COUPLING THE $SO(2)$ SUPERGRAVITY
THROUGH DIMENSIONAL REDUCTION

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ABSTRACT

We construct the pure supergravity theory in five dimensions which after dimensional reduction gives $SO(2)$ supergravity coupled to an $SO(2)$ vector matter multiplet.
Already some years ago, the pure extended SO(2) supergravity theory was constructed by different methods\(^1\). Also, the coupling of this theory to SO(2) matter was found by application of the Noether method\(^2\) and by using U(1) invariance\(^3\). The subsequent discovery of the auxiliary fields and a corresponding tensor calculus for \(N = 2\) \(^4\) made it possible to recover the results of Ref. 2 in a more systematic fashion. Similarly, the superspace approach should lead to a full understanding of the coupling between matter and gravity\(^5\). However, although these methods, in principle, enable us to write down even more general models involving SO(2) multiplets, this has, to date, not been done, presumably because of the large number of auxiliary fields that have to be introduced\(^4\).

In this paper we intend to construct an explicit Lagrangian coupling SO(2) gravity to the SO(2) vector multiplet (which contains one vector, two Majorana spinors (= one Dirac spinor) and two scalars). Our method of construction is that of Cremmer, Julia and Scherk\(^6\). We first derive the action and transformation laws in five dimensions and then dimensionally reduce\(^6\) the theory to four dimensions, thereby obtaining the desired Lagrangian. In many respects this theory can be used as a toy model for SO(8) supergravity to gain familiarity with it.

This method has the advantage of being rather straightforward and we expect that our results can be rederived by use of the above-mentioned methods and are analogous to Ref. 3. As in \(D = 11\), the field content of the five-dimensional theory turns out to be surprisingly simple: a fünfbein, which upon dimensional reduction yields a vierbein \(e^\alpha_\mu\), a vector \(B^5_\mu\) and a scalar \(^*)\( e^5_\mu = \exp P\); a five-dimensional vector \(A_\mathcal{M}\) subject to the gauge transformation

\[
S A_\mathcal{M} = \mathcal{A}_\mathcal{M} \wedge
\]

which gives another four-dimensional vector \(A_\mu\) and a scalar \(A_5\); and finally, a Dirac spinor \(\psi_\mathcal{M}\) which is decomposed into a Dirac gravitino \(\psi_\mu\) and an ordinary Dirac spinor \(\psi_5\). It is easy to see that the physical degrees of freedom are just those of one SO(2) gravity and one SO(2) vector multiplet.

Our conventions are the same as in Ref. 6): as the metric, we take \((+-----)\), and by

\(^*)\) We use the notation \(\mathcal{M}\) for the curved internal index and \(s\) for the flat one.
\[ \Gamma^{\Lambda_1 \ldots \Lambda_k} \equiv \Gamma^{[\Lambda_1 \ldots \Lambda_k]} \] (2)

we denote the fully antisymmetrized product of the 4 \times 4 Dirac matrices \( \Gamma^{\Lambda_1} \ldots \Gamma^{\Lambda_k} \). In particular,

\[ \Gamma^5 \equiv -\Gamma_5 \equiv i\gamma^5 = i\gamma_5 \] (3)

is anti-Hermitean. Tangent space indices are from the beginning of the alphabet while curved ones are from the middle.

The Lagrangian we find reads

\[ L = -\frac{1}{4} \mathcal{E} R(\omega) - \frac{i}{2} \left[ \overline{\psi}_M \Gamma^{mnp} D_N \left( \frac{3\omega - \hat{\omega}}{2} \right) \psi_P + \right. \]
\[ + \left. \overline{\psi}_P D_N \left( \frac{3\omega - \hat{\omega}}{2} \right) \Gamma^{mnp} \psi_M \right] - \frac{1}{4} \mathcal{E} F_{MN} F^{MN} \]
\[ - \frac{\sqrt{3}}{2} i \mathcal{E} \overline{\psi}_M \chi^{mnpq} \psi_N \left( F_{pq} + \hat{F}_{pq} \right) \]
\[ - \frac{1}{6\sqrt{3}} \mathcal{E}^{mnpqr} F_{MN} F_{PO} \hat{A}_R \] (4)

