PLASMA PHYSICS AND INSTABILITIES

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ABSTRACT

These lectures provide an introduction to the theory of plasmas and their instabilities. Starting from the Bogoliubov, Born, Green, Kirkwood, and Yvon (BBGKY) hierarchy of kinetic equations, the additional concept of self-consistent fields leads to the fundamental Vlasov equation and hence to the warm two-fluid model and the one-fluid MHD, or cold, model. The properties of small-amplitude waves in magnetized (and unmagnetized) plasmas, and the instabilities to which they give rise, are described in some detail, and a complete chapter is devoted to Landau damping. The linear theory of plasma instabilities is illustrated by the current-driven electrostatic kind, with descriptions of the Penrose criterion and the energy principle of ideal MHD. There is a brief account of the application of feedback control. The non-linear theory is represented by three examples: quasi-linear velocity-space instabilities, three-wave instabilities, and the stability of an arbitrarily large-amplitude wave in a plasma.
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LISTS OF THE SYMBOLS

N.B. The symbols only used in one chapter, or with a meaning different from the general one, are listed below by chapter.

GENERAL LIST OF SYMBOLS

c  velocity of light

c_{A}  Alfvén speed of a linear wave (= B/\sqrt{\mu_0 \rho})

c_{S}  ion acoustic velocity (= \sqrt{\frac{1}{m_i} T_e})

f[f_{0}, f_{1}]  phase fluid distribution function (f_{0} = equilibrium part, f_{1} = fluctuating part)

j  index of the particle species: = e for electrons
    = i for ions

k  Boltzmann's constant

K  linear perturbation wave vector (components k_x, k_y, k_z)

k  magnitude of the wave vector

m_{j}  mass of the jth species

n_{j}[n_{j1}]  number density of the jth species (n_{j1} = fluctuating part)

p_{0}  isotropic scalar pressure (p_{0} = equilibrium part)

p_{j}  scalar pressure of the jth species

q_{j}  charge of the jth species

t  time

u_{j} or v_{j}  mean velocity of the jth species (v_{j1} = fluctuating part)

V  fluid velocity coordinates (components v_x, v_y, v_z)

\nu_{Tj}  thermal velocity of the jth species (= \sqrt{kT_{j}/m_{j}})

\bar{X}  position coordinates (components x, y, z)

D(\omega,k)  dispersion function of the system

\vec{B}_{0,1}  self-consistent magnetic induction (\vec{B}_{0} = equilibrium part, \vec{B}_{1} = fluctuating part)

\vec{E}_{0,1}  self-consistent electric field (\vec{E}_{1} = perturbation part)

\vec{H}_{0,1}  self-consistent magnetic field (\vec{H}_{1} = perturbation part)

\vec{J}_{0,1}  current density (\vec{J}_{0} = equilibrium part, \vec{J}_{1} = fluctuating part)

T_{j}  kinetic temperature of the jth species

\gamma  ratio of specific heats for one fluid plasma

\gamma_{j}  ratio of specific heats for the jth species

\varepsilon(\omega,k)  dielectric function

\varepsilon_{0}  dielectric constant of the vacuum

\lambda  wavelength of a linear wave

\lambda_{De}  Debye length for electrons (= \nu_{Te}/\omega_{pe})

\mu_{0}  permeability of the vacuum

\rho_{e}  charge density

\rho_{0,1}  mass density (\rho_{0} = equilibrium part, \rho_{1} = fluctuating part)

\phi  electrostatic potential

\omega  frequency of linear perturbation waves

\omega_{cj}  cyclotron frequency of the jth species

\omega_{pj}  plasma frequency of the jth species (= \sqrt{n_{j} e^{2}/\varepsilon_{0} m_{j}})
**MATHMATICAL NOTATIONS**

- $b^*$: complex conjugate of $b$
- $B_z$: complex quantity equal to $B_x + iB_y$
- $\delta(...)$: delta function
- $\nabla$: gradient operator
- $D/ Dt$: total time derivative equal to $\partial/\partial t + \nabla \cdot \mathbf{v}$
- $i$: square root of $-1$
- $I$: unity diagonal tensor
- $\text{Im}(...)$: imaginary part of a complex quantity
- $P(...)$: principal part of an integral
- $\text{Re}(...)$: real part of a complex quantity
- $\text{Tr}(...)$: trace of a tensor

MKSA units are used throughout the report. As mentioned for some variables, the subscripts 0 and 1 denote the equilibrium and fluctuating parts of these variables, respectively.

**LIST OF SYMBOLS FOR CHAPTER I**

- $f$: six-dimensional phase fluid distribution function
- $f_i$: one-particle distribution function
- $f_2$: two-particle distribution function
- $f_j$: three-particle distribution function
- $f_{\lambda_j}$: fluid distribution function of the $j^{th}$ species
- $f_{\lambda_j}$: reduced distribution function
- $n_e$: electron density in the configuration space ($= N/V$)
- $\mathbf{p}_i$: canonical momentum of the $i^{th}$ electron
- $\mathbf{q}_i$: canonical position coordinates of the $i^{th}$ electron
- $\mathbf{u}_i$: mean center of mass velocity
- $\mathbf{x}_i$: six-dimensional phase space coordinates of the $i^{th}$ electron
- $P_N$: N-electron distribution function
- $E_i(\mathbf{q}_i)$: electric field seen by the $i^{th}$ electron at the position $\mathbf{q}_i$
- $H_N$: Hamiltonian function for the N-electron plasma
- $L_D$: Debye shielding distance ($= \varepsilon_0 kT/n_e e^2$)
- $N$: total number of electrons in the plasma
- $\tilde{P}_j$: pressure tensor for the $j^{th}$ species
- $P(\mathbf{x}_1,\mathbf{x}_2)$: two-particle correlation function
- $T(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$: three-particle correlation function
- $V$: volume of the configuration space
- $\phi_{ij}$: electric potential seen by the $i^{th}$ electron, due to the $j^{th}$ electron
- $\phi_{\text{wall}}$: potential barrier at boundary of the N-electron plasma
- $\psi_{\text{ion}}$: smeared-out potential due to the ions

*N.B.* The symbols $m$ and $T$ (without the subscript $e$) are used in this chapter for the mass and the kinetic temperature of the electron.
LIST OF SYMBOLS FOR CHAPTER II

\( n_0 \) equilibrium density of the two fluids (equal and uniform densities of electrons and ions)

\( v \) mean velocity of the plasma (single-fluid model)

\( z \) direction of propagation of the linear wave

\( \beta \) ratio of the square of the acoustic velocity to the square of the Alfvén speed

\( \varepsilon_0, \varepsilon_1 \) dielectric tensor

\( \sigma(\omega, k) \) conductivity tensor

\( \omega_L \) frequency of the longitudinal Langmuir wave

\( \omega_S \) low-frequency branch of the longitudinal modes

\( \omega_T \) frequency of the transverse modes

\( \omega_{\text{HH}} \) upper-hybrid frequency (= \( \sqrt{\omega_p^2 + \omega_e^2} \))

LIST OF SYMBOLS FOR CHAPTER III

\( f(k, v, t) \) two-dimensional distribution function of the electrons

\( f_0 \) equilibrium density function

\( f_\delta \) derivative of the equilibrium density function w.r.t. the velocity amplitude (= \( \delta f_0 / \delta v \))

\( g(k, v) \) initial value of the distribution function [= \( f(k, v, t = 0) \)]

\( p \) complex variable of the Laplace transform (= \( \gamma - i\omega \))

\( p_n(k) \) poles of the electrostatic potential integral

\( v \) velocity amplitude (in the x-direction)

\( x \) direction of propagation of the linear wave

\( C \) Landau's contour in the complex plane for integration

\( R_n(k) \) residues associated with the \( p_n \) poles

\( \sigma \) Bromwich contour for the Laplace transforms

\textit{N.B.} The symbols q, m and T (without the subscript 'e') are used in this chapter for the charge, the mass and the kinetic temperature of the electron.

LIST OF SYMBOLS FOR CHAPTER IV

\( a \) parameter of the instability (e.g. the current density)

\( a_0 \) parameter value at the threshold where \( \varepsilon_h \) is zero

\( f_{s0} \) equilibrium distribution function of the \( j \)th species

\( f_{s1} \) perturbation of the distribution function of the \( j \)th species

\( g(\omega, k) \) response function of the feedback circuit

\( k_s \) wave vector magnitude at which \( \varepsilon_h \) is zero

\( v \) velocity amplitude

\( V_d \) equilibrium drift velocity of the electrons

\( z \) direction of the current flow (relative motion between the ions and the electrons)

\( E \) total energy of the perturbation

\( I(\xi) \) virial associated with plasma displacement

\( K \) total kinetic energy of the plasma

\( N \) number of zeros of the dispersion function in the upper half-\( \omega \)-plane

\( \delta W \) potential energy of the perturbation
\( Z(\zeta) \)    plasma dispersion function or Fried Conté function
\( \alpha(\omega,k) \) function used in the Penrose stability analysis \((= k^2)\)
\( \varepsilon_h(\omega,k) \) Hermitian part of the dielectric function \( \varepsilon \)
\( \varepsilon_a(\omega,k) \) anti-Hermitian part of the dielectric function \( \varepsilon \)
\( \eta \)    plasma displacement
\( \theta \)    phase of the feedback circuit
\( \zeta \)    plasma displacement vector
\( \omega_0 \) real part of the frequency for which \( \varepsilon_h \) is zero
\( \Gamma \)    integration contour in the complex plane of the dispersion \( D \)

N.B. In a limited number of equations, the symbols \( \omega \) and \( \gamma \) are used for the growth rate of the perturbation.

**LIST OF SYMBOLS FOR CHAPTER V**

\( a_L \)    normalized amplitude of the longitudinal Langmuir electric field
\( a_{T0} \) normalized amplitude of the incident (pump) electric field
\( a_{T1} \) normalized amplitude of the scattered electric field
\( f_L \)    amplitude of the distribution function mode \( \vec{k} \) (Fourier analysis)
\( k_L \)    wave number of the longitudinal electromagnetic wave
\( k_{T0} \) wave number of the incident electromagnetic wave
\( k_{T1} \) wave number of the scattered electromagnetic wave
\( n_0 \)    equilibrium density of electrons
\( n_L \)    fluctuating electron density due to the longitudinal wave
\( v_d \)    uniform drift velocity along the z-axis
\( v_L \)    longitudinal velocity associated with the Langmuir wave
\( x \)    direction of propagation of the fluctuating distribution \( f_1 \)
\( z \)    direction of propagation of the large amplitude magnetic wave
\( \hat{x}, \hat{y}, \hat{z} \)    unit vectors in the direction of the coordinates
\( \hat{B}_0 \)    incident magnetic field
\( \hat{D}(\vec{\nu}) \) diffusion tensor in the velocity space
\( E_L \)    longitudinal electric field
\( E_{T0} \) incident electric field
\( E_{T1} \) scattered electric field
\( \hat{\xi}_L \) amplitude of the longitudinal electric field
\( \hat{\xi}_{T0} \) amplitude of the incident electric field
\( \hat{\xi}_{T1} \) amplitude of the scattered electric field
\( \xi_{ext} \) external source of current
\( L \)    longitudinal Langmuir wave
\( T \)    transverse electromagnetic wave
\( \gamma \)    growth rate of longitudinal and transverse fields
\( \gamma_{\vec{k}} \) growth or damping rate of the mode \( \vec{k} \) of the distribution function
\( \gamma_L \) damping rate of the longitudinal field
\( \gamma_T \) damping rate of the transverse field
\( \varepsilon_1 \) real part of the dielectric function \( \varepsilon \)
\( \varepsilon_2 \) imaginary part of the dielectric function \( \varepsilon \)
\( \nu_e \) electron collision frequency
\( \rho_{EL} \)  
total energy density for longitudinal waves

\( \rho_{ET} \)  
total energy density for transverse waves

\( \tau \)  
time interval (\( = t - t_0 \))

\( \psi \)  
frequency mismatch between three interaction waves

\( \omega_K \)  
frequency of the mode \( \hat{K} \) of the distribution function

\( \omega_L \)  
frequency of the longitudinal electromagnetic wave

\( \omega_{T0} \)  
frequency of the incident electromagnetic wave

\( \omega_{T1} \)  
frequency of the scattered electromagnetic wave

\( \delta \omega \)  
correction to the natural frequency of the perturbation

\( \Omega \)  
frequency of the coupled wave amplitude oscillations

\( \text{N.B.} \) In section V.3, the subscript 0 of the variables \( \hat{\delta}, \hat{\beta}_{\text{ext}}, \hat{\nu}, \omega \) and \( k \) means that these quantities are associated with a large amplitude wave.
CHAPTER I

PLASMA MODELS

1.1 BBGKY\textsuperscript{*} THEORY AND THE DERIVATION OF THE VLASOV EQUATION

The most general and powerful description of the plasma state is of course the KINETIC THEORY. However, a plasma presents special difficulties owing to the long-range nature of the Coulomb force. Thus, one can no longer think of two-body interactions acting for a short time and over a small region. Instead, many particles will interact simultaneously with each other. It is a many-body problem par excellence.

We shall give a brief description of the so-called BBGKY\textsuperscript{*} theory, which starts from the probability distribution for all possible states of the plasma, i.e. all the particles. The question of how this distribution is chosen is deliberately left unanswered, and there is no hypothesis of assigning equal probabilities to equal phase-space volumes \textit{a priori}. The aim is to derive Vlasov's equation -- the basis of most modern plasma theory -- from the general considerations just outlined.

We shall keep the problem as manageable as possible by confining our attention to a one-component plasma in which the ions are immobile and simply neutralize the electron charge in equilibrium. Furthermore, we shall ignore magnetic field interactions including only the Coulomb interaction. This simplified model contains all the essential physics and can be generalized to a multi-component plasma with allowance for magnetic field interactions.

Let us begin by writing down the Hamiltonian for an \( N \)-electron plasma (embedded in a background of immobile positive charge) where the \( i \)th electron has six phase-space coordinates \( \vec{z}_i \equiv (\vec{q}_i, \vec{p}_i) \), where the \( \vec{q}_i \) and \( \vec{p}_i \) are canonical Hamiltonian variables,

\[
H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j=1}^{N} \frac{e^2}{4\pi\epsilon_0 |\vec{q}_i - \vec{q}_j|} + \sum_{i=1}^{N} \left[ \phi_{\text{wall}}(\vec{q}_i) + \psi_{\text{ion}}(\vec{q}_i) \right],
\]

(1.1)

where \( \phi_{\text{wall}} \) represents some potential barrier at the boundary of the volume of the system, and \( \psi_{\text{ion}}(\vec{q}_i) \) is the potential of the \( i \)th electron moving in the smeared-out potential due to the ions.

We now introduce the \( N \)-electron probability distribution \( D_N(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N; t) \), normalized to unity. \textit{Liouville's theorem} tells us how \( D_N \) evolves in time, namely

\footnote{Bogoliubov, Born, Green, Kirkwood and Yvon (BBGKY).}
\[
\frac{\partial D_N}{\partial t} + \left\{ D_N; H_N \right\} = 0 ,
\]  
(1.2)

where the Poisson bracket is defined as

\[
\left\{ D_N; H_N \right\} = \sum_{i=1}^{N} \left[ \frac{\partial D_N}{\partial q_i} \frac{\partial H_N}{\partial p_i} - \frac{\partial D_N}{\partial p_i} \frac{\partial H_N}{\partial q_i} \right].
\]  
(1.3)

We should note that we are treating the system classically and that we assume \( D_N \) is symmetric to the interchange of like-particle coordinates (indistinguishable particles). We also assume the system is isolated.

Let us now define a reduced probability distribution \( f_s/V^s \), where \( V \) is the configuration space volume of the system

\[
\frac{f_s}{V^s} \equiv \int D_N \; d\vec{x}_{s+1}, \ldots, d\vec{x}_N .
\]  
(1.4)

Writing out the Poisson bracket in Eq. (1.2) in detail, we obtain

\[
\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \frac{\partial D_N}{\partial q_i} \frac{\vec{p}_i}{m} - \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{|\vec{q}_i - \vec{q}_j|} \frac{\partial D_N}{\partial p_i} - \sum_{i=1}^{N} \frac{e}{\varepsilon_0} \left[ \phi_{\text{wall}}(\vec{q}_i) + \psi_{\text{ion}}(\vec{q}_i) \right] \frac{\partial D_N}{\partial p_i} = 0 .
\]  
(1.5)

This can be expressed in a more useful form as follows:

\[
\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \frac{\vec{p}_i}{m} \frac{\partial D_N}{\partial q_i} - e \sum_{i=1}^{N} \frac{\vec{E}_i}{\varepsilon_0} \frac{\partial D_N}{\partial p_i} - \sum_{i=1}^{N} \frac{e}{\varepsilon_0} \phi_{\text{wall}}(\vec{q}_i) - \sum_{i=1}^{N} \psi_{\text{ion}}(\vec{q}_i) = 0 ,
\]  
(1.6)

where

\[
e\vec{E}_i = \sum_{j=1 \atop j \neq i}^{N} \frac{1}{4\pi\varepsilon_0} \frac{e^2}{|\vec{q}_i - \vec{q}_j|} + \frac{e}{\varepsilon_0} \psi_{\text{ion}}(\vec{q}_i)
\]  
(1.7)

is the electric field seen by the \( i \)th electron at position \( \vec{q}_i \). Now integrate Eq. (1.6) over the \( x_{s+1}, x_{s+2}, \ldots, x_N \) subspace.
\[ \frac{3}{2} \int D_N \, dx_{s+1}, \ldots, dx_N + \sum_{i=1}^{N} \int \frac{p_i}{m} \frac{3D_N}{\partial q_i} \, dx_{s+1}, \ldots, dx_N \]

\[ \sum_{i=1}^{N} \int \frac{E_i}{\partial p_i} \, dx_{s+1}, \ldots, dx_N \]

\[ -e \sum_{i=1}^{N} \int \frac{3D_N}{\partial p_i} \, dx_{s+1}, \ldots, dx_N \]

\[ - \sum_{i=1}^{N} \int \frac{a}{\partial q_i} \, \delta_{\text{wall}}(q_i) \cdot \frac{3D_N}{\partial p_i} \, dx_{s+1}, \ldots, dx_N = 0. \]

Now, using the definition (1.4)

\[ \sum_{i=1}^{N} \int \frac{p_i}{m} \frac{3D_N}{\partial q_i} \, dx_{s+1}, \ldots, dx_N = \sum_{i=s+1}^{s} \frac{p_i}{m} \frac{f_s}{\partial q_i} + \sum_{i=s+1}^{N} \int \frac{p_i}{m} \frac{3D_N}{\partial q_i} \, dx_{s+1}, \ldots, dx_N \]

\[ = \sum_{i=1}^{s} \frac{p_i}{m} \frac{f_s}{\partial q_i} + \sum_{i=s+1}^{N} \left. \frac{p_i}{m} D_N \right|_{-1}^{+1} \]

\[ = \sum_{i=1}^{s} \frac{p_i}{m} \frac{f_s}{\partial q_i} \]

since \( D_N \) is zero at the boundaries. Next

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \int \frac{3q_{ij}}{\partial q_i} \frac{3D_N}{\partial p_i} \, dx_{s+1}, dx_{s+2}, \ldots, dx_N \]

\[ = \sum_{i=1}^{s} \sum_{j=1}^{s} \frac{3q_{ij}}{\partial q_i} \frac{f_s}{\partial p_i} \frac{1}{v^s} + \sum_{i=1}^{s} \sum_{j=s+1}^{N} \int \frac{3q_{ij}}{\partial q_i} \frac{3D_N}{\partial p_i} \, dx_{s+1}, \ldots, dx_N \]

\[ + \sum_{i=s+1}^{N} \sum_{j=1}^{s} \int \frac{3q_{ij}}{\partial q_i} \frac{3D_N}{\partial p_i} \, dx_{s+1}, dx_{s+2}, \ldots, dx_N \]

\[ \left\{ \begin{array}{l}
\text{both ZERO because} \\
\int \frac{3D_N}{\partial p_i} = 0.
\end{array} \right. \]
Now
\[ \sum_{i=1}^{S} \sum_{j=s+1}^{N} \int \frac{\partial \phi_{ij}}{\partial \bar{q}_i} \frac{\partial \bar{D}_N}{\partial \bar{p}_j} dx_{s+1}, \ldots, dx_N \]
\[ = \sum_{i=1}^{S} \int \frac{\partial \phi_{i s+1}}{\partial \bar{q}_i} \frac{\partial \bar{D}_N}{\partial \bar{p}_i} \left\{ \int D_N dx_{s+2}, dx_{s+3}, \ldots, dx_N \right\} dx_{s+1} \]

+ similar terms by the symmetry of $D_N$

\[ = \sum_{i=1}^{S} \int \frac{\partial \phi_{i s+1}}{\partial \bar{q}_i} \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i} (x_1, x_2, \ldots, x_s, x_{s+1}) \, dx_{s+1} \left( N - s \right). \]

