_3, S_\pm] = \pm S_\pm \quad (7.31) \]

Inserting the ansatz
\[ J_+ = S_+ f(\mu, s_3, s), \quad J_- = f(\mu, s_3, s) S_- \quad (7.32) \]

in eqs. (7.29)-(7.30) yields the recursion relation
\[ f(\mu, s_3, s) = \frac{\sin((s_3 + 1)\mu)}{\sin s_3 \mu} \quad (7.33) \]

where $s(s + 1) = (s_+ s_+ + s_- s_-)/2 + (s_3)^2$, as usual. This has as solution [80]
\[ f(\mu, s_3, s) = \frac{1}{\sin \mu} \sqrt{\frac{\sin s_3 [s(s-s_3)] \sin [\mu(s+s_3+1)]}{(s-s_3)(s+s_3+1)}} \quad (7.34) \]

The quadratic ("Casimir") operator commuting with $J_\pm$ and $J_3$ writes here

\[ C = \frac{a}{2} (J_+ J_- + J_- J_+) + \frac{\cos \mu}{\sin^3 \mu} s_3^2 (J_3) \quad (7.35) \]

This deformation of $\mathfrak{g}^2$ has the value
\[ C = \frac{\sin[(s+1)\mu]}{\sin s_3 \mu} \sin s_3 \mu \quad (7.36) \]

In summary eqs. (7.32)-(7.34) explicitly display the SU(2)$_\mu$ quantum group generators in terms of the usual SU(2) generators $S_\pm, S_3$. Moreover, the YBZF generator for the six-vertex case is constructed through eqs. (7.27) or (7.28) for the case $a_\mu = C^2, \eta = C^2 S_3^2$. The case of an arbitrary space $\mathfrak{g}^2$ is considered in refs. [79, 80].

In conclusion, we see that the construction of YBZF representations other than the fundamental ones ($a_\mu = \eta = C^2$ for the six-vertex case) leads to Lie algebra deformations or quantum groups (also called twisted groups). This phenomenon is rather general for trigonometric/hyperbolic R-matrices. The R-matrix (4.1) leads to deformations of SU(N) [81]. More generally, any simple Lie algebra admits an associated trigonometric/hyperbolic R-matrix (as discussed in sec. V) and a quantum group [81, 82]. These quantum groups are equipped with a coproduct that endows them with a Hopf algebra structure (with antipode).

The name quantum groups is not fully appropriate. It is more precise to call them "quantum analog" or "$\mu$-deformation" of the enveloping algebras. These algebras $U_{\mu}(G)$ have as generators
They fulfill the relations \[ [h_i, e_j^\pm] = \pm M_{ij} n_i e_j^\pm, \quad [h_i, h_j] = 0 \]
\[ [e_i^+, e_j^+] = \delta_{ij} \frac{\sin(2\pi n_i)}{\sin(2\pi m_i)}, \quad 1 \leq i, j \leq r \] (7.37)

\[
\sum_{s=0}^{r-n_i} (-1)^s C_s^{(n_i)} e_s^{(n_i)} (e_i^\pm)^s (e_i^\pm)^{r-n_i-s} = 0, \quad 1 \leq i \neq j \leq r
\] (7.38)

where

\[
C_s^{(n_i)} = \frac{1}{s!} \left\{ \sin\left(\nu - k + 1\right) \right\}, \quad \nu = \frac{<\alpha_i, \omega_i>}{2}
\] (7.39)

The \( |\alpha_i> \) are simple roots of the Lie algebra \( G \) and \( M_{ij} \) is the Cartan matrix

\[
M_{ij} = 2 \frac{<\alpha_i, \alpha_j>}{<\alpha_i, \omega_i>}
\] (7.40)