The quartic terms have been absorbed into the supercovariant fields \( \hat{A}, \hat{F} \) to be defined below. We have used the following definitions:

\[ D_N \psi_P = \nabla_N \psi_P + \frac{i}{4} \omega_{MAB} \Gamma^{AB} \psi_P \] (5)

\[ \omega_{MAB} = \omega_{MAB}(e) + K_{MAB} \]

with the contorsion tensor...
\[ K_{MAB} = -\frac{i}{2} \bar{\psi}_o \Gamma^{oR}{}_{MAB} \psi_R + \frac{i}{2} \left[ (\bar{\psi}_m \Gamma_8 \psi_A - \bar{\psi}_A \Gamma_8 \psi_M) + (\bar{\psi}_B \Gamma_m \psi_A - \bar{\psi}_A \Gamma_m \psi_B) - (\bar{\psi}_m \Gamma_A \psi_B - \bar{\psi}_B \Gamma_A \psi_M) \right] \]

(6)

The field strength tensor is

\[ F_{MN} = \omega_{M}{}_{A} - \omega_{N}{}_{A} \]

(7)

and we have also introduced the notation

\[ \chi^{MNPR} = \Gamma^{MNPR} + g^{MP} g^{NR} - g^{MA} g^{NP} \]

(8)

The Lagrangian (4) is invariant under the following supersymmetry transformations:

\[ \delta \epsilon_m = -i \bar{\epsilon} \Gamma^A \psi_m + i \bar{\psi}_m \Gamma^A \epsilon \]

\[ \delta \psi_m = \hat{D}_m (\hat{\omega}) \epsilon \equiv D_m (\hat{\omega}) \epsilon + \frac{1}{\sqrt{3}} (\Gamma_m \rho^o - 4 \delta_m \Gamma^o) \epsilon \hat{F}^o \]

(9)

\[ \delta A_m = -\frac{i \sqrt{3}}{2} (\bar{\epsilon} \psi_m - \bar{\psi}_m \epsilon) \]

We are using second order formalism throughout, so \( \omega_{MAB} \) is obtained from its equation of motion. Its supercovariant extension is given by

\[ \hat{\omega}_{MAB} \equiv \omega_{MAB} + \frac{i}{2} \bar{\psi}_o \Gamma^{\sigma R}{}_{MAB} \psi_R \]

(10)

The supercovariant field strength is
$$\hat{F}_{MN} = F_{MN} + \frac{i \sqrt{2}}{2} (\overline{\psi}_M \gamma^\nu \psi_N - \overline{\psi}_N \gamma^\nu \psi_M)$$  \hspace{2cm} (11)$$

and one readily verifies that

$$\delta \hat{F}_{AB} = -i \sqrt{2} \epsilon^M_{eij} \epsilon^{N\ell} \left( \overline{\psi} \gamma^\mu D_N \psi_M - D_N \psi_M \right)$$  \hspace{2cm} (12)$$

where $D_N \psi$ is defined analogously to $D_N \phi = \overline{D}_N \phi$. To obtain the action and transformation laws we have mimicked the procedure of Ref. 6; the coefficients in the transformation rules, as well as the form of the $\overline{\psi} \gamma^\mu \psi \gamma^\nu \psi$ interaction, are determined by requiring the closure of the algebra on the Bose fields and the cancellation of $\overline{\psi} \gamma^\mu \psi \gamma^\nu \psi^2$ terms. To obtain a full cancellation of all terms of type $\overline{\psi} \gamma^\mu \psi \gamma^\nu \psi^2$ we are forced to add a term proportional to

$$\epsilon^{MNPQR} \hat{F}_{MN} F_{PO} A_R$$  \hspace{2cm} (13)$$

again in complete analogy with the eleven-dimensional theory\textsuperscript{6).} The quartic terms are then fixed so as to reproduce the supercovariant equation of motion

$$\Gamma^{MNP} \overline{D}_N (\hat{\omega}) \psi_P = 0$$  \hspace{2cm} (14)$$

This is a test of consistency and involves the following Fierz rearrangement identity

$$\epsilon^{RCE} \hat{F}^{MNP} \psi_N \cdot \overline{\psi}_P \Gamma_R \psi_Q =$$  \hspace{2cm} (15)$$

$$= \frac{1}{4} \left( \Gamma^{MNP} \psi_N \cdot \overline{\psi}_P \psi_Q - \psi_N \cdot \overline{\psi}_P \Gamma^{MNP} \psi_Q \right)$$