Finally, the terms

\[ \sum_{i=1}^{S} \int \left[ \frac{\partial}{\partial \bar{q}_i} \phi_{\text{wall}}(\bar{q}_i) + \frac{\partial}{\partial \bar{q}_i} \psi_{\text{ion}}(\bar{q}_i) \right] \frac{\partial \bar{D}_N}{\partial \bar{p}_i} \, dx_{s+1}, dx_{s+2}, \ldots, dx_N \]

\[ = \sum_{i=1}^{S} \int \left[ \frac{\partial}{\partial \bar{q}_i} \phi_{\text{wall}}(\bar{q}_i) + \frac{\partial}{\partial \bar{q}_i} \psi_{\text{ion}}(\bar{q}_i) \right] \frac{\partial \bar{f}_S}{\partial \bar{p}_i} \frac{1}{V^S} \]

+ \[ \sum_{i=s+1}^{S} \int \left( \frac{\partial}{\partial \bar{q}_i} \phi_{\text{wall}} + \frac{\partial}{\partial \bar{q}_i} \psi_{\text{ion}} \right) \frac{\partial \bar{D}_N}{\partial \bar{p}_i} \, dx_{s+1}, \ldots, dx_N, \]

and the last term is again zero because $\int (\partial \bar{D}_N/\partial \bar{p}_i) \, d\bar{q}_i = 0$ since $D_N \rightarrow 0$ at the limits of the phase space. Putting all these terms in the Eq. (1.8) we obtain

\[ \frac{1}{V^s} \frac{\partial \bar{f}_S}{\partial t} + \sum_{i=1}^{S} \frac{\bar{p}_i}{m} \frac{\partial \bar{f}_S}{\partial \bar{q}_i} v^S - \sum_{i=1}^{S} \sum_{j=1}^{S} \frac{\partial \phi_{ij}}{\partial \bar{q}_i} \frac{\partial \bar{f}_S}{\partial \bar{p}_j} \frac{1}{V^S} \]

\[ - \frac{(N - s)}{V^S} \sum_{i=1}^{S} \int \frac{\partial \phi_{i s+1}}{\partial \bar{q}_i} \frac{\partial \bar{f}_{s+1}}{\partial \bar{p}_i} (x_1, \ldots, x_{s+1}) \, dx_{s+1} \]

\[ = \sum_{i=1}^{S} \left[ \frac{\partial}{\partial \bar{q}_i} \phi_{\text{wall}}(\bar{q}_i) + \frac{\partial}{\partial \bar{q}_i} \psi_{\text{ion}}(\bar{q}_i) \right] \frac{\partial \bar{f}_S}{\partial \bar{p}_i} \frac{1}{V^S}. \]
This can now be written as

\[
\frac{\partial f_S}{\partial t} + \sum_{i=1}^{S} \frac{p_i}{m} \frac{\partial f_S}{\partial q_i} = \sum_{i=1}^{S} \sum_{j \neq i}^{S} \frac{\partial \phi_{ij}}{\partial q_i} \frac{\partial f_S}{\partial p_j} - \frac{(N - S)}{V} \sum_{i=1}^{S} \int \frac{\partial \phi_{iS+1}}{\partial q_i} \left( x_1, x_2, \ldots, x_S, x_{S+1} \right) \, dx_{S+1} = n_0 \sum_{i=1}^{S} \left( \frac{\partial}{\partial q_i} \psi_{wall} + \frac{\partial}{\partial q_i} \psi_{ion} \right) \frac{\partial f_S}{\partial p_i}.
\]  

(I.9)

We now take the boundary to infinity such that

\[N \to \infty, \quad V \to \infty,\]

but \( n_0 \equiv N/V \) remains finite. In this limit we assume that \( \psi_{wall} \) terms vanish at all finite \( q_1 \). We also drop the \( \psi_{ion} \) terms by assuming that the ion background provides only a constant potential which we may subtract out. However, we must remember to include it in any discussion of equilibria, although it will not affect the dynamics for a non-equilibrium system. With these simplifications, we may write the equation for the reduced distribution \( f_S(x_1, x_2, \ldots, x_S) \) as

\[
\frac{\partial f_S}{\partial t} + \sum_{i=1}^{S} \frac{p_i}{m} \frac{\partial f_S}{\partial q_i} - \sum_{i=1}^{S} \sum_{j \neq i}^{S} \frac{\partial \phi_{ij}}{\partial q_i} \frac{\partial f_S}{\partial p_j} = n_0 \sum_{i=1}^{S} \int \frac{\partial \phi_{iS+1}}{\partial q_i} \frac{\partial f_{S+1}}{\partial p_i} \, dx_{S+1}.
\]  

(I.10)

Equation (I.10) can be seen to give a hierarchy of equations for the reduced distribution functions, where each function is coupled to the one above it in the series. Let us write down the equations for the first two distribution functions, i.e. the one- and two-particle functions:

\[
\frac{\partial f_1}{\partial t} + \frac{p_1}{m} \frac{\partial f_1}{\partial q_1} = n_0 \int \frac{\partial \phi_{12}}{\partial q_1} \frac{\partial f_2}{\partial p_1} \, dx_2.
\]  

(I.11)

\[
\frac{\partial f_2}{\partial t} + \frac{p_2}{m} \frac{\partial f_2}{\partial q_2} = \left( \frac{\partial \phi_{12}}{\partial q_1} \frac{\partial f_1}{\partial p_1} + \frac{\partial \phi_{23}}{\partial q_2} \frac{\partial f_3}{\partial p_2} \right) = n_0 \int \left( \frac{\partial \phi_{13}}{\partial q_1} \frac{\partial f_3}{\partial p_1} + \frac{\partial \phi_{23}}{\partial q_2} \frac{\partial f_3}{\partial p_2} \right) \, dx_3.
\]  

(I.12)

Most measurable quantities can be expressed in terms of the functions \( f_1 \) and \( f_2 \). Clearly, what we want to do is to find out under what conditions we can break the infinite chain of equations so that we can determine \( f_1 \) (and \( f_2 \)) without knowing \( f_3 \) and higher functions. This would obviously represent a vast simplification of the hierarchy of equations, and would enable us to perform most calculations of practical interest. No clear proof of the validity of such a procedure for any non-equilibrium situation has yet been given. Nevertheless, there are physical grounds for truncation of the hierarchy, and in fact plasma physics rests on this truncation (in an even more extreme form!).
In obtaining Eqs. (I.11) and (I.12) the fact that \( \phi_{ij} \) is the Coulomb potential has not yet been made use of.

In equilibrium statistical mechanics the reduced distribution functions are approximated by a form known as the Mayer cluster expansion:

\[
\begin{align*}
    f_1(x_1) &= f_1(x_1) \\
    f_2(x_1, x_2) &= f_1(x_1)f_1(x_2) + P(x_1, x_2) \\
    f_3(x_1, x_2, x_3) &= f_1(x_1)f_1(x_2)f_1(x_3) + f_1(x_1)P(x_2, x_3) + f_1(x_2)P(x_3, x_1) + T(x_1, x_2, x_3) .
\end{align*}
\]  

Clearly, a two-particle distribution \( f_1(x_1)\)\( f_1(x_2) \) means that the particles are uncorrelated, i.e. the probability of particle 1 being in the range \( x_1 \) to \( x_1 + \Delta x_1 \) is independent of the position of particle 2. The function \( P(x_1, x_2) \) therefore gives a measure of the correlation between two particles. From the structure of Eqs. (I.11) and (I.12) and the fact that for a dilute gas one might expect that the correlation functions \( P \) and \( T \) would become progressively smaller, Eqs. (I.13) to (I.15) are taken as the basis for a non-equilibrium perturbation expansion. There is clearly a difficulty in applying this to a system where the interaction is through the long-range Coulomb interaction. One might expect that the correlation function would be as significant as the uncorrelated one. However, it is found that the particles in a plasma move in such a way as to preserve local charge neutrality, i.e. each particle tends to attract a cloud of the oppositely charged particles around it in such a way as to neutralize its own charge. This screening has the effect that over almost all of the two-particle phase space, \( P/f_1f_1 \) is of the order \( g \equiv 1/n_0L_D^2 \), where

\[
L_D^2 = e^2 kT/n_0 e^2 .
\]

The quantity \( g \) is usually very small (e.g. for \( n_0 \sim 10^{14}/\text{cm}^3 \) and \( T \sim 10^6 \text{°K}, n_0L_D^2 \sim 10^4 \). Thus, when these correlations can be assumed to become weaker and weaker, we can hope for a rapidly converging expansion using Eqs. (I.13) to (I.15).

Thus, Eqs. (I.11) and (I.12) are expanded (Rosenbluth and Rostoker) in \( g \), where \( f_1 \) is regarded as zeroth order, \( P \) as first order, and \( T \) as higher order.

The equations can now be written

\[
\begin{align*}
    \frac{\partial f_1}{\partial t} + \frac{\vec{p}_1}{m} \cdot \frac{\partial f_1}{\partial \vec{q}_1} &= n_0 \int \frac{\partial \phi_{12}}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} f_1(x_1)f_1(x_2) \, d\vec{x}_2 + n_0 \int \frac{\partial \phi_{12}}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} P(x_1, x_2) \, d\vec{x}_2 \quad (I.16)
\end{align*}
\]
\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial q_1} + \frac{p_2}{m} \frac{\partial}{\partial q_2} \right) P(x_1, x_2) &= \left( \frac{\partial f_{12}}{\partial q_1} + \frac{\partial f_{21}}{\partial q_2} \right) \left[ f_1(x_1) f_2(x_2) + P(x_1, x_2) \right] \\
&+ n_0 \left[ \int dx_3 f_1(x_3) \frac{\partial f_{13}}{\partial q_1} + (1 \rightarrow 2) \right] P(x_1, x_2) \\
&+ n_0 \left[ \int \frac{\partial f_{13}}{\partial p_1} \int \frac{\partial f_{13}}{\partial q_1} P(x_2, x_3) \, dx_3 + (1 \rightarrow 2) \right] \\
&+ n_0 \int \left[ \frac{\partial f_{13}}{\partial q_1} + (1 \leftrightarrow 2) \right] T(x_1, x_2, x_3) \, dx_3 , \quad (I.17)
\end{align*}
\]

where we have used Eq. (I.16) to eliminate some terms in Eq. (I.17)

\[
\left[ \text{e.g. } \left( \frac{\partial}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial q_1} + \frac{p_2}{m} \frac{\partial}{\partial q_2} \right) f_1(x_1) f_2(x_2) f_1(x_3) , \right.
\]

\[
n_0 \int \frac{\partial f_{13}}{\partial p_1} f_1(x_2) P(x_1, x_1) \, dx_3 , \quad \left. \text{etc.} \right]
\]

The effect of screening is such that the correlations between two particles will be very weak except over a very small region of space whose volume is \( \approx g^2/n_0 \). Correspondingly, three-particle correlations will be even weaker, and so on. We therefore describe the system by means of a smoothed-out distribution which is a good approximation over almost the whole of phase space. This smoothing can be thought of as a limiting procedure in which the charge \( e \), mass \( m \), and \( 1/n_0 \) are made progressively smaller but in such a way that \( e/m \), \( kT/m \), and \( n_0 e^2/m \) remain constant, i.e. the total charge, mass, and energy per unit volume is held fixed.

\[ \text{[N.B. If } (kT/m)^{1/2} \text{ is to be of order unity, and } g = (n_0 e^2/e_0 kT)^{1/2} 1/n_0 \text{ is to be small, then since } n_0 e^2/e_0 m \text{ is to be held constant, } 1/n_0 \text{ must be } O(g), \text{ i.e. a small parameter. Therefore } e^2/m \text{ is } O(g) \text{ and, since } e/m \text{ must remain fixed, } e \text{ is to be treated as } O(g). \text{ Finally, } m \text{ is } O(g). \text{ The quantities } e, m, \text{ and } 1/n_0 \text{ are then discreteness parameters.]}
\]

Assuming that \( T(x_1, x_2, x_3) \) is of higher order than the first order enables the system to be closed. The closed system now consists of the pair of coupled equations for \( f_1 \) and \( P \) \[ \text{[Eqs. (I.16) and (I.17).]}. \] However, even this, very much reduced, system is far too difficult to solve. Many approximate schemes for the solution of this pair of equations have been considered. However, for many purposes the zeroth approximation in which the limit \( g = 0 \) is taken is sufficient! The second equation \[ \text{[Eq. (I.17)]} \] then disappears and the second term on the right-hand side of Eq. (I.16) can also be neglected. The result is the following equation, where we have changed variables from \( \bar{q}, \bar{p} \) to \( \bar{X}, \bar{V} \), i.e. \( f_1(\bar{q}, \bar{p}) \, d\bar{p} = f(\bar{X}, \bar{V}) \, d\bar{V} \):
\[
\frac{\partial \nu}{\partial t} + \nabla \cdot \mathbf{V} = -\frac{n_0}{m} \left[ \int \frac{\partial \phi_{12}}{\partial x} f(\mathbf{x}_2, \mathbf{v}_2) \ d\mathbf{x}_2 \ d\mathbf{v}_2 \right] \frac{\partial \nu}{\partial v_1} = 0.
\] (I.18)

The distribution function \( f \) now describes a six-dimensional (continuous) phase fluid, where the particle characteristics have been smoothed out by the limiting procedure where the charges and masses were repeatedly subdivided until the continuum limit was reached, \( g = 0 \). Instead of a singular particle distribution consisting of a set of \( 5 \)-functions, we have a smooth, continuous (repeatedly differentiable) distribution. Note, however, that contrary to a 'normal' fluid, this six-dimensional phase fluid contains the thermal information arising from the distribution of velocities.

Equation (I.18) is the basis of a great part of high-temperature plasma physics. Let us write it in a physically more transparent form. The electric field acting on the particle at position \( \mathbf{x}_1 \), at time \( t \), can be written,

\[
e^{2} \nabla \cdot \Phi = n_0 \int \frac{\partial \phi_{12}}{\partial x} f(\mathbf{x}_2, \mathbf{v}_2, t) \ d\mathbf{x}_2 \ d\mathbf{v}_2.
\] (I.19)

The equation for \( f \) can now be written,

\[
\frac{\partial \nu}{\partial t} + \nabla \cdot \mathbf{V} = -\frac{e}{m} \nabla \cdot \mathbf{E} = 0.
\] (I.20)

Finally, we can take the divergence of Eq. (I.19) to obtain

\[
e^{2} \nabla \cdot \mathbf{E} = n_0 \int \frac{\partial \phi_{12}}{\partial x} f(\mathbf{x}_2, \mathbf{v}_2, t) \ d\mathbf{x}_2 \ d\mathbf{v}_2
\]

\[= -n_0 e^2 \int d\mathbf{x}_2 \ d\mathbf{v}_2 \ f(\mathbf{x}_2, \mathbf{v}_2, t) \frac{(\mathbf{x}_1 - \mathbf{x}_2)^2}{4 \pi \varepsilon_0 |\mathbf{x}_1 - \mathbf{x}_2|^3}
\]

where we have now made explicit use of the Coulomb nature of \( \phi_{12} \). But

\[
-\frac{(\mathbf{x}_1 - \mathbf{x}_2)^2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} = \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|},
\]

\[
\therefore \ e^{2} \nabla \cdot \mathbf{E} = n_0 e^2 \int d\mathbf{x}_2 \ d\mathbf{v}_2 \ f(\mathbf{x}_2, \mathbf{v}_2, t) \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}.
\]

But

\[
\nabla \cdot \mathbf{E} = 4 \pi \varepsilon_0 (\mathbf{x}_1 - \mathbf{x}_2)
\]

\[
\therefore \ e^{2} \nabla \cdot \mathbf{E} = \frac{n_0 e^2}{4 \pi \varepsilon_0} \int d\mathbf{x}_2 \ d\mathbf{v}_2 \ f(\mathbf{x}_2, \mathbf{v}_2, t) 4 \pi \delta(\mathbf{x}_1 - \mathbf{x}_2)
\]

\[= \frac{n_0 e^2}{\varepsilon_0} \int d\mathbf{v}_2 \ f(\mathbf{x}_1, \mathbf{v}_2, t).
\]
Finally, dropping the indices 1 and 2 for reasons of simplicity, we obtain
\[ \vec{v} \cdot \vec{E}(\vec{x},t) = \frac{\rho(\vec{x},t)}{\varepsilon_0} \]
or
\[ \nabla \cdot \vec{E}(\vec{x},t) = \frac{n\varepsilon \varepsilon_0}{\varepsilon_0} \int f(\vec{x},\vec{v},t) \, d\vec{v} . \]

This equation is now recognized as Poisson's equation. The Vlasov equation (1.20) can now be written as
\[ \begin{align*}
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{e}{m} \vec{E}(\vec{x},t) \cdot \frac{\partial f(\vec{x},\vec{v},t)}{\partial \vec{v}} &= 0 , \\
\nabla \cdot \vec{E}(\vec{x},t) &= \frac{n\varepsilon \varepsilon_0}{\varepsilon_0} \int f(\vec{x},\vec{v},t) \, d\vec{v} .
\end{align*} \]

We can now see the real elegance and beauty of Vlasov's equation. The electric field which appears in this equation is the self-consistent field which is due to all the other particles of the system, moving in an uncorrelated way. This is the reason why the Vlasov equation is so rich in phenomena and so difficult to solve. This is the solution to the paradox of the Coulomb interaction. The long-range nature of the force manifests itself through the presence of the self-consistent field which results from every particle in the system. However, each particle can be treated independently and the force at any point in the plasma is the average of the fields due to all these (uncorrelated) particles. This concept will be familiar to those acquainted with the Hartree model of atoms, where the same notion of a self-consistent field produced by all the other particles of the atom is used.

\((N.B.\) The idea can be illustrated by considering an iterative solution of the Vlasov-Poisson equations, i.e. assume an \( f \), calculate \( \vec{E} \), and then determine the correction to \( f \) and so on until the solutions converge.\)

\[ \text{I.2 FLUID EQUATIONS} \]

The most accurate description of the dynamical behaviour of a plasma is obtained through the Vlasov equation (1.21). In its most general form this can be written
\[ \begin{align*}
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{q_j}{m_j} \left( \vec{E} + \frac{\vec{v} \times \vec{B}}{\varepsilon_0} \right) \cdot \frac{\partial f}{\partial \vec{v}} &= 0 , \\
\end{align*} \]

where we have now included the full electromagnetic interaction due to the presence of magnetic fields. The fields \( \vec{E} \) and \( \vec{B} \) are of course the self-consistent fields due to the presence of all the charged particles making up the plasma. This field is determined by Maxwell's equations. In addition, any external field is also included. The self-consistent fields are determined by coupling the Vlasov equation to Maxwell's equations. There are many examples where the particular physical phenomena under consideration can only be obtained from the Vlasov (or kinetic) model. However, there are also many situations where a
simpler description than the Vlasov model is adequate. This is the case for many (but
certainly not all) of the wave phenomena which can occur in a plasma. It is also the case
that many problems may be complicated by other features (e.g. geometry), so that it is al-
most imperative to have a simpler description of the plasma.

If the distribution function \( f_j(x, v, t) \) is known, then the various physically signif-
ificant variables can be obtained as follows:

\[
n_j(x, t) = \int f_j(x, v, t) \, dv ,
\]

\[
\hat{u}_j(x, t) = \frac{1}{n_j} \int f_j(x, v, t) \, v \, dv ,
\]

where \( \hat{u}_j \) is the mean velocity of the \( j \)th species. The pressure tensor is obtained from

\[
\hat{p}_j(x, t) = m_j \int (\hat{v} - \hat{u}_j) (\hat{v} - \hat{u}_j) f_j \, dv
\]

(1.25)

for the two-fluid case, and

\[
\hat{p}_j(x, t) = m_j \int (\hat{v} - \hat{u}_0) (\hat{v} - \hat{u}_0) f_j \, dv
\]

(1.26)

for the one-fluid case, where \( \hat{u}_0 \) is the centre-of-mass velocity

\[
\hat{u}_0 = \frac{\sum_j m_j n_j \hat{u}_j}{\sum_j n_j m_j} .
\]

(1.27)

The kinetic temperature \( T_j(x, t) \) can be defined in terms of \( \hat{p}_j(x, t) \) as

\[
3n_j k T_j(x, t) = \text{Tr} (\hat{p}_j) \equiv m_j \int \hat{v} \cdot (\hat{v} - \hat{u}_j) f_j \, dv
\]

(1.28)

and so on.