In the \( \rho \rightarrow 0 \) limit, eqs.(7.38)-(7.39) yield the usual Lie algebras. For \( G = SU(2) \), eq.(7.37) becomes eq.(7.30) and eq. (7.38) is satisfied trivially. It must be noticed that all finite dimensional representations of \( U_{\rho}(G) \) follow by deformation of those of \( G \) \([81,83] \). The coproduct for the \( U_{\rho}(G) \) algebras read as follows:

\[
\Delta(e^{\pm h_i}) = e^{\mp h_i} \otimes e^{\pm h_i}
\]
\[
\Delta(e^\pm) = e^\pm \otimes e^{-\pm} + e^{\pm h_i} \otimes e^\pm
\] (7.41)

This is a homomorphism preserving eqs.(7.38)-(7.39) and endowing \( U_{\rho}(G) \) with a Hopf algebra structure. The YBZF algebras enjoy a Hopf algebra structure with antipode as discussed at the end of sec.II.

We arrive to the end of this review. Many important aspects of integrable theories are not treated here as corner transfer matrix calculations of one-point functions \([2,87]\), the fusion or bound-state construction of YBZF generators (bound-state S-matrices) \([1,22,88]\), the temperature Bethe-Ansatz \([89]\) and many others.

Concerning the relativistic QFT listed in sec.I we have not exposed here their detailed resolution except perhaps for the MTM and the Chiral fermion models \((1.9)\) for all simple Lie algebras in their fundamental representation (sec.V). However all the explicit solutions can be worked out with the methods exposed here \( \& \) actually found already in the literature in many cases. Indeed non-trivial work will be needed in some cases like for models \((1.6) \) and \((1.8) \).

Acknowledgements. It is a pleasure to thank C. Destri, L. D. Faddeev, M. Jimbo, M. Karowski, E. K. Sklyanin, T. T. Truong, P. B. Wiegmann, F. Woynarovich and A. B. Zamolodchikov for useful discussions.
APPENDIX

Spin Hamiltonians as Logarithmic Derivatives of Row-to-Row Transfer Matrices.

A graphical proof of the following statement is given:

The logarithmic derivation of the row-to-row transfer matrix with $\mathbf{a} = \mathbf{V}$ at $\theta = 0$ yields an operator coupling nearest neighbors provided eq. (2.21) holds (regular R-matrix).

Eq. (2.21) can be graphically represented as

$$ R_{bc}^d (0) = c \delta_b^d \delta_c^e = a \left[ t^h \right] d_{(0)} (A-1) $$

In an analogous way $T_{ab}^{(0)} \xi \xi $ and $\tau (0)$ can be drawn as follows

$$ T_{ab}^{(0)} \xi \xi = a \left[ t^h \right] \xi \xi (A-2) $$

$$ \tau (0) \xi \xi = \xi \xi (A-3) $$

Here $d_{(0)} = \delta_1$ and $\xi \xi$ is clearly the lattice unit shift operator. Therefore, the momentum operator $\rho$ can be defined as

$$ \rho = \mathbf{a} \log \left[ c^{-N} \tau (0) \tau (0) \right] $$

Similarly

$$ \tau (0) -1 = \begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_N & c_1 & c_2 & c_3 & c_{N-1} & \cdots \end{array} (A-4) $$

Now, if we compute $d/d\theta \tau (\theta)$ from eq. (2.5) we shall obtain $N$ terms, each one containing $d/d\theta t^{(0)}(\theta)$, $1 \leq h \leq N$ and the others $t^{(0)}(\theta)$ not derived. Hence, setting $\theta = 0$ yields

$$ \tau (0) = \begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} (A-5) $$

Here stands for $\tau (0)$

It is now very simply to perform the product $\tau (N) -1 \tau (N)$ just combining eqs. (A-4) and (A-5) with the result

$$ \tau (N) -1 \tau (N) = \sum_{k=1}^{N} \begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} = \sum_{k=1}^{N} \begin{array}{cccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} $$
Therefore $\tau^{[N]}(\theta)^{-1}$ is a sum of terms each one acting as an operator on two neighboring sites. Now, putting all factors

$$
\mathcal{H} = \frac{2}{\partial \theta} \log \tau^{[N]}(\theta) = \sum_{k} \Theta_{k, k+1}
$$