The next step in our construction is the dimensional reduction which allows us to make some contact with the real world and to identify the physical fields of the model. In the simplest scheme, one assumes independence of the fifth coordinate and, again, our derivation is analogous to the one that has been carried out for the $N = 8$ theory by Cremmer and Julia\textsuperscript{7).}
Using the non-diagonal part of local $\text{SO}(1,4)$, we gauge away the field $e_5^a$, so the fünfbein assumes the form

$$\mathbf{e}_\mathbf{M}^a = \begin{pmatrix} e_\mu^a & B_\mu^5 \\ 0 & e_5^a = \exp[P] \end{pmatrix}, \quad e_\mathbf{A}^M = \begin{pmatrix} e_\mu^M & -B_\mu^5 \\ 0 & e_5^a \exp[-P] \end{pmatrix}$$

where the scalar field has been written as $\exp P$ for later convenience.

After the following redefinitions and Weyl rescalings\(^7\) for the bosonic fields

$$e_{\gamma\mu}^a = \exp\left[\frac{1}{2} P\right] e_{\gamma\mu}^a, \quad e_{\gamma\mu}^\nu = \exp \left[-\frac{1}{2} P\right] e_{\gamma\mu}^\nu$$

$$G_{\gamma\mu^5} = 2 \exp[P] e_\gamma^\mu e_\mu^5 \nu \partial_\nu \left( \exp[-P] B_\nu^5 \right)$$

$$A'_\mu = A_\mu - B_\mu^5 A_5, \quad A_5 = e_5^a A_5$$

$$F_{\mu\nu} = 2 \partial_\mu A'_\nu, \quad F_{\mu\nu} = F_{\mu\nu} + G_{\mu\nu}^5 A_5$$

$$F_{\alpha\beta} = \exp[P] F_{\gamma\mu}$$

we obtain the reduced Lagrangian

$$L_{\text{bosonic}} = -\frac{1}{4} e_\gamma R_\gamma (\omega_0) + \frac{3}{8} e_\gamma \left( \partial_\gamma P \right)^2 + \frac{1}{16} e_\gamma \left( G_{\gamma\mu^5} \right)^2 - \frac{1}{4} e_\gamma \left( \exp[P] F_{\mu\nu} F_{\mu\rho\sigma} G_{\sigma\nu}^\mu G_{\rho\gamma}^a \partial_\gamma \partial_\rho A_5 + 2 \delta_5^a \partial_\mu A_5 \partial_\nu A_5 \partial_\gamma \partial_\rho \right)$$

$$- \frac{1}{2\sqrt{3}} e_\gamma \exp[P] \epsilon^{\mu\nu\rho\sigma} \left( F_{\mu\nu} F_{\mu\rho\sigma} + F_{\mu\mu} G_{\rho\sigma}^5 A_5 - \frac{1}{3} G_{\mu\sigma}^5 G_{\rho\sigma}^5 A_5^{2} \right) A_5$$

For the fermionic sector we need the following redefinitions:
\[ X_4 = Y_i^5 e_5 i \exp \left[ \frac{P}{4} \right] \]

\[ X_{4d} = Y_M e_\alpha^d \exp \left[ -\frac{P}{4} \right] \]

\[ \Psi_M = e_\mu^d \left( X_{4d} - \frac{i}{2} \Gamma^5 \gamma_\alpha X_4 \right) \]

(19)