We may now obtain equations relating these macroscopic variables by taking velocity
moments of the Vlasov equation. The procedure is similar to that used in neutral gas dy-
namics but there is an important difference. For high-temperature, tenuous plasmas, col-
lisions are relatively rare and there is no rapid thermalization mechanism. This means
that, in general, a finite number of moments are not sufficient for describing the motion --
there is no general way of closing the hierarchy of moment equations. Nevertheless, despite
the lack of a rigorous closure principle, moment equations are often useful and surprisingly
accurate. Thus, integrating Vlasov's equation over velocity space, we have

\[
\int \frac{\partial f_j}{\partial t} \, dv + \int \hat{v} \cdot \frac{\partial f_j}{\partial \hat{x}} \, dv + \frac{q_j}{m_j} \int (E + \hat{v} \times \hat{B}) \cdot \frac{\partial f_j}{\partial \hat{v}} \, dv = 0 .
\]
Now
\[ \int \frac{\partial f_j}{\partial t} \, d\mathbf{v} = \frac{\partial}{\partial t} \int f_j \, d\mathbf{v} = \frac{\partial}{\partial t} n_j (\mathbf{x}, t) , \]
\[ \int \mathbf{v} \cdot \frac{\partial f_j}{\partial x} \, d\mathbf{v} = \frac{\partial}{\partial x} \int \mathbf{v} f_j \, d\mathbf{v} = \frac{\partial}{\partial x} (n_j \mathbf{\dot{u}}_j) , \]
\[ \int \mathbf{E} \cdot \frac{\partial f_i}{\partial v} \, d\mathbf{v} = \mathbf{E} \cdot \int \frac{\partial f_i}{\partial v} \, d\mathbf{v} = 0 , \]
since \( f \) vanishes at the velocity limits, and
\[ \int (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_i}{\partial v} \, d\mathbf{v} = \int (\mathbf{v} \times \mathbf{B})_i \frac{\partial f_i}{\partial v} \, d\mathbf{v} \]
\[ = (\mathbf{v} \times \mathbf{B})_i \mathbf{f}_i \int - \int f \frac{\partial}{\partial v} (\mathbf{v} \times \mathbf{B})_i \, d\mathbf{v}_i = 0 , \]
since \( f \) vanishes at the limits of integration as before, and \((\partial/\partial v_i) (\mathbf{v} \times \mathbf{B})_i = 0\). We therefore obtain
\[ \frac{\partial n_i}{\partial t} + \mathbf{\nabla} \cdot (n_j \mathbf{\dot{u}}_j) = 0 . \] (I.29)

Next, multiply through by \( m_j \mathbf{\dot{v}} \)
\[ m_j \int \mathbf{v} \cdot \frac{\partial f_i}{\partial t} \, d\mathbf{v} + m_j \int \mathbf{v} \cdot \frac{\partial f_i}{\partial x} \, d\mathbf{v} + \frac{\partial}{\partial x} \int \mathbf{v} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_i}{\partial v} \, d\mathbf{v} = 0 . \]

Clearly
\[ m_j \int \mathbf{v} \cdot \frac{\partial f_i}{\partial t} \, d\mathbf{v} = \frac{\partial}{\partial t} (n_j \mathbf{\dot{u}}_j m_j) \]
\[ \int \mathbf{v} \cdot \frac{\partial f_i}{\partial x} \, d\mathbf{v} = \frac{\partial}{\partial x} m_j \int \mathbf{v} \cdot f_j \, d\mathbf{v} . \]

Now
\[ \mathbf{\dot{p}}_j = m_j \int (\mathbf{v} - \mathbf{\dot{u}}_j) (\mathbf{v} - \mathbf{\dot{u}}_j) f_j \, d\mathbf{v} \]
\[ = m_j \int \mathbf{v} f_j \, d\mathbf{v} - m_j \int \mathbf{\dot{u}}_j \cdot \mathbf{v} f_j \, d\mathbf{v} - m_j \int \mathbf{v} \cdot \mathbf{\dot{u}}_j f_j \, d\mathbf{v} + m_j \int \mathbf{\dot{u}}_j \cdot \mathbf{\dot{u}}_j f_j \, d\mathbf{v} \]
\[ = m_j \int \mathbf{v} f_j \, d\mathbf{v} - m_j \int \mathbf{\dot{u}}_j n_j \mathbf{\dot{u}}_j - m_j \int \mathbf{\dot{u}}_j n_j \mathbf{\dot{u}}_j + m_j \int \mathbf{\dot{u}}_j n_j \mathbf{\dot{u}}_j \]
\[ = m_j \int \mathbf{v} f_j \, d\mathbf{v} - m_j \int \mathbf{\dot{u}}_j n_j \mathbf{\dot{u}}_j . \]
\[ \therefore \quad m_j \int \mathbf{v} f_j \, d\mathbf{v} = \mathbf{\dot{p}}_j - m_j \int \mathbf{\dot{u}}_j n_j \mathbf{\dot{u}}_j . \]
The \((r m)\) element of this tensor is
\[
\frac{3}{\partial x^m} \left( m_j \int n_m f_j \, d\vec{v} \right) = \frac{3}{\partial x^m} P_{rm} + \frac{3}{\partial x^m} m_j u_r n_j u^m = \vec{\nabla} \cdot \vec{P}_{j} + m_j u_r \frac{3}{\partial x^m} (n_j u^m) + n_j u^m \frac{3}{\partial x^m} (m_j u^m).
\]

\[
\therefore \quad m_j \frac{3}{\partial x^m} \int \vec{v} \cdot \vec{v} f_j \, d\vec{v} = \vec{\nabla} \cdot \vec{P}_{j} + m_j u_j \vec{\nabla} \cdot (n_j u_j) + n_j u_j \vec{\nabla} \cdot (m_j u_j).
\]

Now combining the first and second terms,
\[
\frac{3}{\partial t} (n_j m_j u_j) + \vec{\nabla} \cdot \vec{P}_{j} + m_j u_j \vec{\nabla} \cdot (n_j u_j) + n_j m_j (\vec{u}_j \cdot \vec{v}) u_j
\]
\[
= m_j u_j \frac{3n_j}{3t} + m_j n_j \frac{3u_j}{3t} + \vec{\nabla} \cdot \vec{P}_{j} + m_j u_j \vec{\nabla} \cdot (n_j u_j) + n_j m_j (\vec{u}_j \cdot \vec{v}) u_j.
\]

But
\[
m_j u_j \left\{ \frac{3n_j}{3t} + \vec{\nabla} \cdot (n_j u_j) \right\} = 0.
\]

The first two terms of the momentum equation following (1.29) therefore give
\[
m_j n_j \frac{3u_j}{3t} + n_j m_j (\vec{u}_j \cdot \vec{v}) u_j + \vec{\nabla} \cdot \vec{P}_{j}.
\]

Finally, consider the last term of the same momentum equation
\[
\int \vec{v} (\vec{E} + \vec{\nabla} \times \vec{B}) \cdot \frac{3f_j}{3\vec{v}} \, d\vec{v}.
\]

Now
\[
\int \vec{v} (\vec{E} + \vec{\nabla} \times \vec{B}) \cdot \frac{3f_j}{3\vec{v}} \, d\vec{v} = \int v_n E_m \frac{3f_j}{3\vec{v}_m} \, d\vec{v}
\]
\[
= v_n E_m f_j \int E_m f_j \delta_{nm} \, d\vec{v}
\]
\[
= -E_n n_j.
\]

Similarly,
\[
\int \vec{v} (\vec{v} \times \vec{B}) \cdot \frac{3f_j}{3\vec{v}} \, d\vec{v} = -(\vec{u}_j \times \vec{B}) n_j.
\]

Finally, the momentum equation is
\[
m_j n_j \frac{3u_j}{3t} + n_j m_j (\vec{u}_j \cdot \vec{v}) u_j + \vec{\nabla} \cdot \vec{P}_{j} = q_j n_j \vec{E} + q_j n_j (\vec{u}_j \times \vec{B}).
\]
This is usually written

$$\frac{\partial \hat{u}_j}{\partial t} + (\hat{u}_j \cdot \nabla) \hat{u}_j + \frac{\nabla \cdot \vec{p}_j}{n_j m_j} = \frac{c_j}{n_j} (\hat{E} + \hat{u}_j \times \vec{B}) \ . \tag{1.30}$$

Notice that the equation for each moment contains a term involving the next higher moment -- the problem of closure. Thus the equation for $n_j$ contained $\hat{u}_j$. Similarly, the equation for $\hat{u}_j$ contains a term in $\vec{p}_j$.

As already mentioned, there is no rigorous way of terminating this string of moment equations. Instead, one usually appeals to physical intuition in order to justify truncating the series. The next equation in the series is the energy equation obtained from the moment of $\frac{1}{2} n_j \hat{v} \cdot \hat{v}$. This is usually as far as one goes with the moment method and we shall not write down the general energy equation. Instead, we shall briefly consider the various physical approximations which are used in a fluid description. Possibly, the simplest model is the so-called COLD plasma approximation. Here we neglect the term $\hat{v} \cdot \vec{p}_j$ in Eq. (1.30) thus closing the set after the first two moments. This model of the plasma describes it as two cold interpenetrating fluids. Such an approximation is often used in problems of wave propagation and may be expected to give a reasonable approximation when the phase velocity of the wave is much greater than the thermal velocity of the particles $\nu T_j = (kT_j/m_j)^{1/2}$, e.g., electromagnetic waves.

An extension of this approximation is that of two warm fluids in which the tensor $\vec{p}_j$ is now included but is assumed to be diagonal, and an isotropic pressure is assumed. Thus,

$$\vec{p}_j = p_j \hat{I}$$

and

$$p_j = n_j kT_j \ .$$

This set can be closed by making the ISOTHERMAL approximation ($T_j = \text{const}$), which is commonly done, or the next moment (the energy equation) can be included. This warm two-fluid model represents a slightly more sophisticated viewpoint and gives rise to new effects through the presence of the fluid pressures. It also exhibits one of the characteristic features of a high-temperature plasma, namely the slow thermalization behaviour, which enables a plasma to persist in a state with widely different temperatures for the individual species. This fact can often have important consequences.

The final fluid approximation we would like to mention is the ONE-FLUID or magnetohydrodynamic (MHD) model. This is appropriate at low frequencies, i.e., where $\omega << \omega_{ci}$, and long wavelengths $\lambda >> \nu T_j / \omega_{ci}$ (where $\omega_{ci}$ is the ion cyclotron frequency). In this range the electrons and ions of the plasma are very strongly coupled and therefore tend to move together (N.B. for the ideal MHD model, the conductivity is assumed to be infinite so that one has the condition that $E \cdot B = 0$). Adding the continuity equations for the different species gives the equation of continuity for mass flow:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \hat{u}_0) = 0 \ , \tag{1.31}$$
where

\[ p \equiv \sum_j n_j m_j, \]

and

\[ \vec{u}_0 = \sum_j n_j \vec{u}_j m_j. \]

Adding the momentum equations gives the equation for the centre-of-mass motion,

\[ \frac{\partial \vec{u}_0}{\partial t} + (\vec{u}_0 \cdot \nabla) \vec{u}_0 = -\frac{1}{\rho} \nabla p + \vec{J} \times \vec{B}, \tag{I.32} \]

where

\[ \vec{p} \equiv \vec{p}_i + \vec{p}_e \]

and the pressure tensor refers to the centre-of-mass velocity as is appropriate to a one-fluid description. Again the pressure term is assumed to be diagonal and \( \vec{J} \) is given by the usual expression:

\[ \vec{J} = \sum_j n_j q_n \vec{u}_j. \tag{I.33} \]

The one-fluid model is usually closed by making the adiabatic approximation which assumes a local Maxwellian so that

\[ \left( \frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla \right) \left( \frac{\vec{p}}{\rho^\gamma} \right) = 0 \quad (\gamma \text{ is the ratio of specific heats}), \tag{I.34} \]

e.g. the heat-flow term has been neglected; then under the assumption of perfect conductivity (\( \vec{E} + \vec{u}_0 \times \vec{B} = 0 \)) Eq. (I.34) follows.
CHAPTER II

PLASMA WAVES

II.1 INTRODUCTION

A magnetized plasma is one of the richest wave media yet to have been studied. It owes this richness to the long-range nature of the Coulomb interaction, and it is this long-range interaction which makes the many-body or collective effects in a plasma so subtle. One of the most important concepts arising from the collective effects is that of the self-consistent electromagnetic field, which is the field which a given plasma particle experiences owing to the presence of all the other particles. This self-consistent field is crucial to the theory of waves in the plasma medium.

The idea is as follows. Suppose there is a small electromagnetic field present in the plasma. This produces forces on the plasma particles, resulting in currents and charge perturbations which act as source terms for further electromagnetic fields, which will then produce further plasma motions, and so on. This system of field perturbations and particle motions are iterated until the assumed electromagnetic field is itself produced by the resulting plasma motion. For the theory of linear waves in a plasma, the self-consistency condition gives the dispersion relation which contains all the information concerning the wave motion.

In this chapter, we shall only be concerned with the properties of linear waves. In order to discuss linear waves we must first of all define the equilibrium state. We shall confine ourselves, almost entirely, to the highly artificial model of an infinite uniform plasma. It may be asked what possible relevance the results obtained from this model could have for any realistic situation. We shall return to this point in a moment. Now, having specified the equilibrium, the linear waves give the possible modes of oscillation about this equilibrium when it is subjected to a small perturbation. The perturbation is assumed to be small enough so that the equations describing the model can be linearized, i.e. products of perturbations are ignored; this means, physically, that we neglect such effects as the generation of heat waves (waves at the sum and difference of two frequencies \( \omega_1 \pm \omega_2 \)), harmonics, the excitation of pairs of waves by a finite amplitude wave, etc. It is not clear from the theory of linear waves how small a perturbation must be for this approximation to be a reasonable one. Stated another way, the linear theory of waves (in a plasma) does not contain any criterion from which the wave amplitude can be determined. This can only be calculated by means of more accurate, non-linear theories.

Let us now make a few remarks concerning the relevance of our idealized model, bearing in mind that most plasmas are either finite, non-uniform, varying in time, or any combination
of these! The first justification is on the grounds of simplicity. A study of the simplest of all models serves to identify many of the basic phenomena and then forms a framework for more complicated situations. If the wavelength is less than the plasma radius \((\lambda << a)\) or the characteristic scale length \(L\) of the non-uniformity \((\lambda << L)\), or if the frequency of the wave is much greater than the inverse of the plasma lifetime \((\omega >> \tau^{-1})\), then these complications will produce only corrections to the simple theory.

Another noteworthy feature of the simple theory is that no modes are lost by the simplification -- the more realistic models only modify the properties of existing modes of the simple model.

Since most waves which are observed in laboratory (or natural) plasmas have reached some non-linear level, one might again wonder about the relevance of the linear theory. Nevertheless, it appears that for most cases such waves occur at frequencies and wavelengths which are determined quite closely by the linear theory and quite often by the infinite uniform model (with perhaps some straightforward correction term). In view of the simplicity of the model, it is surprising how much of the information concerning the linear waves for an infinite uniform plasma, is of use in a realistic situation.

The waves which occur in a magnetized plasma can be summarized briefly as three high-frequency branches which involve only the electrons, and three low-frequency branches which involve both electrons and ions. In this chapter I shall describe the properties of all these waves, and shall adopt the procedure of using the simplest model which will describe the mode under discussion.

One final comment concerning the importance of waves to plasma physics. In very many problems, one is interested in equilibria that are unstable. These instabilities are invariably due to one or possibly two waves which can draw on the free energy available in the plasma. A knowledge of the various waves which can exist in a plasma can then serve as a guide in unstable situations and in their subsequent development.

Let us now consider the simple example of waves in an infinite field-free, uniform plasma -- a model of interest to a laser plasma rather than to a confined plasma.

### II.2 WAVES IN A FIELD-FREE PLASMA

We use the TWO-FLUID model to describe the motion of the plasma. The equations of the model are the fluid equations (I.29) and (I.30) together with Maxwell's equations

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{II.1}
\]

\[
\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e + \nabla \left( \frac{kT_e}{m_e} \right) = - \frac{e}{m_e} \mathbf{E} - \frac{e}{m_e} \mathbf{v}_e \times \mathbf{B} \tag{II.2}
\]

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \tag{II.3}
\]

\[
\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i = \frac{e}{m_i} \mathbf{E} + \frac{e}{m_i} \mathbf{v}_i \times \mathbf{B} \tag{II.4}
\]
\[ \nabla \times \vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (II.5) \]

\[ \nabla \times \vec{E} = -\nu_e \frac{\partial \vec{H}}{\partial t} , \quad (II.6) \]

where we have used MKS units; \( n_e, n_i, v_{e}^{i}, \) and \( v_{e} \) are the electron and ion number densities and the electron and ion fluid velocities respectively. We have assumed a cold ion fluid (\( T_i = 0 \)) and a finite-temperature electron fluid. The current density in Eq. (II.5) is given by

\[ \vec{J} = n_i \vec{v}_i - n_e \vec{v}_e . \]

(II.7)

In order to describe the small-amplitude (strictly infinitesimal) linear waves of this model, we must linearize the equations about the equilibrium. In this case, the equilibrium is described by the two fluids with zero flow, uniform and equal densities, and uniform pressure. All variables are separated into an equilibrium part plus a small perturbation, varying in both time and space. The equations are then linearized, and since the coefficients of the resulting equations are constants, the variables can be Fourier analysed in space and time. We may therefore assume that all perturbed quantities vary as

\[ \exp \left( i(kz - \omega t) \right) . \]

We have chosen the \( z \)-direction as the direction of propagation. This is perfectly legitimate since there is no preferred direction in the model. The set of simultaneous linear partial differential equations is now reduced to the following set of algebraic equations:

\[ -i\omega n_{e1} + ikn_{e1} v_{e1z} = 0 \quad (II.8) \]

\[ -i\omega v_{e1} + \gamma_e \frac{kT_e}{n_e m_e} \frac{ikn_{e1}}{n_e} = -\frac{e}{m_e} \vec{E}_1 \quad (II.9) \]

\[ -i\omega n_{i1} + ikn_{i1} v_{i1z} = 0 \quad (II.10) \]

\[ -i\omega v_{i1} = \frac{e}{m_i} \vec{E}_1 \quad (II.11) \]

\[ ik \times \vec{H}_1 = n_e e (\vec{v}_{i1} - \vec{v}_{e1}) - i\omega_e \vec{H}_1 \quad (II.12) \]

\[ ik \times \vec{E}_1 = i\omega_e \vec{H}_1 . \quad (II.13) \]

where perturbed variables are denoted by a subscript 1.

The SELF-CONSISTENCY condition on the fields and particle motions is now obtained by requiring that the above set of equations be simultaneously satisfied. This condition is given by the vanishing of the determinant of the coefficients, and results in the dispersion relation for the system which is usually written as

\[ D(\omega, k) = 0 . \quad (II.14) \]
The solutions $\omega(k)$ of this equation then give the propagation characteristics of the waves of the system.

The dispersion relation can be obtained from Eqs. (II.8) to (II.13) in a variety of ways. However, some ways are more enlightening than others. A route we would like to emphasize is the following. Solving for the fluid velocities in terms of the components of $\vec{v}_i$ and substituting these into the equation for the current $J_i$, we obtain the conductivity tensor defined by

$$\tilde{\sigma}(\omega, k) = \tilde{\sigma}(\omega, \vec{k}) \vec{E}_1,$$  \hspace{1cm} (II.15)

where

$$\tilde{\sigma}(\omega, \vec{k}) = \begin{pmatrix}
    i \frac{n_e e^2}{\omega m_i} + i \frac{n_e e^2}{\omega m_e} & 0 & 0 \\
    0 & i \frac{n_e e^2}{\omega m_i} + i \frac{n_e e^2}{\omega m_e} & 0 \\
    0 & 0 & i \frac{n_e e^2}{\omega m_i} + i \frac{n_e e^2}{\omega m_e} \left(1 - \frac{\gamma_e (k^2 v_T^2 / \omega^2)}{n_e^2}ight)
\end{pmatrix},$$

where $v_T^2 \equiv kT_e / m_e$. Now from Eqs. (II.12) and (II.13),

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = -\frac{\omega^2}{c^2} \tilde{\varepsilon}(\omega, k) \vec{E}_1,$$  \hspace{1cm} (II.16)

where

$$\tilde{\varepsilon}(\omega, k) \equiv \tilde{\varepsilon} + \frac{i}{\omega \varepsilon_0} \tilde{\sigma}(\omega, k);$$  \hspace{1cm} (II.17)

$\tilde{\varepsilon}(\omega, k)$ is the dielectric tensor of the plasma. The dispersion relation follows immediately from Eq. (II.16) and is

$$\begin{vmatrix}
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} & 0 & 0 \\
0 & -k^2 + \frac{\omega^2}{c^2} \varepsilon_{yy} & 0 \\
0 & 0 & -k^2 + \frac{\omega^2}{c^2} \varepsilon_{zz}
\end{vmatrix} = 0.$$  \hspace{1cm} (II.18)

We immediately see from Eq. (II.18) that the longitudinal ($\vec{E} \parallel \vec{k}$) modes are decoupled from the transverse ($\vec{E} \perp \vec{k}$) modes.