(A-6)

where the matrix elements of $h$ reads

$$
\langle c_{k} d_{k} | h_{k, k+1} | c_{k+1} d_{k+1} \rangle = \frac{1}{c} \delta_{k, k+1} \langle c_{k}, c_{k+1} \rangle
$$

(A-7)

More generally the $n$-th derivative of log $\tau(\theta)$ at $\theta = 0$ is an operator that couple $(n+1)$ neighboring sites$^{[39]}$.

In the case of the six-vertex model, one finds from eqs. (3.1), (A-1) and (A-7)

$$
\mathcal{H} = \frac{1}{2 \sin \gamma} \left[ \cos \gamma + \sigma_{1} \otimes \sigma_{1} + \sigma_{2} \otimes \sigma_{2} - \cos \gamma \sigma_{3} \otimes \sigma_{3} \right]^{(A-8)}
$$

Then

$$
\mathcal{H}_{XX} = \frac{N}{2} \cos \gamma + \frac{1}{2 \sin \gamma} \sum_{\alpha \beta} \left[ \sigma_{x}^{\alpha} \otimes \sigma_{x}^{\alpha+1} + \sigma_{y}^{\beta} \otimes \sigma_{y}^{\beta+1} + \sigma_{x}^{\alpha+1} \sigma_{x}^{\alpha} \right]
$$

(A-9)

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[84] See for example the first ref. in [51].


\begin{table}
\begin{tabular}{ll}
\hline
\chi & symmetry group \\
\hline
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\begin{table}
\begin{tabular}{ll}
\hline
\phi & \theta\text{-dependence in } R_{ab}^{\phi}(\theta) \\
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\begin{table}
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discrete & \mathbb{Z}_q \\
\hline
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\begin{table}
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\hline
continuous abelian & \mathbb{U}(1)^\mathbb{R} \\
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\begin{table}
\begin{tabular}{ll}
\hline
continuous non-abelian & \mathbb{U}(q), \mathbb{O}(q), \\
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\begin{table}
\begin{tabular}{ll}
\hline
\text{elliptic} & trigonometric or hyperbolic \\
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\text{rational} & \\
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\end{tabular}
\end{table}
TABLE II

<table>
<thead>
<tr>
<th>Lie Algebra</th>
<th>Dynkin's diagram</th>
<th>( \kappa )</th>
<th>( m\kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n )</td>
<td>( 1 \ 2 \ 3 \ \ldots \ n )</td>
<td>( 2 \ \pi/(n+1) )</td>
<td>( \sin(\pi \ k/(n+1)), \ 1 \leq k \leq n )</td>
</tr>
<tr>
<td>( B_n )</td>
<td>( 1 \ 2 \ 3 \ \ldots \ n )</td>
<td>( \pi/(2n-1) )</td>
<td>( \sin(\pi \ k/(2n-1)), \ 1 \leq k \leq n-1 )</td>
</tr>
<tr>
<td>( C_n )</td>
<td>( 1 \ 2 \ 3 \ \ldots \ n )</td>
<td>( \pi/(n+1) )</td>
<td>( \sin(\pi \ k/2(n+1)), \ 1 \leq k \leq n )</td>
</tr>
<tr>
<td>( D_n )</td>
<td>( 1 \ 2 \ 3 \ \ldots \ n )</td>
<td>( \pi/(n-1) )</td>
<td>( \sin(\pi \ k/2(n-1)), \ 1 \leq k \leq n-2 )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( 1 \ 2 \ 3 \ 4 \ 5 \ 6 )</td>
<td>( \pi/6 )</td>
<td>( m_1 = m_5 = m_6/2 = \sqrt{3}/2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( m_2 = m_4 = (3 + \sqrt{3})/2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( m_3 = (3 + \sqrt{3})/\sqrt{2} )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 )</td>
<td>( \pi/9 )</td>
<td>(*)</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>( 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 )</td>
<td>( \pi/15 )</td>
<td>(*)</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>( 1 \ 2 \ 3 \ 4 )</td>
<td>( \pi/9 )</td>
<td>(*)</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( 1 \ 2 )</td>
<td>( \pi/6 )</td>
<td>(*)</td>
</tr>
</tbody>
</table>