For instance, the last redefinition diagonalizes the kinetic term for the fermions which now takes the form

\[ -\frac{i}{2} e_4 \left[ \bar{\Psi}_M \gamma^\mu \nu \rho \sigma D_\nu (\omega_0) \Psi_\rho + \bar{\Psi}_\rho \Gamma_{\nu \rho \sigma} (\omega_0) \gamma^\mu \nu \rho \Psi_\mu \right. \]

\[ + \frac{3}{2} \bar{X}_4 \gamma^\mu D_\nu (\omega_0) X_4 + \frac{3}{2} \bar{X}_4 \Gamma_{\nu \rho \sigma} \gamma^\mu X_4 \]

(20)

The interaction terms arise from

\[ -\frac{ie}{8} \bar{\Psi}_M \left\{ \Gamma^{\mu \nu \rho}, \Gamma^{\alpha \beta} \right\} \Psi_\rho \omega_0^{\alpha \beta} - \frac{\sqrt{3}}{4} ie \bar{\Psi}_M \chi^{\mu \nu \rho \sigma} \chi_\rho \Gamma_0 \]

(21)

which leads to the reduced vector interaction

\[ \mathcal{L}^\text{int} = ie_4 \exp \left[ \frac{P}{2} \right] \left( -\frac{\sqrt{3}}{2} F_\mu \chi_\rho + \frac{i}{4} \tilde{G}_{\alpha \beta} \right) \]

\[ \cdot \left\{ \left( \bar{\Psi}_\mu \gamma_\rho - \frac{i}{2} \bar{X}_4 \gamma^\rho \Gamma^5 \Psi_\mu + \frac{i}{2} \bar{\Psi}_\mu \Gamma^5 \gamma^\rho \chi_4 \right) \right. \]

\[ - \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \left( \bar{X}_4 \gamma_5 \Psi_\rho - \frac{i}{2} \bar{X}_4 \gamma_5 \Psi_\rho + \frac{i}{2} \bar{\Psi}_\rho \gamma_5 \chi_4 \right) \}

(22)

The reduction of the quartic terms is straightforward but tedious, and we have only checked that the spin 3/2 quartic terms coincide with those of pure SO(2) supergravity.
We are now able to identify the $SO(2)$ submultiplets. A non-trivial mixing, besides the one in (19) between spin 3/2 and spin 1/2, only occurs for the vector fields where the two multiplets "overlap". We find

$$
{F}_{\mu \nu} (\text{gravity}) = -\frac{\sqrt{2}}{2} F_{\mu \nu} + \frac{2}{\epsilon_q} \frac{D^2}{\delta G_L^{\mu \nu}}
$$

$$
{G}_{\mu \nu} (\text{vector}) = \frac{1}{2} F_{\mu \nu} + \frac{2 \sqrt{3}}{\epsilon_q} \frac{D^2}{\delta G_L^{\mu \nu}}
$$

where

$$
\frac{2}{\epsilon_q} \frac{D^2}{\delta G_L^{\mu \nu}} = \frac{1}{4} G_{\mu \nu} + \frac{i}{2} \left( \langle \overline{\chi}_q \gamma_\alpha \chi_q \rangle - \frac{1}{2} \overline{\chi}_q \gamma_\alpha \gamma^\beta \gamma^5 \chi_q \right) + \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \left( \overline{\chi}_q \gamma_5 \gamma^\beta \gamma^\rho \gamma^\sigma \gamma^5 \chi_q \right)
$$

A curious, although not unexpected, feature is that the identification (23) is possible on shell only because it is only there that $F_{\mu \nu}$ and $G_{\mu \nu}$ can be expressed as curls of vector fields. The duality transformation (23) automatically leads to the correct sign for the kinetic terms of the vector fields because

$$
-\frac{1}{4} \left( F_{\mu \nu}^2 + G_{\mu \nu}^2 \right) = -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{16} (G_{\mu \nu}^5)^2
$$

(This is also true for the $N = 8$ theory.)