**II.2.1 Longitudinal modes**

The dispersion relation for these modes is

$$\frac{\omega^2}{c^2} \varepsilon_{zz} \equiv \frac{\omega^2}{c^2} \left(1 + i \frac{\sigma_{zz}}{\omega \varepsilon_0}\right) = 0.$$  \hspace{1cm} (II.19)
This has two solutions, a high-frequency branch and a low-frequency one. The high-frequency branch

$$\omega_L^2 = \omega_p^2 \left( 1 + \frac{\gamma_e e^2}{\omega_0^2} \right) + \omega_p^2,$$

(II.20)

where $\omega_p^2 = n_e e^2 / \epsilon_0 m_j$ and $j = i$ or $e$, is the LANGMUIR (or plasma) wave. The low-frequency branch is

$$\omega_S^2 = \frac{k^2 c_s^2 \omega_L^2}{(1 + \gamma_e e^2)}$$

(II.21)

where $c_s^2 = \gamma_e kT_e / m_i$ and $\lambda_D^2 = v_f^2 / \omega_p^2$. The low-frequency branch cuts off as $\omega \rightarrow \omega_p^2$, and we have

$$\frac{\omega_S}{\omega_L^2} \lesssim \left( \frac{m_e}{m_i} \right)^{1/2}.$$

These modes $L$ and $S$ are illustrated by a dispersion diagram (Fig. II.1).

![Dispersion diagram](image)

**Fig. II.1**

### II.2.2 Transverse modes

There are two independent modes, both linearly polarized. The dispersion relations are

$$-k^2 + \frac{\omega_L^2}{c^2} \varepsilon_{xx} = 0,$$

(II.22)

$$-k^2 + \frac{\omega_L^2}{c^2} \varepsilon_{yy} = 0;$$

(II.23)

$\varepsilon_{xx} = \varepsilon_{yy}$ since the plasma is isotropic and both transverse modes have the same propagation characteristics. Equations (II.22) and (II.23) give

$$\omega_T^2 = \omega_p^2 + c^2 k^2.$$

(II.24)
These are also high-frequency modes involving only the electrons. For \( \omega < \omega_p e \), 
\[
k^2 < 0,
\]
and the wave is evanescent. The transverse modes \( T \) are also represented in Fig. II.1.

II.3 WAVES IN A PLASMA IN THE PRESENCE OF A UNIFORM, CONSTANT, MAGNETIC FIELD

For the moment we shall consider only the high-frequency modes. These involve the electrons but not the ions, which are assumed to remain stationary.

The linearized momentum equation for the electron fluid is now written
\[
\frac{\partial \vec{v}_{e1}}{\partial t} + \gamma_e \frac{\vec{k} \cdot \vec{v}_{e1}}{n_e m_e} \vec{v}_{e1} = - \frac{e}{m_e} (\vec{E}_1 + \vec{v}_{e1} \times \vec{B}_0),
\]
where \( \vec{B}_0 \) is assumed to point in the \( z \)-direction, i.e. \( \vec{B}_0 = (0,0,B_0) \). Looking for waves varying as
\[
\exp \left( i \vec{k} \cdot \vec{x} - \omega t \right)
\]
we must solve the following set of equations:
\[
-i \omega \vec{v}_{e1} + \gamma_e \frac{k \cdot \vec{v}_{e1}}{n_e m_e} \vec{v}_{e1} = - \frac{e}{m_e} (\vec{E}_1 + \vec{v}_{e1} \times \vec{B}_0)
\]
(II.26)
\[
-i \omega n_{e1} + i n_e (\vec{k} \cdot \vec{v}_{e1}) = 0
\]
(II.27)
\[
\vec{k} \times \vec{H}_1 = n_e e \vec{v}_{e1} - i \omega \vec{E}_1
\]
(II.28)
\[
\vec{i} \vec{k} \times \vec{E}_1 = i \omega n_e \vec{H}_1.
\]
(II.29)

Without loss of generality we choose \( \vec{k} = (0,k_y,k_z) \). Even without the ion motion, the derivation of the dielectric tensor is rather lengthy. We shall indicate the necessary steps. If \( n_{e1} \) and \( \vec{H}_1 \) are eliminated from Eqs. (II.26) to (II.28) then \( \vec{v}_{e1} \) can be solved in terms of \( \vec{E}_1 \) to give
\[
\vec{v}_{e1} = \begin{bmatrix}
E_{1x} \\
E_{1y} \\
E_{1z}
\end{bmatrix}
= \frac{e}{m_e} \begin{bmatrix}
\omega \gamma_e \\
-k \frac{k^2 \nu_y^2}{\gamma_T e} \\
\frac{k^2 \nu_z^2}{\gamma_T e}
\end{bmatrix}
\begin{bmatrix}
0 \\
-i \omega \left( 1 - \gamma_e \frac{k^2 \nu_y^2}{\omega^2} \right) \\
-i \omega \left( 1 - \gamma_e \frac{k^2 \nu_z^2}{\omega^2} \right)
\end{bmatrix}
\]
(II.30)
\[ \varphi_{e1y} = -\frac{e}{m_e} \text{Det.} \begin{vmatrix} -i\omega & E_{1x} & 0 \\ -\omega c_e & E_{1y} & i\gamma_e \frac{k_y k_z v_z^2}{\omega} \\ 0 & E_{1z} & -i\omega \left( 1 - \frac{k_y^2 v_y^2}{\omega^2} \right) \end{vmatrix} \]  

(II.31)

\[ \varphi_{e1z} = -\frac{e}{m_e} \text{Det.} \begin{vmatrix} -i\omega & \omega c_e & E_{1x} \\ -\omega c_e & -i\omega \left( 1 - \frac{k_y^2 v_y^2}{\omega^2} \right) & E_{1y} \\ 0 & i\gamma_e \frac{k_y k_z v_z^2}{\omega} & E_{1z} \end{vmatrix} \]  

(II.32)

where

\[ \text{Det.} = \begin{vmatrix} -i\omega & \omega c_e & 0 \\ -\omega c_e & -i\omega \left( 1 - \frac{k_y^2 v_y^2}{\omega^2} \right) & i\gamma_e \frac{k_y k_z v_z^2}{\omega} \\ 0 & i\gamma_e \frac{k_y k_z v_z^2}{\omega} & -i\omega \left( 1 - \frac{k_y^2 v_y^2}{\omega^2} \right) \end{vmatrix} \]  

(II.33)

These expressions for the components of \( \varphi_{e1} \) in terms of the components of \( \overrightarrow{E}_1 \) enable us to construct the conductivity tensor \( \overrightarrow{\sigma}(\omega, k) \) from

\[ \overrightarrow{J}_1 = -n_e e \overrightarrow{v}_1 \quad \text{giving} \quad \overrightarrow{J}_1 = \overrightarrow{\sigma} \overrightarrow{E}_1 , \]

and hence

\[ \overrightarrow{\varepsilon}(\omega, k) = \overrightarrow{1} + \frac{i\overrightarrow{\sigma}(\omega, k)}{\omega e}. \]

Assuming now that \( \overrightarrow{\varepsilon}(\omega, k) \) is known, we then obtain from Maxwell's equations

\[ \overrightarrow{k} \times (\overrightarrow{k} \times \overrightarrow{E}_1) + \frac{\omega^2}{c^2} \overrightarrow{\varepsilon}(\omega, k) \overrightarrow{E}_1 = 0 , \]  

(II.34)

which we have already obtained for the field-free case. This time, however, for oblique propagation all the elements of \( \overrightarrow{\varepsilon} \) are non-zero. The condition that the three equations given by Eq. (II.34) shall have a non-trivial solution is that the determinant of the coefficients
vanishes giving the dispersion relation for the general case of oblique propagation in a magnetic field. The dispersion relation is formally

\[
\begin{pmatrix}
-k^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} & \frac{\omega^2}{c^2} \varepsilon_{xy} & \frac{\omega^2}{c^2} \varepsilon_{xz} \\
\frac{\omega^2}{c^2} \varepsilon_{yx} & -k_z^2 + \frac{\omega^2}{c^2} \varepsilon_{yy} & k_y k_z + \frac{\omega^2}{c^2} \varepsilon_{yz} \\
\frac{\omega^2}{c^2} \varepsilon_{zx} & k_y k_z + \frac{\omega^2}{c^2} \varepsilon_{zy} & -k^2 + \frac{\omega^2}{c^2} \varepsilon_{zz}
\end{pmatrix} = 0 .
\tag{II.35}
\]

Let us now consider propagation parallel to \( \vec{B}_g \).

II.3.1 Propagation parallel to the magnetic field

For this case \( (k_y = 0) \), it is easily verified from Eqs. (II.30) to (II.32) that

\[
\sigma_{zx} = \sigma_{xz} = \varepsilon_{xz} = 0 = \varepsilon_{xx} , \\
\sigma_{zy} = \sigma_{yz} = \varepsilon_{yz} = 0 = \varepsilon_{yy} .
\]

Equation (II.35) then reduces to

\[
\left[ \left( -k_z^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} \right) \left( -k_z^2 + \frac{\omega^2}{c^2} \varepsilon_{yy} \right) - \frac{\omega^2}{c^2} \varepsilon_{xy} \varepsilon_{xy} \right] \frac{\omega^2}{c^2} \varepsilon_{zz} = 0 .
\tag{II.36}
\]

This immediately shows that the transverse and longitudinal modes again decouple. The dispersion relation for the longitudinal waves is

\[
\frac{\omega^2}{c^2} \varepsilon_{zz} = 0 ,
\tag{II.37}
\]

which yields the same result as Eq. (II.20) (Langmuir waves).

Now consider the transverse modes. Since \( \varepsilon_{xy} \) and \( \varepsilon_{yx} \) are non-zero, \( E_{1x} \) and \( E_{1y} \) are no longer independent. From Eqs. (II.30) and (II.31) we find

\[
\varepsilon_{yx} = -\varepsilon_{xy} \\
\varepsilon_{xx} = \varepsilon_{yy} .
\]

The transverse dispersion relation can then be written

\[
\left( -k_z^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} \right)^2 + \frac{\omega^2}{c^2} \varepsilon_{xy}^2 = 0
\]

\[
\therefore -k_z^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} = \pm i \frac{\omega^2}{c^2} \varepsilon_{xy} .
\tag{II.38}
\]

Using this result in Eq. (II.34) we obtain

\[
E_{1x} = \pm i E_{1y} .
\tag{II.39}
\]
In other words, the transverse electromagnetic waves propagating along a uniform magnetic field are circularly polarized.

To obtain the dispersion relation, we solve Eqs. (II.30) and (II.31) and use Eq. (II.33) to obtain

\[ \varepsilon_{xx} = 1 - \frac{\omega^2 P_e}{(\omega^2 - \omega_{ce}^2)} \]  

(II.40)

and

\[ \varepsilon_{xy} = \frac{i \omega^2 P_e \omega_{ce}}{\omega (\omega^2 - \omega_{ce}^2)} . \]  

(II.41)

Using these expressions in Eq. (II.38) we now obtain the explicit dispersion relation

\[ \frac{c^2 k^2}{\omega^2} = n^2 = 1 - \frac{\omega^2 P_e}{\omega (\omega^2 - \omega_{ce}^2)} . \]  

(II.42)

We note that the right-hand and left-hand circularly polarized waves propagate with different phase velocities, which leads to the possibility of FARADAY ROTATION -- a plasma diagnostic to measure \( n_0 \) if \( B_0 \) is known. The right-hand mode resonates with the electrons, i.e. the fields drive the electrons in the same sense as their gyro-motion.

The dispersion diagram obtained from Eq. (II.42) is shown in Fig. II.2. The cut-off frequencies of the RH and LH circularly polarized waves are no longer given by \( \omega_{Pe} \) but

\[ \omega_{1,2} = \frac{\omega_{ce}}{2} \left( 1 \pm \sqrt{1 + \frac{4 \omega^2 P_e}{\omega_{ce}^2}} \right) . \]  

(II.43)

However, the most striking effect is the existence of a second branch to the RH mode which can propagate for \( \omega \ll \omega_{ce} \) and \( \omega \ll \omega_{Pe} \).

![Dispersion Diagram](image_url)
Let us see how this second branch arises. For \( \omega \ll \omega_{ce} \) and \( \omega_{pe} \gg \omega \omega_{ce} \), Eq. (II.42)

\[
\frac{c^2 k_z^2}{\omega^2} \approx \frac{\omega_{pe}^2}{\omega \omega_{ce}},
\]

\[ \text{(II.44)} \]

... the LH wave is evanescent and the RH wave propagates. We can write the solution as

\[
\frac{\omega^2}{c^2 k_z^2} = \frac{\omega_{pe}^2}{\omega_{ce}^2}.
\]

\[ \text{(II.45)} \]

This lower branch is known as the WHISTLER or HELICON branch. It has been observed in the ionosphere, in a plasma discharge, and in a rod of indium at liquid-helium temperature! For \( \omega_{pe}^2 / \omega \omega_{ce} \gg 1, \nu_{\text{phase}} / c \ll 1 \).

II.3.2 Propagation perpendicular to the magnetic field

We now take \( k_z = 0 \), which again results in

\[ \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0. \]

Using these results and \( k_z = 0 \) reduces the general dispersion relation of Eq. (II.55) to the form

\[
\left[ \left( -k_y^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} \right) \frac{\omega^2}{c^2} \varepsilon_{yy} - \frac{\omega^2}{c^2} \varepsilon_{xy} \varepsilon_{yx} \right] \left( -k_y^2 + \frac{\omega^2}{c^2} \varepsilon_{zz} \right) = 0.
\]

\[ \text{(II.46)} \]

From Eq. (II.46) we see that we have two factors. The second one gives

\[ k_y^2 = \frac{\omega^2}{c^2} \varepsilon_{zz}. \]

\[ \text{(II.47)} \]

From Eqs. (II.32) and (II.33) we find

\[ \varepsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2}, \]

and so Eq. (II.47) results in the familiar equation for electromagnetic waves in a field-free plasma

\[ \frac{c^2 k_y^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}. \]

This is, of course, only to be expected since these are transverse waves linearly polarized with the \( \vec{E} \) vector along \( \vec{B}_0 \) -- hence the electron motion is unaffected by \( \vec{B}_0 \). These transverse waves propagating across a uniform magnetic field with \( \vec{E} \) along \( \vec{B}_0 \) are called ORDINARY waves.

Now consider the other factor,

\[
\left( -k_y^2 + \frac{\omega^2}{c^2} \varepsilon_{xx} \right) \frac{\omega^2}{c^2} \varepsilon_{yy} - \frac{\omega^2}{c^2} \varepsilon_{xy} \varepsilon_{yx} = 0.
\]

\[ \text{(II.48)} \]
For simplicity, let us consider only the special case when \( v_T = 0 \). Equation (II.46) describes a coupling between transverse and longitudinal modes so that the waves described by Eq. (II.48) are a mixture, i.e. HYBRID modes. From Eqs. (II.50), (II.31), and (II.33) we obtain

\[
\varepsilon_{xx} = \varepsilon_{yy} = 1 - \frac{\omega^2_p}{\omega^2 - \omega_c^2}
\]

\[
\varepsilon_{yx} = \varepsilon_{xy} = \frac{-i\omega^2_p c}{\omega(\omega^2 - \omega_c^2)}.
\]

Substituting these expressions into Eq. (II.48) gives the following dispersion relation:

\[
k_y^2 = \frac{\omega^2_c}{c^2} \left[ \frac{(\omega^2 - \omega_e^2)^2 - \omega^2_c \omega^2_e}{(\omega^2 - \omega_e^2 - \omega^2_c)} \right].
\] (II.49)

We may again plot the dispersion diagram (Fig. II.3), making use of the cut-offs \( k_y = 0 \) and resonances \( k_y = \infty \). From this figure, \( \omega_{\text{UH}}^2 = \omega_e^2 + \omega_c^2 \) (upper hybrid frequency); \( \omega_1 \) and \( \omega_2 \) are the same cut-offs as for the case of parallel propagation. We can look at this diagram as the removal of degeneracy by the magnetic field.

![Dispersion Diagram](image)

**Fig. II.3**

The upper and lower branches in Fig. II.3 are neither pure longitudinal nor pure transverse modes. From Maxwell's equations [Eq. (II.34)] we can calculate the ratio of the longitudinal and transverse components of \( E_1 \):

\[
\frac{E_{1y}}{E_{1x}} = \frac{\varepsilon_{yx}}{\varepsilon_{yy}} = \frac{\omega_c}{\omega} \frac{\omega_e^2}{\omega^2 - \omega_{\text{UH}}^2}.
\] (II.50)

For \( \omega \gg \omega_c \), \( E_{1y} \ll E_{1x} \) and the wave is almost transverse, whereas when \( \omega = \omega_{\text{UH}} \), \( E_{1y} \gg E_{1x} \) and the wave is almost longitudinal.
II.4 LOW-FREQUENCY WAVES

Let us now consider low-frequency waves in a magnetic field. By low frequency, we mean \( \omega \ll \omega_c \). For these low-frequency modes, the ions and the electrons move together and the plasma maintains quasi-neutrality. For wavelengths that are long compared with the ion and electron Larmor radii, the particles of the plasma can be described to a good approximation by a single-fluid model. This is the MHD model of the plasma (ideal or non-ideal, i.e. zero resistivity or finite resistivity). Since one can describe the three basic low-frequency modes of a plasma in a magnetic field with the aid of the ideal MHD equations, we shall use this model. The equations for this model are the following:

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{II.51}
\]

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = -\vec{\nabla} p + \vec{J} \times \vec{B} \tag{II.52}
\]

\[
\vec{E} + \vec{v} \times \vec{B} = 0 \tag{II.53}
\]

\[
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{II.54}
\]

\[
\vec{\nabla} \times \vec{H} = \vec{J} \tag{II.55}
\]

Equation (II.51) is the continuity equation (I.31), where \( \rho \) is the mass density of the fluid and \( \vec{v} \) its mean velocity. Equation (II.52) is the momentum equation (I.32) for the fluid in which a current \( \vec{J} \) flows, so that \( \vec{J} \times \vec{B} \) is the body force on the fluid when a magnetic field is present. The pressure will be assumed isotropic. Equation (II.53) expresses the fact that the conductivity is infinite, and so there can be no \( E \), and the magnetic field is "frozen-in" to the plasma, i.e. the plasma moves across the magnetic field with velocity \( \vec{v} = (\vec{E} \times \vec{B})/|\vec{B}|^2 \). Equations (II.54) and (II.55) are Maxwell's equations with the displacement current neglected since we are dealing with very low frequencies and usually low-phase velocities compared with \( c \). Equations (II.51) and (II.52) can be obtained from the two-fluid equations (cf. Spitzer and Section 1.2).

It is convenient to reduce the above set of five equations to three by substituting (II.55) into (II.52) and (II.53) into (II.54). The linearized version of these equations can then be written

\[
\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla} p + (\vec{v} \times \vec{B}) \times \frac{\vec{B}_0}{\mu_0} \tag{II.56}
\]

\[
\frac{\partial \vec{B}_1}{\partial t} = \vec{v} \times (\vec{v}_1 \times \vec{B}_0) \tag{II.57}
\]

\[
\frac{\partial \rho_0}{\partial t} + \vec{v}_1 \cdot (\rho_0 \vec{v}_1) = 0 \tag{II.58}
\]

In order to close this set of equations, we must add an equation of state, i.e. a relation between \( p \) and \( \rho \). We can take either the adiabatic model (I.34)

\[
\frac{d}{dt} (p \rho^{-\gamma}) = 0 ,
\]
or the isothermal model (Section I.2, p. 13)

\[ \frac{d}{dt} \left( \frac{\rho}{\rho} \right) = 0. \]

It makes only a trivial difference to the properties of the linear waves, which can propagate, and so we choose the isothermal model. For the linearized quantities, this reduces to

\[ p_1 = c_s^2 \rho_1, \quad (II.59) \]

where \( c_s^2 = p_0/\rho_0 \). We again take the uniform, constant, magnetic field to point in the 
\( z \)-direction, and have assumed an infinite, stationary, and uniform plasma when linearizing the

\[ \exp(i(k \cdot \hat{x} - \omega t)), \]

and again take \( \hat{k} = (0, k_y, k_z) \). Substituting into Eqs. (II.56) to (II.58) and using Eq. (II.59)
we obtain

\[ -i \omega_0 \hat{V}_1 = -ikc_s^2 \rho_1 + i(k \times \hat{B}_1) \times \frac{\hat{B}_0}{\mu_0} \quad (II.60) \]

\[ -i \omega \hat{B}_1 = i(k \times (\hat{\nu}_1 \times \hat{B}_0)) \quad (II.61) \]

\[ -i \omega_0 + i \rho_0 (k \cdot \hat{v}_1) = 0. \quad (II.62) \]

Splitting these equations into their Cartesian components, we find that \( \nu_{1x} \) and \( B_{1x} \) are independent of all the other variables. Expressing \( \nu_{1x} \) in terms of \( B_{1x} \) we obtain the dis-

\[ \omega^2 = \frac{B_x^2}{\rho_0 \mu_0} k_z^2 \quad (II.63) \]

or

\[ \omega^2 = c_A^2 k_z^2, \]

where \( c_A^2 = B_0^2/(\rho_0 \mu_0) \), i.e. (Alfvén speed)$^2$.