(*) These values can be extracted from refs. (64) and (31).

FIGURE CAPTIONS

**Fig. 1** A \( N \times M \) two dimensional lattice. The local states of horizontal (vertical) bonds belong to the vector space \( \mathfrak{sl}(V) \).

**Fig. 2** The local statistical weights \( w(\alpha \beta, a \ b) \) depend on the states of the four bonds joining at a vertex.

**Fig. 3** The YBZF generator \( T_{ab}(\theta) \) associated to a horizontal line of the lattice.

**Fig. 4** The R-matrix is associated to a vertex where all four bonds belong to \( \mathfrak{sl} \).

**Fig. 5** The YBZF algebra in graphical form for local vertices \( \mathfrak{sl}(V) \), \( \mathfrak{sl}(V) \) and \( \mathfrak{sl}(\mathfrak{sl}(R)) \). Up to down in the drawings correspond to matrix multiplication from left to right (see eq. (2.33)).

**Fig. 6** The YBZF algebra for the \( T_{ab}(\theta) \) generator (fig. 3) easily follows from graphical representations.

**Fig. 7** The YB equations for the R-matrix.

**Fig. 8** The 'unitary relation' \( R(\theta) \ R(-\theta) = 1 \) in graphical form.
Fig. 9 Inhomogeneous YBZG generators. The intersection angles $\Theta - \alpha_j$ depend upon the site.

Fig. 10 $S_{a,b}^{c,d}(\theta_1 - \theta_2) = R_{a,b}^{c,d}(\theta_1 - \theta_2)$ is interpreted as the scattering amplitude from the particle states $(a,b)$ to $(c,d)$ with relative rapidity $\theta_1 - \theta_2$.

Fig. 11 Allowed configurations in the six-vertex model and their statistical weights (see eq.(3.1))

Fig. 12 Graphical representation of the $R$-matrices for $\theta = \pm \infty$. They provide braid group generators.

Fig. 13 A braid from $B_n$.

Fig. 14 The elementary operations $b_i$ and $b_i^{-1}$ from the braid group $B_n$.

Fig. 15 Discretized Minkowski space-time. Sites are world events joined by world lines of the bare particle propagation.

Fig. 16 The six non-zero microscopic transition amplitudes. They coincide with the weights of fig. 11.

Fig. 17 Fermion lattice operators associated to the links stemming upwards from each site.

Fig. 18 Allowed configurations in the q(2q - 1) vertex model (1sa, bsq) and their statistical weights for the antiferroelectric regime I.
\[ R_{ii}^{ii} = R_{22}^{22} = a(0,y) \quad R_{42}^{24} = R_{24}^{22} = b(0) \]

\[ R_{42}^{12} = R_{24}^{12} = c(r) \]

Fig. 11

Fig. 12
\[ \psi_{L,n} \quad \psi_{R,n} \]

**Fig. 17**

\[ \exp[\Theta \text{sign}(a-b)] \quad \text{where } \chi = C_{a-b}(0, \lambda) \]
\[ a \neq b \]

\[ \begin{align*}
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{b} \\
\text{a} & \quad \text{a} \\
\text{a} & \quad \text{a}
\end{align*} \]

\[ b(0) = \min b, \Theta \]
\[ a \neq b \]

\[ a(0, \chi) = \min (\chi - \Theta) \]

**Fig. 18**