The supersymmetry transformation rules become, after the reduction (dropping indices $\alpha, \beta$ so, e.g., $e, \xi e$, etc.)

$$
\delta \epsilon_\mu \alpha = -i \overline{\xi} \gamma_\alpha \chi_\mu + i \overline{\psi}_\mu \gamma_\alpha \chi + i \Omega_\alpha^\beta \epsilon_\mu \beta
$$

$$
\Omega_\alpha^\beta = \text{compensating gauge rotation}
$$

$$
\delta \mathcal{A}_5^- = -\frac{i \sqrt{3}}{2} (\overline{\chi} \chi - \overline{\chi} \xi \chi) \exp [-\mathcal{P}]
$$
\[ \delta \mathcal{P} = -i \left( \bar{\mathcal{E}} \gamma^5 \chi - \bar{\chi} \gamma^5 \mathcal{E} \right) \]

\[ \delta \chi = \frac{1}{2} \gamma^\mu \gamma^5 \varepsilon \cdot \hat{D}_\mu \mathcal{P} - \frac{i}{\sqrt{3}} \exp[-i \mathcal{P}] \gamma^\nu \varepsilon \cdot \hat{D}_\nu \mathcal{A}_5 \]

\[ + \frac{i}{2 \sqrt{3}} \exp[-i \mathcal{P}] \gamma^5 \gamma^\alpha \varepsilon \cdot \mathcal{G}_{\alpha \beta} \]

\[ \delta \Psi_\mu = D_\mu \varepsilon - \frac{i}{4} \gamma^{\rho \sigma} \gamma_\mu \varepsilon \cdot \mathcal{F}_{\rho \sigma} + \ldots \]

where

\[ \hat{D}_\mu \mathcal{P} = \partial_\mu \mathcal{P} - \frac{i}{2} \left( \bar{\mathcal{E}} \gamma^5 \chi - \bar{\chi} \gamma^5 \Psi_\mu \right) \]

\[ \hat{D}_\mu \mathcal{A}_5 = \partial_\mu \mathcal{A}_5 - \frac{i \sqrt{3}}{2} \left( \bar{\Psi}_\mu \chi - \bar{\chi} \Psi_\mu \right) \]

To get the right parity assignments we perform a further (chiral) redefinition

\[ \varepsilon \rightarrow (\gamma^5)^{1/2} \varepsilon, \quad \chi \rightarrow (\gamma^5)^{-1/2} \chi, \quad \Psi_\mu \rightarrow (\gamma^5)^{-1/2} \Psi_\mu \]

(28)

(to preserve the proper transformation rule for the vierbein). Moreover, all factors of \( \sqrt{3} \) disappear if one rescales

\[ \chi \rightarrow \frac{1}{\sqrt{3}} \chi, \quad \mathcal{P} \rightarrow \frac{2}{\sqrt{3}} \mathcal{P} \]

(29)

so as to obtain the canonical normalization of the corresponding kinetic terms.

Splitting \( \chi \) into \( 1/\sqrt{2}(\chi_1 + i \chi_2) \) where \( \chi_1, \chi_2 \) are Majorana spinors, we get, for instance, using (3)

\[ \delta \chi^i = \frac{1}{2} \gamma^{\mu \nu} \varepsilon^i \mathcal{G}_{\mu \nu} - i \varepsilon^i j (\gamma^\mu \varepsilon_j \partial_\mu \mathcal{A}_5 + i \gamma^\mu \gamma_5 \varepsilon_j \partial_\mu \mathcal{P}) \]

\[ + \ldots \]

(30)

which is just the expected transformation law \(^{2,3}\). It is not difficult to check that the truncations to the respective \( \text{SO}(2) \) submultiplets are consistent. The above results are easily seen to agree with Ref. 3).
A physically somewhat more interesting situation is obtained in a generalized dimensional reduction\textsuperscript{8)} if one retains a dependence on the fifth coordinate in the form \( \exp \left[ \mathrm{i} m x_5 \right] \). The supersymmetry is then spontaneously broken because \( \partial_5 \neq 0 \) and the central charge\textsuperscript{9)} which is gauged by the \( A_5 \) field no longer vanishes but is rather proportional to the mass. This may in turn lead to anti-gravity\textsuperscript{10)} and we are presently investigating whether this interesting phenomenon indeed occurs for the model of this paper.

ACKNOWLEDGEMENTS

We are grateful to J. Ellis for a critical reading of the manuscript.
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