From the equations for the other variables we obtain

\[ (\omega^2 - k_x^2 c_A^2)(\omega^2 - k_y^2 c_A^2 k_z^2 c_A^2) - k_z^2 k_x^2 c_A^2 = 0. \quad (II.64) \]

For \( k_y = 0 \), i.e. \( \hat{k} \parallel \hat{B}_y \), we have three independent modes -- two transverse and one longitu-

\[ \omega^2 = k_x^2 c_A^2 \quad \text{(Transverse)} \]

\[ \omega^2 = k_x^2 c_A^2 \quad \text{(Transverse)} \]

\[ \omega^2 = k_z^2 c_A^2 \quad \text{(Longitudinal)} \quad \rho_0, \nu_{1z}. \quad (II.65) \]

\[ \left\{ \begin{array}{l} (\nu_{1x}, B_{1x}) \\ (\nu_{1y}, B_{1y}) \\ \rho_0, \nu_{1z} \end{array} \right\} \]
Note that all three modes are non-dispersive. For the two-fluid case, the ion acoustic wave was dispersive at the higher frequencies \((\omega \sim \omega_p)\). However, here we are restricted to low frequencies, so that there is no conflict -- only a more restricted model.

For oblique propagation, we still have one transverse mode \((v_{1x}, B_{1x})\) which is called the SHEAR ALFVÉN wave. (The Langmuir wave and the shear Alfvén wave are probably the two best-known plasma waves.)

We can solve Eq. (II.64) for the case of oblique propagation since the equation is biquadratic. Noting that

\[
\frac{c_s^2}{c_A^2} = \frac{\mu_0 v_B^2}{B_0^2},
\]

which is a small quantity for almost all laboratory plasmas and is known as \(\beta\) (a fundamental quantity for fusion research), we may write the two solutions of Eq. (II.64):

\[
\omega^2 = k^2 c_A^2 \left( 1 + \frac{c_s^2}{c_A^2} \right), \quad (II.67)
\]

\[
\omega^2 = k^2 c_s^2 \left( 1 + \frac{c_A^2}{c_s^2} \right). \quad (II.68)
\]

These two modes are referred to as the fast and slow magneto-acoustic waves, respectively. They are also referred to as the compressional Alfvén and ion acoustic waves. Both modes are hybrid waves. (See Fig. II.4.) The three waves are sometimes referred to as the fast, intermediate, and slow waves (for obvious reasons).
II.5 ANALOGY OF SHEAR ALFVÉN WAVES WITH THE VIBRATIONS OF A STRING UNDER TENSION

The magnetic field lines act like strings under tension, and when they are displaced the plasma moves with them [the plasma is like a set of beads on the strings! (frozen-in condition)].

There is a striking illustration of this (Fig. II.5) which comes from a famous experiment carried out by Melde in the nineteenth century. Transverse oscillations of a stretched string were excited by Melde when he clamped one end of the string rigidly and connected the other end to a vibrating tuning fork (frequency $\omega_p$) such that the tension of the string was modulated periodically. Even though the tuning fork only induced a force along the string, transverse oscillations of frequency $\omega_p/2$ were excited!

![Diagram of string vibrations](image)

Fig. II.5

Now, in a similar manner, it can be shown mathematically (and has been demonstrated experimentally) that when a uniform magnetic field is modulated sinusoidally, i.e.

$$\vec{B}_0 = \vec{B}_0 (1 + \epsilon \sin \omega_p t),$$

transverse Alfvén waves at $\omega_p/2$ are excited.

One final remark concerning these low-frequency waves. We note that only the fast magneto-acoustic (or compressional Alfvén) wave can propagate perpendicularly to $\vec{B}_0$. A plot of the phase velocity surfaces is shown in Fig. II.4.
CHAPTER III

LANDAU DAMPING

The phenomenon of Landau damping is probably the best-known result from the whole of plasma physics, and Landau's 1946 paper on the subject must surely be the most widely quoted paper in this field. Its consequences are far reaching and underlie a vast body of plasma theory.

Let us begin by considering the electrostatic oscillations of a fully ionized plasma. We shall consider the simplest possible situation of one-dimensional oscillations in an infinite, uniform plasma with no external magnetic field. In problems of this type, where the natural oscillations of a continuous medium are being studied, the standard procedure is to Fourier-analyse the motion in space and time, i.e. assume disturbances which vary as \( \exp i(kx - \omega t) \), and hence find the relation between \( \omega \) and \( k \), i.e. the linear dispersion relation. The equations describing the problem (I.21) are now (the ions are assumed to be immobile and simply neutralize the electrons in equilibrium):

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{qE}{m} \frac{\partial f}{\partial v} = 0
\]  \hspace{1cm} (III.1)

\[
\frac{\partial E}{\partial x} = \frac{q}{\varepsilon_0} \int_{-\infty}^{\infty} f(v_x) \, dv_x,
\]  \hspace{1cm} (III.2)

where

\[
f(v_x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v_x, v_y, v_z) \, dv_y \, dv_z.
\]

Equation (III.1) has been linearized, and we wish to solve for the perturbations \( f \) and \( E \). Assuming these vary as \( \exp i(kx - \omega t) \), we obtain

\[
i(\omega - kv)f = \frac{qE}{m} \frac{\partial f}{\partial v}
\]  \hspace{1cm} (III.3)

\[
ike = \frac{q}{\varepsilon_0} \int f(v) \, dv
\]  \hspace{1cm} (III.4)
Solving for \( f \) from Eq. (III.3) and substituting into Eq. (III.4) we obtain the dispersion relation for the oscillations,

\[
1 + \frac{q^2}{\varepsilon_b \kappa_m} \int_0^0 \frac{f'(v) dv}{(\omega - kv)} = 0 ,
\]

(III.5)

where \( f'(v) \) stands for \( \delta f/\delta v \). This is the required dispersion relation for the longitudinal oscillations of an electron plasma. However, there is one problem -- the integral in Eq. (III.5) can be singular for those particles with velocity equal to the phase velocity of the wave. Such a singularity is, of course, unacceptable physically. However, it is also an indication that something of interest is to be expected for these particles.

The integral in Eq. (III.5) is therefore not defined, and we cannot evaluate the "dispersion relation" until it has been defined. There are various possibilities for dealing with the singularity. We could integrate above it (in the complex \( v \)-plane) or below it, or we could do as Vlasov did and take the principal part. At the moment there is no way of deciding this question. It is a great irony that Vlasov, who has given his name to the most celebrated equation in plasma physics, should have missed the most significant application of his famous equation by treating this singularity too lightly! In effect, Vlasov ignored it and so failed to predict one of the most interesting phenomena in plasma physics.

Before going on to Landau's method of solving this problem, let us say a few words about the physical content of Vlasov's equation. A fundamental property is that the equation is time reversible, i.e. if all velocities were reversed and time was made to run backwards, all previous motions would be reproduced -- the equation is in fact invariant under the transformation

\[
v \rightarrow -v , \quad x \rightarrow x , \quad t \rightarrow -t .
\]

This is because the irreversible particle collisions have been left out of the equation, i.e. the two-body collisions which introduce randomness into the interactions. The particle interactions are included only through the self-consistent electric field \( \vec{E} \) which is averaged over all the particles. For this reason, Vlasov's equation is sometimes referred to as the collisionless Boltzmann equation. [It can be shown that Vlasov's equation satisfies an H-theorem,

\[
\frac{dH(t)}{dt} = 0 ,
\]

where \( H(t) = \int f \ln f \, d^3v \, d^3x . \]

] One final property is that the equilibrium solutions of Vlasov's equation do not single out the Maxwell-Boltzmann distribution. Any \( f_{eq}(v) \) will be an equilibrium solution provided only

\[
\sum_j q_j \int f_{eq}(\vec{v}) \, d\vec{v} \, d\vec{x} = 0 .
\]

Let us now describe Landau's approach to this problem. Since the model to be analysed is an infinite, uniform plasma (with no boundaries), the most natural approach is to treat it as an initial value problem, i.e. to calculate the response to a given initial value of
the perturbed electron distribution. The perturbations may therefore be Fourier-analysed in space (as above), but the time behaviour must now be treated by means of a Laplace transform. Starting again from Eqs. (III.1) and (III.2), and assuming that the perturbations vary in space as \( \exp(ikx) \), we find

\[
\frac{\partial f}{\partial t} + ikv f - i \frac{kq}{m} \phi \frac{\partial f}{\partial v} = 0 \quad (\text{III.6})
\]

\[
k^2 \phi = \frac{q}{\varepsilon_0} \int f \, dv \quad ,
\]

(III.7)

where we have introduced the electrostatic potential through \( \vec{E} = -\nabla \phi \).

We now introduce the Laplace transforms in time, as follows:

\[
f(k,v,p) = \int_0^\infty e^{-pt} f(k,v,t) \, dt \quad (\text{III.8})
\]

\[
\phi(k,p) = \int_0^\infty e^{-pt} \phi(k,t) \, dt \quad ,
\]

(III.9)

where \( \text{Re} \, (p) > 0 \) for these integrals to have a sense. The inverse Laplace transforms are

\[
f(k,v,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} f(k,v,p) e^{pt} \, dp \quad (\text{III.10})
\]

\[
\phi(k,t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \phi(k,p) e^{pt} \, dp \quad ,
\]

(III.11)

where the path of integration in the complex \( p \)-plane, defined by \( \sigma \), is such that it is to the right of all singularities of \( f(k,v,p) \) or \( \phi(k,p) \), and to the right of the imaginary axis, i.e. the path is chosen to the right of all singularities of the transform. This then guarantees causality, since for \( t < 0 \) we can close the contour with a semicircle in the right half-plane (\( H.B. \) \( e^{pt} \rightarrow 0 \) as \( p \rightarrow -\infty \) for \( t < 0 \)). The integral around this contour must be zero since there are no singularities enclosed by the contour (by definition) and the integral around the semicircle vanishes as its radius \( \rightarrow \infty \). Since the integral around the semicircle is zero, the integral along the Bromwich contour defined by \( \sigma \) must also be zero.

Laplace-transforming Eqs. (III.6) and (III.7) we therefore obtain

\[
(p + ikv)f(k,v,p) - i \frac{kq}{m} \phi(k,p) \frac{\partial f}{\partial v} = g(k,v) \quad (\text{III.12})
\]

\[
k^2 \phi(k,p) = \frac{q}{\varepsilon_0} \int_{-\infty}^\infty f(k,v,p) \, dv \quad ,
\]

(III.13)
where
\[ g(k,v) \equiv f(k,v_0, t=0) \]  
represents the perturbation to the \( k^{th} \) component of the velocity distribution function at the initial time \( t = 0 \). We may now solve Eqs. (III.12) and (III.13) for \( f(k,v,p) \) and \( \phi(k,p) \). We obtain
\[ f(k,v,p) = \frac{1}{(p + i(kv))} \left\{ g(k,v) + i \frac{kq}{m} \phi(k,p) f_0'(v) \right\}. \]  
(III.15)

Substituting for \( f(k,v,p) \) into Eq. (III.13) we have
\[ k^2 \phi(k,p) = \frac{q}{\varepsilon_0} \int_{-\infty}^{\infty} g(k,v) dv \left( \frac{f_0'(v)}{p + i(kv)} \right) + i \frac{kq}{m} \phi(k,p) \int_{-\infty}^{\infty} f_0'(v) dv. \]

Solving for \( \phi(k,p) \) we obtain
\[ k^2 \left\{ 1 - i \frac{q^2}{k^2 \varepsilon_0 m} \int_{-\infty}^{\infty} f_0'(v) dv \right\} \phi(k,p) = \frac{q}{\varepsilon_0 k^2} \int_{-\infty}^{\infty} g(k,v) dv. \]
\[ : \quad \phi(k,p) = \frac{\frac{q}{\varepsilon_0 k^2} \int_{-\infty}^{\infty} g(k,v) dv}{1 - i \frac{q^2}{k^2 \varepsilon_0 m} \int_{-\infty}^{\infty} f_0'(v) dv}. \]  
(III.16)

The explicit expression for \( f(k,v,p) \) may now be obtained on substitution of \( \phi(k,p) \):
\[ f(k,v,p) = \frac{g(k,v)}{(p + i(kv))} + i \frac{kq}{m} \frac{f_0'(v)}{(p + i(kv))} \left\{ \frac{q}{\varepsilon_0 k^2} \int_{-\infty}^{\infty} g(k,v) dv \right\} \left[ 1 - i \frac{q^2}{k^2 \varepsilon_0 m} \int_{-\infty}^{\infty} f_0'(v) dv \right]. \]  
(III.17)

Let us concentrate on the time behaviour of the electrostatic potential \( \phi(k,t) \). Notice that all the integrals in Eqs. (III.16) and (III.17) are now well defined since \( p \) is a complex variable. The formal expression for the electrostatic potential is obtained from the inverse Laplace transform given by Eq. (III.11). Let us write this down again, since we must now consider this expression in some detail:
\[ \phi(k,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(k,p) e^{pt} dp, \]  
(III.18)

where \( \phi(k,p) \) is now given by Eq. (III.16). We emphasize that the above expression automatically satisfies the causality condition that \( \phi(k,t) = 0 \) for \( t < 0 \), i.e. the disturbance is
switched on at \( t = 0 \). Now the integral \( \phi(k, t) \) will be evaluated with the aid of Cauchy's theorem. It is also convenient to estimate \( \phi \) asymptotically, i.e. \( t \to \infty \). In order to do this, we must deform the Bromwich contour into the left half-p-plane. Since \( \phi(k, p) \) is only defined in the right half-p-plane, we can only do this when we have analytically continued \( \phi(k, p) \) into the left half-p-plane. It is clear from the above that if we are to evaluate the integral \( \phi(k, t) \) we must know as much as possible about the functions \( g(k, v) \) and \( f^p_t(v) \). Guided by his desire to obtain the asymptotic expression for \( \phi(k, t) \), Landau noticed that if he assumed \( g(k, v) \) was an entire function of \( v \) (i.e. it is finite for all finite values of \( v \)), the integral

\[
\int_{-\infty}^{\infty} \frac{g(k, v) dv}{(p + ikv)},
\]

continued analytically into the left half-p-plane, defines an entire function of \( p \).

The final step in the argument is to find how to accomplish this analytic continuation. This is done by noting the movement of the pole \( v = ip/k \) as \( p \) moves from the right half-plane to the left half-plane. As already noted, for \( \text{Re} \( p \) > 0 \) the integral (III.19) is well defined, since the pole is in the upper half-plane. As \( p \) moves to \( \text{Re} \( p \) < 0 \), the pole \( v = ip/k \) crosses the axis and moves into the lower half-plane (i.e. we assume \( k > 0 \)). The \( v \)-integration must now be deformed into the complex \( v \)-plane to ensure that the path of integration always lies below the pole \( v = ip/k \) (Fig. III.1). The assumption that \( g(k, v) \) is an entire function has now led us to a procedure for dealing with the singularity -- we must always integrate below the singularity in the complex \( v \)-plane. The same argument also applies to the other integral in Eq. (III.16):

\[
\int_{-\infty}^{\infty} \frac{f^p_t(v) dv}{(p + ikv)}.
\]

\( v \)-plane

\[\begin{array}{c}
\text{\bullet pole} \\
\text{Re}(p) > 0 \quad \text{Re}(p) = 0 \quad \text{Re}(p) < 0
\end{array}\]

Fig. III.1

Since \( f^p_t(v) \) is an entire function, it is usually taken to be the Maxwell-Boltzmann distribution,

\[
f_0(v) = n_0 \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} e^{-mv^2/2kT}.
\]

Therefore, as noted by Landau, the function \( \phi(k, t) \) is now composed of a ratio of two entire functions of the complex variable \( p \) and is therefore an entire function in itself.
We may therefore deform the Bromwich contour into the left half-p-plane provided we ensure that all poles of the integrand are to the left of the deformed contour. These poles can now only arise from the zeros of the denominator:

\[
1 - i \frac{q^2}{\kappa c \omega m} \int_C \frac{f_0^2(\nu) d\nu}{(p + iK \nu)} = 0 ,
\]

where \(C\) represents the LANDAU contour, defined above. We note, in passing, that it is straightforward to show that there are no roots of this equation for which

\[
\text{Re } (p) > 0 \quad \text{(see Appendix, p. 39)}
\]

when \(f_0\) is the Maxwell-Boltzmann distribution, i.e. there are no growing solutions (instabilities) for this case. This is just as well, although it was not a foregone conclusion since, as already noted, the equilibrium Vlasov equation does not single out the Maxwell-Boltzmann distribution.

Since there are no poles in the right half-plane, we may deform the contour as shown in Fig. III.2. On the deformed contour, the electrostatic potential may now be evaluated as follows:

\[
\phi(k,t) = \sum_n R_n(k) e^{p_n(k)t} + \int_{-\infty}^{\infty} \phi(k,p) e^{pt} dp ,
\]

where \(p_n(k)\) are the zeros of Eq. (III.22), and \(-\alpha\) is to the left of all \(p_n(k)\). Now, let \(t \to \infty\) and the integral on the right-hand side of Eq. (III.23) vanishes leaving only the contribution from the poles \(p_n(k)\). Of these, the rightmost one will dominate as \(t \to \infty\). We conclude, therefore, that only in the limit \(t \to \infty\) do we have a dispersion relation

\[
D(p,k) = 0 .
\]

At earlier times there is evidently no unique relation between \(p\) and \(k\). However, for many applications, Eq. (III.22) is interpreted as the dispersion relation for the electrostatic oscillations of a plasma. We should not forget, however, that this will only be the case when, after a sufficiently long time, the contributions from the integral have damped away.
Let us now obtain the Landau solution for the vibration modes of the electron plasma in the limit of long wavelengths (and, of course, large times). We make the assumption that for small \( k \)

\[
\text{Re} \left( \frac{F_n}{p_n} \right) \rightarrow 0 \quad \text{as} \quad k \to 0
\]

and \( \text{Im} \left( \frac{F_n}{p_n} \right) \) remains finite as \( k \to 0 \) (this will be verified by the solution). With this assumption, the pole in the complex \( v \)-plane will be just below the real axis. We therefore assume

\[
p = -i \omega + \gamma,
\]

where \( |\gamma| << \omega \). (N.B. We have chosen the sign of \( \omega \) such that the wave propagates \( (k > 0) \) in the positive \( x \)-direction. Remember, also, that we previously remarked that any zeros of Eq. (III.22) were necessarily in the left half-\( p \)-plane.) In order to find them, we must first evaluate the integral occurring in that equation. We rewrite it as

\[
1 - \frac{q^2}{k^2 \varepsilon_0 m} \int_{C} \frac{f_1(v) dv}{v - i \frac{P}{k}} = 0, \quad (III.24)
\]

where the contour is taken to be

\[
v = i \frac{P}{k}
\]

\[
\text{Im} \ (v) = 0.
\]

We have two contributions to the integral: that from the real axis, and that from the small semicircle around the pole. Thus

\[
\int_{C} \frac{f_1(v) dv}{v - i \frac{P}{k}} = P \left[ \int_{C} \frac{f_1(v) dv}{v - i \frac{P}{k}} \right] + \text{i} \pi f_1' \left( v = i \frac{P}{k} \right), \quad (III.25)
\]

We can now see the error in Vlasov's treatment of the problem. The integration around the semicircle gives a small imaginary term, the other being Vlasov's principal part contribution [noted as \( P \) in Eq. (III.25)]. The principal part is easily evaluated if we remember our original assumption concerning the real and imaginary parts of \( p_k \). Thus, the pole \( v = ip/k \) will occur for large values of \( v \), where \( f_0(v) \) is very small, so that we may treat the integral as being along the real axis. Furthermore, we may expand the integrand in powers of \( k \) (small \( k \) approximation). We then have

\[
P \left[ \int_{-\infty}^{\infty} \frac{df_0}{dv} \left( \frac{dv}{v - i \frac{P}{k}} \right) \right] = \frac{k}{(-1p)} \left[ \int_{-\infty}^{\infty} \frac{df_0}{dv} \left( 1 - i \frac{kv}{p} \cdots \right) \ dv \right] = \frac{k}{(-1p)} \left[ \int_{-\infty}^{\infty} \frac{df_0}{dv} \left( f_0 \ v \right) \right] _{-\infty}^{\infty} = -\frac{k^2 n_0}{p^2}, \quad (III.26)
\]
where the definition (I.23) is used for the equilibrium density $n_e$. Substituting Eqs. (III.25) and (III.26) into Eq. (III.24), we find

$$1 + \frac{q^2 n_e^2}{\varepsilon_m \omega_p^2} - \frac{q^2}{\varepsilon_m k^2} i \frac{\omega_p}{k} f'\left|_{\nu = i \frac{\omega_p}{k}} \right| = 0.$$  \hspace{1cm} (III.27)

This can now be solved by successive approximations assuming

$$p \approx -i \omega + \gamma,$$

where $|\gamma| \ll \omega$. The first approximation is obtained by neglecting $\gamma$ and the small imaginary part of Eq. (III.27). We then find

$$1 - \frac{\omega_p^2}{\omega^2} = 0$$

and thus $\omega = \omega_p$. \hspace{1cm} (III.28)

Now, including $\gamma$ we find

$$1 + \frac{\omega_p^2}{\left(-i\omega_p + \gamma\right)^2} - \frac{i q^2}{\varepsilon_m k^2} f'\left|_{\nu = \frac{\omega_p}{k}} \right| = 0$$

and

$$1 + \frac{\omega_p^2}{\left(-i\omega_p - 2i\omega_p \gamma\right)} - \frac{i q^2}{\varepsilon_m k^2} f'\left|_{\nu = \frac{\omega_p}{k}} \right| = 0,$$

leading to

$$1 - \left(1 - \frac{2i\gamma}{\omega_p}\right) - \frac{i q^2}{\varepsilon_m k^2} f'\left|_{\nu = \frac{\omega_p}{k}} \right| = 0$$

and

$$\gamma = \frac{\omega_p}{2} \frac{\omega_p^2}{\omega^2} f'\left|_{\nu = \frac{\omega_p}{k}} \right|.$$  \hspace{1cm} (III.29)

For a Maxwellian distribution $f' < 0$, and Eq. (III.29) represents damping (it has already been mentioned that there are no growing waves for a Maxwellian). Substituting for $f'\left(\omega_p/k\right)$, we have

$$\gamma = \frac{\omega_p}{2} \frac{\omega_p^2}{\omega^2} n_e \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left(\frac{m}{kT}\right) \exp\left(-\frac{mv^2}{2kT}\right)\bigg|_{\nu = \omega_p/k}$$

and

$$\gamma = \left(\frac{3}{8}\right)^{\frac{3}{2}} \frac{\omega_p}{\omega^2} \frac{\omega_p^2}{k} \exp\left(-\frac{\omega_p}{2k^2 v^2}\right).$$

Putting $\lambda_{Te} = v_{Te}/\omega_p$, we finally obtain Landau's expression

$$\gamma = -\omega_p \left(\frac{3}{8}\right)^{\frac{3}{2}} \frac{1}{k^3 \lambda_{Te}^3} e^{-1/(2k^2 \lambda_{Te}^2)},$$  \hspace{1cm} (III.30)
The results (III.28) and (III.30) giving $\omega$ and $\gamma$ for the electron plasma oscillations confirm the original assumption that $\gamma \ll \omega$.

We can obtain another simple approximation for the damping in the short wavelength regime, i.e. $\lambda \ll \lambda_d$. In this limit it is assumed (and again this is borne out by the final result) that $\omega/\gamma \ll 1$ for large $k$. This means that wavelengths much shorter than $\lambda_d$ are damped out in a fraction of a wavelength and therefore there can be no collective oscillations for wavelengths shorter than $\lambda_d$.

The expression given by Eq. (III.29) is the famous formula, first derived by Landau in 1946, for the "collisionless damping" of electrostatic oscillations in an electron plasma. It occurs in the absence of any collisions. This result has been the subject of a great deal of controversy partly because of its unexpectedness and partly because of the rather abstract manner in which it was derived; the crucial step in the argument being the assumption that the function $g(k,v) \equiv f(k,v,0)$, i.e. the initial perturbation to the velocity distribution, should be an entire function in the complex velocity plane! However, the result has now been verified experimentally.

Various physical interpretations have been suggested to explain Landau damping. It would take one or two lectures to describe them fully. Here we shall give only a very brief description of the one which is currently more favoured. This is due to Dawson. The expression for the damping given by Eq. (III.28) shows that it is due to a small group of electrons travelling at the phase velocity of the wave. The interpretation of this is that these "resonant particles" are the only ones which interact strongly with the wave. Those particles which travel slightly slower than the wave will be accelerated by it and will extract energy from it, whereas those particles travelling slightly faster than the wave will give energy to it. For a velocity distribution where $f_0'(\omega/k) < 0$ (e.g. a Maxwellian) there are more particles travelling slightly more slowly than the wave than there are particles travelling just a little faster than it. The net result is that the resonant particles absorb power and the wave is damped. This resonant power transfer was described in detail by Dawson. He also showed that the mechanism he proposed was also frame-independent -- clearly a desirable property of any proposed interpretation. We shall have more to say on this property when we go on to consider growing wave solutions or instabilities.

Finally, we should mention one other property, which was also noted by Landau. We have concentrated on the asymptotic behaviour of the electrostatic potential. We could, however, also obtain the asymptotic particle distribution function from Eq. (III.17) and the inversion formula

$$f(k,v,t) = \frac{1}{2\pi i} \int_{\sigma-i=\infty}^{\sigma+i=\infty} f(k,v,p) e^{pt} dp.$$  

We see from Eq. (III.17) that, in contrast to the expression for $\Phi(k,p)$, the pole $(p + ikv)^{-1}$ occurs in the expression for $f(k,v,p)$. This means that for large times $t \rightarrow \infty$ the behaviour of the distribution function is purely oscillatory, i.e. undamped. Thus Landau damping is only damping in configuration space and the information in the velocity distribution is not lost. This shows that the effect is not a true dissipative effect. In view of our remarks at the foot of page 31 concerning the conservation of entropy, this result is not unexpected but is a further demonstration that Landau's result is the correct one.
Proof that for a Maxwellian (or single-humped distribution) there are no zeros of

\[ 1 - \frac{q^2}{k_e^2 m} \int_C \frac{f_0^4(v) dv}{(kv - 1p)} = 0 \]  \hspace{1cm} (A.1)

in the right half-p-plane.

To prove this, assume that there is a zero in the right half-plane, then

\[ p = x + iy \quad \text{with} \quad x > 0. \]

Since the pole is in the right half-plane, we can carry out the integral along the real v axis:

\[ \therefore 1 - \frac{q^2}{k_e^2 m} \int_{-\infty}^{\infty} \frac{f_0^4(v) dv}{(kv - 1x + y)} = 0 \]  \hspace{1cm} (A.2)

\[ \therefore 1 = \frac{q^2}{k_e^2 m} \int_{-\infty}^{\infty} \frac{(kv + y)f_0^4(v) dv}{(kv + y)^2 + x^2} \]  \hspace{1cm} (A.3)

\[ \frac{q^2 x}{k_e^2 m} \int_{-\infty}^{\infty} \frac{f_0^4(v) dv}{(kv + y)^2 + x^2} = 0. \]  \hspace{1cm} (A.4)

Now assume that \( f_0(v) \) is single-humped: for \( v < U, \partial f_0/\partial v > 0 \); and for \( v > U, \partial f_0/\partial v < 0 \). Multiplying Eq. (A.4) by \( (ku + y)/x \) and subtracting it from Eq. (A.3) gives

\[ 1 = -\frac{q^2}{k_e^2 m} \int_{-\infty}^{\infty} \frac{k(U - v)f_0^4(v) dv}{(kv + y)^2 + x^2}. \]

Now, owing to the assumptions concerning \( f_0(v) \) the integrand is positive definite, and so the right-hand side is negative definite. We therefore have a contradiction and there can be no zeros for which \( x > 0 \).
CHAPTER IV

LINEAR STABILITY THEORY

IV.1 CURRENT-DRIVEN ELECTROSTATIC INSTABILITIES

There are many modes of instability in hot collisionless plasmas. This is partly a reflection of the large number of degrees of freedom of a plasma and the number of different types of waves it can support; and it is partly due to the variety of ways in which a plasma can depart from thermal equilibrium because of the rarity of particle collisions. Thus the energy to drive an instability can be stored in currents (or particle streams), in non-uniform spatial distributions of particle numbers or energy, in anisotropic velocity distributions, in twisted magnetic field lines, and so on. Here we shall just consider one simple example of a plasma instability, i.e. that due to a current or stream of particles. However, this simple example will exhibit a number of the characteristic features of plasma instabilities. It will also serve to illustrate other points in the behaviour of plasmas that we have already mentioned briefly.

The two-stream instability which occurs when there is relative motion between the ions and electrons in equilibrium, i.e. a current is flowing, has already been described (J.D. Lawson). The problem can be described in one dimension. We shall generalize the treatment given by Dr. Lawson by using the Vlasov model. Thus we make the normal linearization approximation and look for the behaviour of perturbations assumed to vary as

$$\exp i(kz - \omega t),$$

where $z$ is the direction of current flow. We assume that the waves are electrostatic, so that we need only Poisson’s equation. The linearized Vlasov equation (1.21) for the $j$th species is

$$\frac{\partial f_{1j}}{\partial t} + v \cdot \frac{\partial f_{1j}}{\partial x} + \frac{q_j}{m_j} E \cdot \frac{\partial f_{1j}}{\partial v} = 0 .$$  \hspace{1cm} \text{(IV.1)}

Assuming the above one-dimensional variation, we have

$$-i\omega f_{1j} + ikv_z f_{1j} = -\frac{q_j}{m_j} E_z \frac{\partial f_{1j}}{\partial v_z} .$$  \hspace{1cm} \text{(IV.2)}

Introducing the electrostatic potential $\bar{E} = -\nabla \phi$, we can solve for $f_{1j}$,

$$-i(\omega - kv_z)f_{1j} = i \frac{kq_j}{m_j} \phi \frac{\partial f_{1j}}{\partial v_z} ,$$  \hspace{1cm} \text{(IV.3)}
\[ f_{1j} = - \frac{\kappa q_j}{m_j} \frac{\delta f_{0j}/\delta v_z}{(\omega - k v_z)} \]  

We must couple this to Poisson's equation,

\[ \nabla \cdot E = \frac{1}{\varepsilon_0} \sum_j q_j \int f_{1j} \, dv_z . \]  

Substituting Eq. (IV.4) into (IV.5), we obtain

\[ k^2 = - \frac{1}{\varepsilon_0} \sum_j \frac{k q_j^2}{m_j} \int \frac{(\delta f_{0j}/\delta v_z) dv_z}{(\omega - k v_z)} . \]  

Equation (IV.6) is the dispersion relation. Note that we again have the singular integral we encountered in the problem of Landau damping. However, we now know how to deal with it by means of Landau's prescription, so that we need not go through all the details of Laplace transforms but can Fourier-analyse and then use the Landau contour to perform the integration.

We now assume that the electron and ion equilibrium distributions are Maxwellian (III.21),

\[ f_{0e} = n_0 \left( \frac{m_e}{2\pi k T_{ke}} \right)^{1/2} \exp \left[ -\frac{m_e (v_z - v_{d})^2}{2k T_{ke}} \right] \]  

\[ f_{0i} = n_0 \left( \frac{m_i}{2\pi k T_{ki}} \right)^{1/2} \exp \left[ -\frac{m_i v_z^2}{2k T_{ki}} \right] . \]  

Substituting these distribution functions into Eq. (IV.6) we can write the dispersion relation in the form

\[ 1 + \frac{\omega_p^2}{k^2} \left( \frac{m_e}{2\pi k T_{ke}} \right)^{1/2} \int_{-\infty}^{\infty} \frac{v_z}{(\omega/k) - v_z} \exp \left[ -\frac{m_e (v_z - v_{d})^2}{2k T_{ke}} \right] dv_z = 0 . \]  

Now the above integrals, where the equilibrium distribution functions are Maxwellian, occur frequently, and the corresponding function has been named the Plasma Dispersion Function usually denoted by \( \Pi \) (but sometimes \( h \)). Thus

\[ 2(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \zeta} \, dx , \]  

where the contour runs along the real axis when \( \text{Im} \, (\zeta) > 0 \) but drops below when \( \text{Im} \, (\zeta) < 0 \) (the Landau prescription). Tables of \( 2(\zeta) \) exist, but for analytical work one can make use of small and large argument expansions. Thus
\[ Z(\zeta) \approx i\sqrt{\pi} e^{-\zeta^2} - 2\zeta \left( 1 - \frac{2}{3} \zeta^2 + \frac{4}{15} \zeta^4 + \ldots \right) \quad \text{for } \zeta \ll 1 , \]  

(IV.11)  

and  

\[ Z(\zeta) \approx i\sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta} \left( 1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \ldots \right) \quad \text{for } \zeta \gg 1 . \]  

(IV.12)  

Strictly the imaginary term should be $i\sqrt{\pi} \sigma e^{-\zeta^2}$, where $\sigma = 0$ for $\text{Im}(\zeta) > 0$, $\sigma = 1$ for $\text{Im}(\zeta) = 0$, and $\sigma = 2$ for $\text{Im}(\zeta) < 0$. The dispersion equation can now be written

\[
1 - \frac{\omega_p^2}{k^2 v_f^2} \int_{-\infty}^{\infty} \frac{(v_z - v_d) \exp \left[-m_e(v_z - v_d)^2/2kT_e\right]}{[\omega/k - v_z]} dv_z \]

\[- \frac{\omega_p^2}{k^2 v_f^2} \int_{-\infty}^{\infty} \frac{v_z \exp \left(-m_i v_z^2/2kT_i\right)}{[\omega/k - v_z]} dv_z = 0 , \]

Now, putting $(v_z - v_d)/\sqrt{2} v_f e = x$ we obtain

\[
\int_{-\infty}^{\infty} \frac{(v_z - v_d) \exp \left[-m_e(v_z - v_d)^2/2kT_e\right]}{[\omega/k - v_z]} dv_z = \sqrt{2} v_f e \int_{-\infty}^{\infty} \frac{x e^{-x^2}}{\sqrt{2kT_e}} dx
\]

\[
= \sqrt{2} v_f e \sqrt{\pi} \left[ 1 + \frac{\omega - kv_d}{\sqrt{2kT_e}} \left( \frac{\omega - kv_d}{\sqrt{2kT_e}} \right) \right] \]

\[
= \sqrt{2} v_f e \sqrt{\pi} \left[ 1 + \frac{\omega - kv_d}{\sqrt{2kT_e}} \left( \frac{\omega - kv_d}{\sqrt{2kT_e}} \right) \right] \]

The dispersion relation can now be written

\[
1 + \frac{\omega_p^2}{k^2 v_f^2} \left[ 1 + \frac{\omega - kv_d}{\sqrt{2kT_e}} \left( \frac{\omega - kv_d}{\sqrt{2kT_e}} \right) \right] + \frac{\omega_p^2}{k^2 v_f^2} \left[ 1 + \frac{\omega - kv_d}{\sqrt{2kT_i}} \left( \frac{\omega - kv_d}{\sqrt{2kT_i}} \right) \right] = 0 . \]  

(IV.13)  

Let us first demonstrate that the simple fluid result is contained in this equation. We expect the fluid picture to hold when the thermal velocities are less than the phase velocity, i.e.

\[ \omega \gg kv_f \]

or, more precisely, $\omega - kv_d \gg kv_f$. 

This implies larger drift velocities such that \( v_d \gg v_{Te} \). Under these conditions we use the asymptotic expansion for both ions and electrons. We then have

\[
1 + \frac{\omega^2 p_e}{k^2 v_{Te}^2} \left[ 1 - \frac{\zeta_e}{\xi_e} \left( 1 + \frac{1}{2 \zeta_e^2} + \ldots \right) \right] + \frac{\omega^2 p_i}{k^2 v_{Ti}^2} \left[ 1 - \frac{\zeta_i}{\xi_i} \left( 1 + \frac{1}{2 \zeta_i^2} + \ldots \right) \right] = 0,
\]

where \( \xi_e \equiv (\omega - kv_d)/\sqrt{2kv_{Te}} \) and \( \zeta_i \equiv \omega/\sqrt{2kv_{Ti}} \).

\[
1 - \frac{\omega^2 p_e}{k^2 v_{Te}^2} \frac{1}{2} \frac{2k^2 v_{Te}^2}{(\omega - kv_d)^2} - \frac{\omega^2 p_i}{k^2 v_{Ti}^2} \frac{1}{2} \frac{2k^2 v_{Ti}^2}{\omega^2} = 0
\]

\[
1 - \frac{\omega^2 p_e}{(\omega - kv_d)^2} - \frac{\omega^2 p_i}{\omega^2} = 0. \quad \text{(IV.14)}
\]

This is the required equation obtained previously from a fluid model. For the case of two electron streams of equal density and velocities \( v_1 \) and \( v_2 \), it is easy to show that instability results from the coupling between the slow wave of the faster beam (negative energy mode) with the fast wave on the slower beam (positive energy mode).

Now let us depart from the fluid picture and see what new information (if any) the kinetic model can give us. Suppose we either increase the electron temperature, or reduce \( v_d \), such that \( v_d \ll v_{Te} \). We now have the condition

\[
\omega - kv_d \ll kv_{Te}.
\]

Let us assume, however, that we can still satisfy \( \omega \gg kv_{Ti} \) -- we shall find that the condition for this to be the case is that \( T_e \gg T_i \). Thus we now assume solutions satisfying

\[
k_{x, i} v_{Ti} \ll \omega \ll kv_{Te}
\]

and use the small argument expansion for the electrons and the asymptotic expansion for the ions, giving the result

\[
1 + \frac{\omega^2 p_e}{k^2 v_{Te}^2} \left[ 1 + \frac{(\omega - kv_d)}{\sqrt{2kv_{Te}}} \right] - \frac{\omega^2 p_i}{k^2 v_{Ti}^2} \left[ \frac{\omega}{\sqrt{2kv_{Ti}}} \right] = 0. \quad \text{(IV.15)}
\]

We can write this as

\[
1 + \frac{\omega^2 p_e}{k^2 v_{Te}^2} \frac{\omega^2 p_i}{k^2 v_{Ti}^2} - \frac{\omega^2 p_e}{k^2 v_{Te}^2} \frac{\omega}{\sqrt{2kv_{Ti}}} \frac{\omega^2 p_i}{k^2 v_{Ti}^2} \exp \left( -\frac{\omega^2}{2k^2 v_{Ti}^2} \right) = 0. \quad \text{(IV.16)}
\]
For the sake of simplicity, let us consider long wavelengths such that:

\[ \frac{\omega^2}{P_e} \gg \frac{1}{k^2 \nu_I^2} \]

that is

\[ \lambda^2 \gg \lambda_{\parallel e}^2 \]

The equation can then be written in the physically revealing form

\[ \omega^2 - k^2 c_S^2 = -i \frac{\omega}{\sqrt{2k\nu_p}} (\omega - \nu_d) - i \sqrt{\frac{\omega}{\nu_p}} \frac{T_e}{T_1} \exp \left( -\frac{\omega^2}{2k^2 \nu_p} \right) \]  

(IV.17)

We now recognize this as the dispersion relation for low-frequency ion-acoustic waves, which we previously encountered as one of the electrostatic modes to come out of the warm fluid plasma model (Section II.2.1). However, we now see that there is a small, imaginary correction term. Let us now solve this equation by a perturbation solution. The zero-order solution is, of course,

\[ \omega = \pm k c_S \]

Now put

\[ \omega = k c_S + \delta \omega \]

and we obtain

\[ \delta \omega = -i \sqrt{\frac{k}{\nu_p}} \left( \frac{m_p}{m_i} \right)^{\frac{1}{2}} k \left[ c_S - \nu_d + c_S \left( \frac{T_i \nu_p}{T_e \nu_m} \right)^{\frac{1}{2}} \exp \left( -\frac{T_e}{2T_1^2} \right) \right] \]  

(IV.18)

For \( T_e \gg T_i \) we can neglect the last term on the right-hand side (in practice this means \( T_e / T_i > 10 \)). However, note that the damping is weak even for \( T_e / T_i \approx 6 \). Thus for \( \nu_d > c_S \), \( \text{Im} (\omega) > 0 \), and we have the ION ACOUSTIC INSTABILITY.

Notice that as \( T_e / T_1 \) decreases, the phase velocity of the wave \( c_s \) approaches \( \nu_1 \left[ c_s = (kT_e/m_i)^{\frac{1}{2}} \right] \) and ion Landau damping becomes stronger and stronger. For \( \nu_d = 0 \) the ion acoustic wave becomes heavily damped by the ions as \( T_i \rightarrow T_e \), but for \( T_e > T_i \) the wave is only lightly attenuated. \( \text{Im} (\omega) / \text{Re} (\omega) \approx (m_e/m_i)^{\frac{1}{2}} \) so that the wave is driven unstable by a relatively low drift velocity, i.e.

\[ \nu_d > c_s \]

We note that the other root of the equation \( \omega = -k c_s \) is always damped, with or without \( \nu_d \). It is only the wave which propagates in the direction of the current flow, or stream velocity, which is driven unstable, i.e. it is only this wave which sees a positive slope on the electron distribution or finds more fast resonant electrons than slow ones.

We can now make an interesting check on the stability analysis which sheds further light on the instability mechanism. The physical result of instability cannot depend on
what frame we choose to use -- similarly, we cannot effect the damping of a wave by changing frames. Let us therefore write down the above dispersion relation in the rest frame of the electrons. We can easily obtain this from Eq. (IV.13), and we have

\[
1 + \frac{\omega^2 P_e}{k^2 v_T^2 e} \left[ 1 + \frac{\omega}{\sqrt{2} k v_T e} \left( \frac{\omega}{\sqrt{2} k v_T e} \right) \right] + \frac{\omega^2 P_i}{k^2 v_T^2 i} \left[ 1 + \frac{\omega + k|v_d|}{\sqrt{2} k v_T i} \left( \frac{\omega + k|v_d|}{\sqrt{2} k v_T i} \right) \right] = 0. \quad (IV.19)
\]

We previously had the ions at rest and the electrons drifting to the right, whereas we now have the electrons at rest and the ions drifting to the left. For \(\omega/\sqrt{2} k v_T e << 1\) and \((\omega + k|v_d|)/\sqrt{2} k v_T i >> 1\) we obtain

\[
1 + \frac{\omega^2 P_e}{k^2 v_T^2 e} + \frac{\omega^2 P_e}{k^2 v_T^2 i} \frac{\omega}{\sqrt{2} k v_T e} \frac{i\sqrt{\pi}}{k^2 v_T^2 i} \left( \frac{k^2 v_T^2 i}{(\omega + k|v_d|)^2} \right) = 0, \quad (IV.20)
\]

where we assume that \(T_e >> T_i\) and so neglect the effect of resonant ions. Again assuming long wavelengths, we have

\[
(\omega + k|v_d|)^2 - k^2 c_s^2 = -i\sqrt{\pi} \frac{\omega}{\sqrt{2} k v_T e} (\omega + k|v_d|)^2. \quad (IV.21)
\]

Clearly, in this frame the electrons are at rest so that there will be more slow electrons than fast electrons. Can we therefore have an instability?

Solving Eq. (IV.21) by a perturbation procedure we assume

\[
\omega = -k|v_d| + kc_s + \delta\omega \quad \text{ (where } k > 0) \quad (IV.22)
\]

to obtain

\[
\delta\omega = i\sqrt{\pi} \frac{kc_s}{2} \frac{(|v_d| - c_s)}{\sqrt{2} v_T e}, \quad (IV.23)
\]

which is the same result as before -- as it should be. The reason is simple -- the slow-ion acoustic wave

\[
\omega = -k(|v_d| - c_s)
\]
is a negative energy wave. Since there are more slow electrons than fast ones, the electrons extract energy from the slow-ion acoustic wave in this frame thus causing it to grow. One can verify that the wave has indeed negative energy since

\[
c_h(\omega, k) = 1 + \frac{\omega^2 P_e}{k^2 v_T^2 e} - \frac{\omega^2 P_i}{(\omega + k|v_d|)^2},
\]
where \( \varepsilon_h(\omega,k) \) is the Hermitian part of the dielectric function, and the wave energy is proportional to \( \frac{\partial}{\partial \omega} \left[ \omega \, c_h(\omega,k) \right] |E_z|^2 \).

Finally, we can compare the two-stream instability (the fluid limit) with the ion acoustic instability. In the former case, the ion wave (positive energy) is driven unstable by the negative energy electron wave -- there is an exchange of energy between two collective modes of the plasma, and both waves grow exponentially. Clearly, all the electrons are active in this instability mechanism, and the details of the distribution functions are not important. On the other hand, the ion acoustic instability results when a small group of resonant electrons (travelling around the phase velocity of the ion acoustic wave) feed energy to the ion wave. In this case, only a fraction of the electrons are active in causing the growth of the ion acoustic wave. Obviously, this instability is quite sensitive to the details of the electron distribution function. If \( f_{\text{se}} \) happened to be flat in the vicinity of \( v = c_s \), there would be no instability even though the electrons as a whole were drifting! We shall return to this point when we come to consider the quasi-linear theory of electrostatic oscillations.

### IV.2 THE PENROSE STABILITY CRITERION

There are so many instabilities known in plasma physics that it is helpful to consider general stability criteria. One such criterion was derived by Penrose in 1960 and refers to electrostatic plasma oscillations. The method used by Penrose is related to the Nyquist technique, which is familiar from the stability analysis of electrical circuits. Suppose the dispersion relation is given by

\[
D(\omega,k) = 0 . \tag{IV.24}
\]

We now wish to find whether there are roots of this equation such that

\[
\text{Im} \, (\omega) > 0 .
\]

If there are such roots, then the plasma will be unstable. The Nyquist method consists in mapping the upper half-\( \omega \)-plane into the complex \( D \)-plane (Fig. IV.1).

Thus, consider the integral

\[
\frac{1}{2\pi i} \oint_C \frac{D}{\partial \omega} \, d\omega ,
\]

where the contour \( C \) consists of the upper half-\( \omega \)-plane. Now, assuming there are no poles of \( D \) in the \( \omega \)-plane, the value of this integral is determined by the zeros of \( D(\omega,k) = 0 \). The value is simply

\[
\frac{1}{2\pi i} \oint_C \frac{D}{\partial \omega} \, d\omega = N , \tag{IV.25}
\]
where $N$ is the number of zeros of $D$ in the upper half-plane (double roots give a contribution of 2, etc.). If we express this integral in terms of the corresponding contour $\Gamma$ in the complex $D$-plane, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{dD}{D} = N.$$  \hspace{1cm} \text{(IV.26)}

We now have the simple result that the number of zeros (i.e. unstable roots) in the upper half-$\omega$-plane is given by the number of times that the contour $\Gamma$ encircles the origin.

Let us now see how Penrose applied this technique to the stability of electrostatic oscillations in a plasma. The dispersion relation is obtained from the Landau prescription and can be written

$$D(\omega, k) \equiv 1 - \sum_{j} \frac{\omega_j^2}{k^2} \int_{L} \frac{(3f_{d}/3\nu)d\nu}{[\nu - (\omega/k)]}, \quad \text{where} \quad k > 0,$$

and $L$ denotes that the Landau contour is to be used. We now write this equation in the form

$$k^2 = \sum_{j} \omega_j^2 \int_{L} \frac{(3f_{d}/3\nu)d\nu}{[\nu - (\omega/k)]} = \alpha(\omega/k).$$  \hspace{1cm} \text{(IV.27)}

We now wish to discover whether there are complex values of $\omega$ in the upper half-plane which satisfy Eq. (IV.28), i.e. whether for $\text{Im} \ (\omega) > 0$ the integral on the right-hand side of Eq. (IV.28) can take on positive real values. Instead of mapping $D(\omega, k)$ in the upper half-$\omega$-plane, Penrose considered the related mapping of the right-hand side of Eq. (IV.28). The mapping of the upper half-$\omega$-plane by this integral $[\alpha(\omega/k)]$ is determined by $\omega$ moving from $-\infty$ to $+\infty$ along the upper side of the real axis $\omega + i0$, since the integral vanishes over the whole of the semicircle in the upper half-plane. Assuming that $\int_{-\infty}^{+\infty} |f_{d}(\nu)| d\nu < \infty$, $\alpha(\omega/k)$ traces out a directed curve (anticlockwise) in the $\omega$-plane, which starts and finishes at the origin, i.e. a closed curve. Now, by the argument principle (Carpenter, p. 119), for any point $\alpha_0$ not on this contour $\alpha[(\omega + i0)/k]$, the curve $\alpha$ winds round this point anticlockwise the same number of times as $\alpha(\omega/k)$ takes on this value $\alpha_0$ for $\omega$ in the upper half-$\omega$-plane. Thus, $\alpha(\omega/k)$ takes on real values only if $\alpha[(\omega + i0)/k]$ encloses part of the positive real $\alpha$-axis. Clearly, this can only happen if the curve $\alpha[(\omega + i0)/k]$ crosses the positive real $\alpha$-axis. Since the $\alpha$-curve is traced anticlockwise, this means that the crossing will take place from $\text{Im} \ (\alpha) < 0$ to $\text{Im} \ (\alpha) > 0$. For $\omega$ moving just above the real axis,

$$\alpha(\omega/k) = \sum_{j} \omega_j^2 \int_{-\infty}^{+\infty} \left\{ \left[ \frac{(3f_{d}/3\nu)d\nu}{[\nu - (\omega/k)]} \right] \right\} \left[ \int_{-\infty}^{+\infty} \left(3f_{d}/3\nu\right) d\nu \right] + \sum_{j} i\omega_j^2 f_{d}(\nu) \left|_{\nu=\omega/k} \right. \left. \right.$$  \hspace{1cm} \text{(IV.29)}

This means that the rightmost crossing of the real $\alpha$-axis corresponds to a minimum of $f_{d}(\nu)$. Using Eq. (IV.29) one can easily see that for a Maxwellian distribution (or in fact any
single-humped distribution), the curve $\alpha$ will be of the form shown in Fig. IV.2. Clearly no part of the Re $(\alpha)$-axis is enclosed, and the plasma is stable. An unstable curve may be of the form shown in Fig. IV.3.

To complete the Penrose criterion, we require

$$P \left\{ \int_{-\infty}^{\infty} \frac{f_0(\omega/k)}{[v-(\omega/k)]} \right\} > 0 .$$

The Penrose criterion can now be stated as follows. Exponentially growing modes exist if, and only if, there is a minimum of $f_0(v)$ at $v = \omega_0/k$ such that

$$\int_{-\infty}^{\infty} \frac{f_0(\omega/k)}{[v-(\omega_0/k)]} > 0 . \quad (IV.30)$$

We do not need the principal value since $f_0(\omega_0/k) > 0$. This last condition was given by Penrose in another form. Thus,

$$P \left\{ \int_{-\infty}^{\infty} \frac{f_0(\omega/k)}{[v-(\omega/k)]} \right\} = \lim_{\varepsilon \to 0} \left\{ \int_{-(\omega/k)+\varepsilon}^{(\omega/k)-\varepsilon} \frac{f_0(v) - f_0(\omega/k)}{[v-(\omega/k)]^2} \frac{dv}{v-(\omega/k)} \right\} = \int_{-\infty}^{\infty} \frac{f_0(v) - f_0(\omega/k)}{[v-(\omega_0/k)]^2} \frac{dv}{[v-(\omega_0/k)]^2}$$

where again we do not need the principal value since $f_0(\omega/k) = 0$. 
IV.3 THE ENERGY PRINCIPLE

In their original demonstration of the energy principle for ideal MHD stability theory, Bernstein et al. found it necessary to assume that the eigenfunctions of the linearized operator formed a complete set. A further property that this operator is required to possess is that of self-adjointness. The energy principle then follows. However, it was later pointed out that the property of completeness does not always hold, so that it became necessary to find a proof of the energy principle which did not assume this property. This was accomplished in a very elegant analysis by Laval et al., who assumed only that the linearized operator was self-adjoint. Now, the property of self-adjointness is not in doubt, but its proof can be rather involved and is, physically, not very revealing. However, this property is only necessary in order to prove conservation of energy and to show that the functional which arises in the theory is indeed the potential energy of the system.

In this section we shall extend the method of Laval et al. but without the need to assume (or demonstrate) the self-adjointness property. Instead, we shall demonstrate the conservation of energy directly from the linearized MHD equations. It might be thought that since energy is conserved it is unnecessary to have to show this. However, as pointed out by Sturrock, for discussions of linear stability it is not the physical energy that matters but the small signal energy formed only from the fields which occur in the linearized analysis. This point of view originates from the small signal power theorem of L.J. Chu, which was applied to the theory of microwave tubes. It is this small signal energy that we must show to be conserved. We shall do this by proving a generalized Poynting theorem for the linearized MHD model.

Let us consider a plasma in which the pressure is isotropic and which is bounded by a rigid, perfectly conducting wall, where the usual boundary conditions hold (Bernstein et al.). The linearized equations of the ideal MHD model (Section I.2) are:

\[ \rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{V}p + \vec{J} \times \vec{B}_0 + \vec{J}_0 \times \vec{B} \]  \hspace{1cm} (IV.31)

\[ \frac{\partial \vec{E}}{\partial t} + \vec{V} \cdot (\rho_0 \vec{V}) = 0 \]  \hspace{1cm} (IV.32)

\[ \vec{E} + \vec{V} \times \vec{B}_0 = 0 \]  \hspace{1cm} (IV.33)

\[ \frac{\partial p}{\partial t} + (\vec{V} \cdot \vec{V})p_0 = \frac{\gamma}{\rho_0} \left( \frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{V} \rho_0 \right) \]  \hspace{1cm} (IV.34)

\[ \vec{V} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \]  \hspace{1cm} (IV.35)

\[ \vec{V} \times \vec{H} = \vec{J} \]  \hspace{1cm} (IV.36)

\[ \vec{V} \cdot \vec{H} = 0 \]  \hspace{1cm} (IV.37)
where fields with subscript zero are equilibrium quantities and the linearized variables are written without subscripts. As usual there is no equilibrium flow of the plasma. We now scalar multiply Eq. (IV.31) by \( \mathbf{v} \) to obtain

\[
\rho_0 \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \mathbf{v} \mathbf{p} + \mathbf{v} \cdot (\mathbf{j} \times \mathbf{B}_0) + \mathbf{v} \cdot (\mathbf{j}_0 \times \mathbf{B}) .
\]  

(IV.38)

Using the relation

\[
\mathbf{v} \cdot (\mathbf{v} \mathbf{p}) = p \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \mathbf{p} ,
\]

Eq. (IV.38) can be written in the form

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v} \right) = -\mathbf{v} \cdot (p \mathbf{v}) + p \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\mathbf{j} \times \mathbf{B}_0) + \mathbf{v} \cdot (\mathbf{j}_0 \times \mathbf{B}) .
\]

(IV.39)

Next, scalar multiply Eq. (IV.36) by \( \mathbf{E} \), Eq. (IV.35) by \( \mathbf{H} \), and subtract to obtain

\[
\mathbf{E} \cdot \mathbf{v} \times \mathbf{H} - \mathbf{H} \cdot \mathbf{v} \times \mathbf{E} = \frac{\partial}{\partial t} \left( \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H} \right) + \mathbf{j} \cdot \mathbf{E} .
\]

This can be written as

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H} \right) + \mathbf{v} \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{j} \cdot \mathbf{E} = 0 .
\]

(IV.40)

Substituting for \( \mathbf{E} \) from Eq. (IV.33) into the term \( \mathbf{j} \cdot \mathbf{E} \), Eq. (IV.40) becomes

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H} \right) + \mathbf{v} \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{v} \cdot (\mathbf{j} \times \mathbf{B}_0) ,
\]

(IV.41)

where we have made use of the vector identity \( \mathbf{j} \cdot (\mathbf{v} \times \mathbf{B}_0) = \mathbf{v} \cdot (\mathbf{B}_0 \times \mathbf{j}) \). Adding Eqs. (IV.39) and (IV.41), we obtain

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H} \right) + \mathbf{v} \cdot (\mathbf{E} \times \mathbf{H} + p \mathbf{v}) = p \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\mathbf{j}_0 \times \mathbf{B}) .
\]

(IV.42)

Now consider the term \( p \mathbf{v} \cdot \mathbf{v} \). With the aid of Eqs. (IV.32) and (IV.34), we find

\[
\mathbf{v} \cdot \mathbf{v} = - \left[ \frac{1}{\gamma p_0} \frac{\partial p}{\partial t} + \frac{1}{\gamma p_0} (\mathbf{v} \cdot \mathbf{v}) p_0 \right]
\]

(IV.43)

Substituting Eq. (IV.43) into (IV.42), we have

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \mu_0 \mathbf{H} \cdot \mathbf{H} + \frac{1}{2} \frac{p^2}{\gamma p_0} \right) + \mathbf{v} \cdot (\mathbf{E} \times \mathbf{H} + p \mathbf{v}) = \mathbf{v} \cdot (\mathbf{j}_0 \times \mathbf{B}) - \frac{p}{\gamma p_0} (\mathbf{v} \cdot \mathbf{v}) p_0 .
\]

(IV.44)

In order to obtain the final form of the conservation equation, we must introduce the linear displacement vector \( \xi \) defined by

\[
\frac{\partial \xi}{\partial t} = \mathbf{v} .
\]

(IV.45)
This enables us to integrate Eq. (IV.35) in time to obtain \( \vec{B} \) in terms of \( \xi \), thus

\[
\vec{B} = \vec{\gamma} \times (\xi \times \vec{B}_0) ,
\]

where we have, of course, made use of Eq. (IV.33). We can also obtain \( p \) in terms of \( \xi \) by integrating Eq. (IV.43) in time to obtain

\[
p = -(\xi \cdot \vec{\gamma}) p_0 - \gamma p_0 \vec{\gamma} \cdot \xi .
\]

Now consider the two terms on the right-hand side of Eq. (IV.44). First,

\[
\vec{\nu} \cdot (\vec{J}_0 \times \vec{B}) = -\vec{J}_0 \cdot \left( \frac{\partial \vec{B}}{\partial t} \right) .
\]

Since \( \vec{B} \) depends linearly on \( \xi \), and \( \vec{J}_0 \) is independent of the time, we may write

\[
\vec{J}_0 \cdot \left( \frac{\partial \vec{B}}{\partial t} \right) = \frac{\partial}{\partial t} \left\{ \frac{1}{2} \vec{J}_0 \cdot (\xi \times \vec{B}) \right\}
\]

so that

\[
\vec{\nu} \cdot (\vec{J}_0 \times \vec{B}) = -\frac{\partial}{\partial t} \left\{ \frac{1}{2} \vec{J}_0 \cdot (\xi \times \vec{B}) \right\} .
\]

Next, consider the second term,

\[
\frac{p}{\gamma p_0} \vec{\nu} \cdot \vec{v} p_0 = \frac{p}{\gamma p_0} \xi \cdot \vec{v} p_0 .
\]

Since \( p \) depends linearly on \( \xi \) through Eq. (IV.47), and \( p_0 \) is independent of \( t \), we may again write

\[
\frac{p}{\gamma p_0} \vec{\nu} \cdot \vec{v} p_0 = \frac{\partial}{\partial t} \left( \frac{1}{2} \frac{p}{\gamma p_0} \xi \cdot \vec{v} p_0 \right) .
\]

We may now substitute Eqs. (IV.48) and (IV.49) into Eq. (IV.44) to obtain

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 \vec{v} \cdot \vec{v} + \frac{1}{2} \mu_0 H_0^2 + \frac{1}{2} \frac{p}{\gamma p_0} + \frac{1}{2} \vec{J}_0 \cdot \vec{B} + \frac{1}{2} \frac{p}{\gamma p_0} \xi \cdot \vec{v} p_0 \right) + \vec{\nu} \cdot (\vec{v} \times \vec{B} + p \vec{v}) = 0 .
\]

This is the generalized Poynting theorem for linearized ideal MHD. The term inside the first set of brackets is the total energy density of the perturbation per unit volume, and the second set of brackets represents the energy flux out of this unit volume. The energy flux terms will be easily recognized as the Poynting vector and the convection of the plasma energy by the perturbed plasma motion.

Equation (IV.50) is not yet in its most familiar form. Let us combine the pair of terms containing the perturbed pressure

\[
\frac{1}{2} \frac{p}{\gamma p_0} + \frac{1}{2} \frac{p}{\gamma p_0} \xi \cdot \vec{v} p_0 = \frac{1}{2} \frac{p}{\gamma p_0} (p + \xi \cdot \vec{v} p_0) .
\]
Substituting for $p$ from Eq. (IV.47), we obtain

$$
\frac{1}{2} \frac{\rho_0^2}{\gamma p_0} + \frac{1}{2} \frac{\rho_0}{\gamma p_0} \xi \cdot \dot{\xi} = \frac{1}{2} \xi \cdot \ddot{\xi} p_0 \dot{\xi} + \frac{1}{2} \gamma p_0 (\dot{\xi} \cdot \dot{\xi})^2 .
$$

(IV.51)

Substituting Eq. (IV.51) into (IV.50), we obtain the final form for the small signal energy conservation equation:

$$
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho_0 \ddot{V} \cdot \ddot{V} + \frac{1}{2} \mu_0 \ddot{H} \cdot \ddot{H} + \frac{1}{2} \gamma p_0 (\ddot{\xi} \cdot \ddot{\xi})^2 + \frac{1}{2} \ddot{\xi} \cdot \ddot{p}_0 \ddot{\xi} \cdot \ddot{\xi} \right] + \ddot{\chi} \cdot (\ddot{E} \times \ddot{H} + \ddot{p}_0 \ddot{V}) = 0 .
$$

(IV.52)

We now integrate this equation over the whole plasma to obtain

$$
\frac{\partial}{\partial t} \int \left[ \frac{1}{2} \rho_0 \ddot{V} \cdot \ddot{V} + \frac{1}{2} \mu_0 \ddot{H} \cdot \ddot{H} + \frac{1}{2} \gamma p_0 (\ddot{\xi} \cdot \ddot{\xi})^2 + \frac{1}{2} \ddot{\xi} \cdot \ddot{p}_0 \ddot{\xi} \cdot \ddot{\xi} \right] dV + \int (\ddot{E} \times \ddot{H} + \ddot{p}_0 \ddot{V}) \cdot dS = 0 .
$$

Since we have assumed a perfectly conducting rigid wall in contact with the plasma, the boundary conditions ensure that the surface integral vanishes. We are then left with the result that

$$
\frac{\partial}{\partial t} (K + \delta W) = 0 ,
$$

(IV.53)

where $K \equiv \int \frac{1}{2} \rho_0 \ddot{V} \cdot \ddot{V} dV$ is the total kinetic energy of the plasma, and $\delta W$ is immediately identified as the potential energy of the perturbation and is given by

$$
\delta W = \frac{1}{2} \int \left\{ \mu_0 \ddot{H} \cdot \ddot{H} + \gamma p_0 (\ddot{\xi} \cdot \ddot{\xi})^2 + \ddot{\chi} \cdot \ddot{p}_0 \ddot{\xi} \cdot \ddot{\xi} \right\} dV .
$$

(IV.54)

The form of $\delta W$ given in Eq. (IV.54) is in agreement with that first given by Bernstein et al.

At this stage of the analysis we can already draw a number of conclusions with the aid of the equation for the conservation of energy. Since we assumed zero equilibrium flow, it follows that $K$ is always positive definite. Taking as a definition of instability an unbounded increase of $K$ in time, it is clear from Eq. (IV.53) that if $\delta W > 0$ the system must be absolutely stable. It is also clear from Eq. (IV.53) that, in order to have instability, $\delta W < 0$ such that $|\delta W|$ grows in time so as to balance exactly the increase in $K$. To complete the proof of the energy principle, we now follow the argument given by Laval et al. in order to show that $\delta W < 0$ always results in a growing solution.

Equation (IV.51) can be written as

$$
\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi) ,
$$

(IV.55)

where

$$
F(\xi) = \ddot{V}(\xi \cdot \ddot{\xi} p_0) + \ddot{V}(\gamma p_0 \ddot{\xi} \cdot \ddot{\xi}) + (\ddot{V} \times \ddot{B}) \times \ddot{B}_0 + \ddot{\xi} \times \ddot{P} .
$$

(IV.56)
Laval et al. make use of the virial \( I(\vec{\xi}) \) defined by

\[
I(\vec{\xi}) = \frac{1}{2} \int \rho_0 \vec{\xi} \cdot \vec{\xi} \, dV .
\]  

(IV.57)

Differentiating \( I \) with respect to time, we find

\[
\ddot{I} = \int (\rho_0 \ddot{\vec{\xi}} \cdot \vec{\xi} + \rho_0 \dot{\vec{\xi}} \cdot \ddot{\vec{\xi}}) \, dV .
\]  

(IV.58)

The first term on the right-hand side is twice the kinetic energy, and the second term can be written in terms of \( \vec{F}(\vec{\xi}) \) with the aid of Eq. (IV.55), giving

\[
\ddot{I} = 2K + \int \vec{\xi} \cdot \vec{F}(\vec{\xi}) \, dV .
\]  

(IV.59)

With the aid of some vector algebra, it is straightforward to show that (see Kulsrud)

\[
\int \vec{\xi} \cdot \vec{F}(\vec{\xi}) \, dV = -2\delta W ,
\]  

(IV.60)

where \( \delta W \) is the potential energy defined in Eq. (IV.54). The final form of Eq. (IV.59) can now be written

\[
\ddot{I} = 2K - 2\delta W .
\]  

(IV.61)

We now assume that there is some displacement \( \vec{\eta} \) such that \( \delta W(\vec{\eta}) < 0 \). We write this explicitly as

\[
\delta W(\vec{\eta}) = -\omega^2 I(\vec{\eta}) ,
\]  

(IV.62)

where \( \omega > 0 \), and \( I(\vec{\eta}) \) is, of course, positive definite. The displacement vector \( \vec{\xi} \), which is a solution of Eq. (IV.55), is now chosen to satisfy the following initial conditions:

\[
\vec{\xi} = \vec{\eta} , \quad \dot{\vec{\xi}} = \omega \vec{\eta} .
\]  

(IV.63)

Since the total energy \( E \) of the perturbation must be a constant, we may calculate its value with the aid of Eqs. (IV.63). Thus

\[
E = K + \delta W
\]

\[
= I(\vec{\xi}) + \delta W(\vec{\xi})
\]

\[
= \omega^2 I(\vec{\eta}) + \delta W(\vec{\eta}) ,
\]

so that \( E = 0 \). Using this result we may eliminate \( \delta W \) from Eq. (IV.61) to obtain

\[
\ddot{I} = 4K .
\]  

(IV.64)
Now

\[ \dot{i}^2 = \left( \int_{\dot{0} \xi}^{\hat{\eta}} \xi^2 \, d\nu \right)^2 \]
\[ = \left[ \int \frac{1}{2} \dot{\eta} \cdot \dot{\xi} \, d\nu \right]^2. \]

Using Schwarz's inequality, we may write:

\[ \left[ \int \frac{1}{2} \dot{\eta} \cdot \dot{\xi} \, d\nu \right]^2 \leq \left( \int \dot{0} \dot{0} \xi \cdot \dot{\xi} \, d\nu \right) \left( \int \dot{0} \dot{0} \xi \cdot \dot{\xi} \, d\nu \right), \]

and therefore \( \dot{i}^2 \leq 4IK. \) With the aid of Eq. (IV.64), the inequality becomes

\[ \dot{i}^2 \leq II. \]  

(IV.65)

Now \( \ddot{I} > 0, \) and integrating Eq. (IV.64) we obtain \( \dot{I} = \int 4Kdt + \text{const.} \) We may evaluate the constant with the aid of the initial conditions. Thus

\[ \dot{i}_0 = \int \dot{\rho}_0 \xi \cdot \dot{\xi} \, d\nu = \omega \int \dot{\rho}_0 \hat{\eta} \cdot \hat{\eta} \, d\nu \]
\[ = 2\omega I_0 > 0, \]  

(IV.66)

where the subscript 0 indicates that the quantity has been evaluated at \( t = 0. \) We therefore find that \( \text{const} > 0 \) for \( t > 0, \) so that \( \dot{I} > 0 \) for \( t > 0. \) Since \( II > 0 \) for \( t > 0 \) we do not alter the sense of inequality (IV.65) by dividing throughout by this quantity. We then have

\[ \frac{\ddot{I}}{\ddot{I}} \leq \frac{\ddot{I}}{\ddot{I}}. \]  

(IV.67)

Integrating for \( t > 0 \) we have

\[ \ln \frac{I}{I_0} \leq \ln \frac{\dot{I}}{\dot{I}_0}. \]  

(IV.68)

Using inequality (IV.66) we obtain

\[ \frac{\dot{I}}{\dot{I}} \geq 2\omega. \]  

(IV.69)

Again integrating for \( t > 0 \) we have

\[ I \geq I_0 e^{2\omega t}. \]

Thus \( I \) grows at least as fast as \( \exp (2\omega t), \) and since \( I \) is quadratic in \( \hat{\xi} \) it follows that the displacement \( \hat{\xi} \) will grow at least as fast as \( \exp (\omega t). \) This then completes the proof that for any displacement which makes \( \delta W < 0 \) instability will always occur.
IV.4 FEEDBACK STABILIZATION

The occurrence of instabilities in magnetic confinement devices suggests the application of the method of feedback control in order to stabilize the system. In fact the basic equilibrium of the plasma position in the toroidal discharge of a Tokamak is now controlled by a feedback circuit through the influence of the vertical field coils.

However, in this lecture we shall consider the effect of feedback on electrostatic instabilities which can occur in plasmas. Such instabilities are often short wavelength ones and may result in an enhanced loss rate from the confinement device.

The problem of feedback control of plasma instabilities is somewhat different from conventional control theory. In the first place the system to be stabilized is open-loop unstable, and in the second place it is an essentially distributed system so that we must deal not just with differential equations in time but with partial differential equations in time and space.

We shall represent the effect of a feedback on the system by a function \( g(\omega, k) \), which is the Fourier transform in time and space of the suppressing signal fed back to the plasma. The function \( g(\omega, k) \) gives the response of the feedback circuit to the sensing signal and must therefore obey a causality condition, i.e. Kramers-Krönig conditions such that \( g(-\omega, k) = \gamma^* g(\omega, k) \), and \( \gamma \) real positive constant as \( \omega \rightarrow \infty \). The amplitude of the feedback signal is \( |g| \) and \( \arg g(\omega, k) \) is its phase.

We can now write the dispersion equation for electrostatic oscillations of the plasma as

\[
\varepsilon(\omega, k) + g(\omega, k) = 0 .
\]  

The method to be discussed can also be applied to electromagnetic perturbations. Here, \( \varepsilon(\omega, k) \) is the longitudinal part of the dielectric function.

In order to discuss the stability properties of Eq. (IV.70) we must distinguish two classes of instability. The first class of instability occurs when a natural mode of oscillation of the plasma is driven unstable by a flow of power between the oscillation and the plasma. The power absorption in a plasma is, of course, given by the anti-Hermitian part of the dielectric function. We therefore write Eq. (IV.70) as

\[
\varepsilon_h(\omega, k) + i \varepsilon_a(\omega, k) + g(\omega, k) = 0 .
\]  

These instabilities are usually characterized by \( |\varepsilon_a|/|\varepsilon_h| \ll 1 \) and therefore \( \gamma/\omega \ll 1 \) where the subscripts 'h' and 'a' denote Hermitian and anti-Hermitian, and \( \gamma \) and \( \omega \) the imaginary and real parts of the frequency.

We can now obtain a perturbation solution of (IV.71) by assuming that both \( \varepsilon_a \) and \( g \) are small compared with \( \varepsilon_h \) so that the real part of the frequency, say \( \omega_0 \), is determined by \( \varepsilon_h \). Thus

\[
\varepsilon_h(\omega_0, k_0) = 0 .
\]  

Now expanding about \( \omega_0 \) and assuming \( \omega = \omega_0 + \delta \omega \), we find

\[
\varepsilon_h(\omega_0, k_0) + \delta \omega \frac{\partial \varepsilon_h}{\partial \omega} \bigg|_{\omega_0, k_0} + i \varepsilon_a(\omega_0, k_0) + g(\omega_0, k_0) = 0 .
\]  

(IV.73)
Using (IV.72), the solution of Eq. (IV.73) can be written

\[ \delta \omega = \frac{-|g| \cos \theta}{(\frac{\partial \epsilon_h}{\partial \omega})_{\omega_0, k_0}} \]  

\[ \gamma = \gamma_0 - \frac{|g| \sin \theta}{(\frac{\partial \epsilon_h}{\partial \omega})_{\omega_0, k_0}} \]  

where \( \gamma_0 \equiv -\frac{e_e}{(\frac{\partial \epsilon_h}{\partial \omega})_{\omega_0, k_0}} \). Equation (IV.74) gives the correction to the frequency but the important information is contained in Eq. (IV.75). This shows that the instability will be stabilized provided

\[ \frac{|g| \sin \theta}{(\frac{\partial \epsilon_h}{\partial \omega})_{\omega_0, k_0}} > \gamma_0 , \]

Clearly, there is an optimum phase for stabilization, and stabilization can be achieved within a range of phase angles equal to \( \pi \).

Finally, let us consider the second class of plasma instabilities. These instabilities are known as reactive instabilities and do not involve any power flow to the plasma. The anti-Hermitian part of the dielectric coefficient may therefore be ignored. For these cases, two natural oscillations of the plasma become degenerate in their frequency at the threshold condition corresponding to a coupling between oscillations whose energies are of opposite sign. The instability takes place through an interchange of energy between the two oscillations thus causing both to grow. Mathematically, the threshold condition corresponds to the occurrence of a double root of the dispersion equation, so that

\[ \frac{\partial \epsilon_h}{\partial \omega} = 0 . \]  

(IV.76)

Let us suppose that some parameter \( a \) (e.g. the current density) determines the threshold. If \( a_0 \) is the value of this parameter at threshold, we may assume \( \omega = \omega_0 + \delta \omega \) as before and expand the dielectric function about the threshold (where the frequency is \( \omega_0 \)), giving

\[ \epsilon_h(\omega_0, k_0, a_0) + \frac{\partial \epsilon_h}{\partial \omega} \bigg|_{\omega_0, k_0, a_0} \delta \omega + \frac{\partial^2 \epsilon_h}{\partial \omega^2} \bigg|_{\omega_0, k_0, a_0} \frac{(\delta \omega)^2}{2} + \]

\[ + \frac{\partial \epsilon_h}{\partial a} \bigg|_{\omega_0, k_0, a_0} \delta a + g(\omega_0, k_0) \approx 0 . \]  

(IV.77)

As before, the zero-order solution is given by

\[ \epsilon_h(\omega_0, k_0, a_0) = 0 . \]

However, we also have the threshold condition given by Eq. (IV.76). The solution of Eq. (IV.77) can therefore be written,

\[ (\delta \omega)^2 = -\gamma_0^2 + |g| \cos \theta + i |g| \sin \theta , \]  

(IV.78)
where the growth rate in the absence of feedback is given by

\[ \gamma_0 = \left[ \frac{2(3\epsilon_h / 3a) \delta a}{(\delta^2 c_h / 3\omega^2)} \right]^{\frac{1}{2}}. \]  

(IV.79)

It can be seen from Eq. (IV.78) that the requirement for stabilization of reactive instabilities is much more stringent, since only a single value of the phase \( \theta \) (i.e. zero) will achieve this.
CHAPTER V

NON-LINEAR THEORY

V.1 QUASI-LINEAR THEORY

Let us now consider some of the attempts to understand the non-linear behaviour of waves and instabilities in plasmas. We shall again focus attention on a resonant particle instability described by Landau's procedure. We have seen that the driving mechanism for such an instability comes from a small group of resonant particles travelling close to the phase velocity of the wave. It is natural, therefore, to ask whether some non-linear modification of the equilibrium distribution function can be produced which would switch off the instability. This was the idea of Vedenov et al., and of Drummond and Pines when they proposed the quasi-linear theory. We shall follow an approach given by Fukai and Harris.

As for the linear problem, we separate the distribution function into an equilibrium part and a fluctuating part:

\[ f(\hat{x},\hat{v},t) = f_0(\hat{v},t) + f_1(\hat{x},\hat{v},t) \]  \hspace{1cm} (V.1)

Notice that we allow the equilibrium distribution to be a function of time. This is the required slow time dependence which is expected to result from the reaction of the instability back on the background. We assume that the system is in a box so that we can express the fluctuations as a sum:

\[ f_1(\hat{x},\hat{v},t) = \sum_{\hat{k} \neq 0} f_{\hat{k}}(\hat{v},t) \exp (i\hat{k} \cdot \hat{x}) \]  \hspace{1cm} (V.2)

\[ \tilde{E}(\hat{x},t) = \sum_{\hat{k} \neq 0} \tilde{E}_{\hat{k}}(t) \exp (i\hat{k} \cdot \hat{x}) \]  \hspace{1cm} (V.3)

where \( \tilde{E} \) is assumed to be a longitudinal (or electrostatic) electric field. We also assume that the ions form only a neutralizing background of positive charge and do not take part in the perturbed motion. The theory to be presented is intended to apply to the type of electron distribution function (equilibrium) shown in Fig. V.1, often referred to as the bump-on-tail distribution. We can see from Penrose that this is likely to be unstable. Since we shall retain certain non-linear terms, we must be careful to ensure that we deal with real fields. The requirement that \( f \) and \( \tilde{E} \) are real gives
\[ f_k = f_k^* \] \hfill (V.4)
\[ \hat{f}_k = \hat{f}_k^* \] \hfill (V.5)

and we note also that since \( \hat{p} \) is longitudinal,
\[ \hat{p}_k = \frac{\hat{k}}{k} E_k \] \hfill (V.6)

We must now substitute the expansions given by Eq. (V.2) and (V.3) into Vlasov's equation (I.21)
\[ \frac{\partial f}{\partial t} + \hat{v} \cdot \frac{\partial f}{\partial \hat{v}} = \frac{e}{m_e} \frac{\partial \hat{p}}{\partial \hat{v}}, \] \hfill (V.7)

This gives
\[ \frac{\partial f_k}{\partial t} + i \hat{k} \cdot \hat{v} f_k = \frac{e}{m_e} \hat{p}_k \cdot \frac{\partial f_k}{\partial \hat{v}} + \frac{e}{m_e} \sum_{k'} \hat{p}_{k'} \cdot \frac{\partial f_{k'}}{\partial \hat{v}} \] \hfill (V.8)
\[ \frac{\partial f_k}{\partial t} = \frac{e}{m_e} \sum_{\hat{k} \neq 0} \hat{p}_{-\hat{k}} \cdot \frac{\partial f_{\hat{k}}}{\partial \hat{v}}. \] \hfill (V.9)

The non-linear term in Eq. (V.8) represents the effect of mode coupling between all the fluctuations present in the system, whereas the non-linear term in Eq. (V.9) describes the effect of all these fluctuations on the background distribution. The simplest non-linear step that one can take is to treat the fluctuations linearly and therefore neglect the effect of mode coupling. The only non-linear effect left is the reaction of the fluctuations on the distribution function. This is the QUASI-LINEAR approximation. Let us now work out the consequences.

Assume that for a sufficiently short time interval we may represent the time behaviour of \( \hat{E}_k(t) \) as follows:
\[ \hat{E}_k(t) = \hat{E}_k(t_0) \exp \left[ \left( i \omega_k - i \Delta k \right) (t - t_0) \right], \] \hfill (V.10)
where \( \gamma_K \) and \( \omega_K \) are assumed to be constant over this time interval. Note that \( \gamma_K \) can be either positive (growth) or negative (damping). Substituting Eq. (V.10) into Eq. (V.8) (neglecting the non-linear term) we can then integrate Eq. (V.8) over the time interval \( \tau = t - t_0 \) to obtain

\[
f_K(V, t) = f_K(V, t - \tau) \exp \left(-iK \cdot V \tau\right) + \frac{e}{m_e} \frac{\partial f_o}{\partial V} \frac{1 - \exp \left(-\frac{\gamma_K + i(K \cdot V - \omega_K)}{K}ight)}{\gamma_K + i(K \cdot V - \omega_K)} \, (V.11)
\]

In performing this integration we have neglected the time dependence of \( f_o \) since, from Eq. (V.9) this time dependence can be seen to be of second order in small quantities \( f_K \) and \( \frac{\partial f_o}{\partial V} \).

We now substitute this solution for \( f_K \) into the equation for \( f_o \) [Eq. (V.9)] giving

\[
\frac{\partial f_o}{\partial t} = \frac{\partial}{\partial V} \left[ \frac{\partial}{\partial V} f_o^0 + \frac{\partial f_o}{\partial V} \right] \sum_{K \neq 0} \frac{\partial f_K}{\partial V}(t) f_K(t - \tau) e^{-iK \cdot V \tau}, \quad (V.12)
\]

where we have combined the \( K \) and \( -K \) terms and used Eqs. (V.4) and (V.5) and the fact that \( \omega_{-K} = -\omega_K \) and \( \gamma_{-K} = \gamma_K \). The diffusion tensor \( \frac{\partial}{\partial V} \) is given by

\[
\frac{\partial}{\partial V} \frac{\partial}{\partial V} = \frac{n}{m_e} \sum_{K \neq 0} \frac{\partial f_K}{\partial V}(t) \left| \frac{\partial f_K}{\partial V} \right|^2 (K \cdot V - \omega_K, \tau), \quad (V.13)
\]

where

\[
\delta(x, \tau) \equiv \frac{1}{\pi} \frac{\gamma_K - \gamma_K \exp \left(-\gamma_K \tau\right) \cos x \tau \cos x + x \exp \left(-\gamma_K \tau\right) \sin x \tau}{\gamma_K^2 + x^2}, \quad (V.14)
\]

where

\[ x = K \cdot V - \omega_K. \]

The final step in the argument is to assume that we can choose a time interval \( \tau \) which is sufficiently short that \( \gamma_K, \omega_K \), and \( f_o \) do not change appreciably, and yet sufficiently long that the function \( \exp \left(-iK \cdot V \tau\right) \) performs rapid oscillations so that the positive and negative contributions in the sums of \( K \) cancel each other in the second term on the right-hand side of Eq. (V.12). This means that we keep only the response of the perturbed distribution function to the driving electric field \( E_K \), i.e. the electric field resulting from the collective motion of the electrons. The neglected part of the distribution is the part which was responsible for the preservation of the information in the Landau damping problem.

Evidently it is at this point in the approximations that the physical content of the theory undergoes a qualitative change. Equation (V.12) can now be written

\[
\frac{\partial f_o}{\partial t} = \frac{\partial}{\partial V} \left[ \frac{\partial}{\partial V} f_o^0 + \frac{\partial f_o}{\partial V} \right] \quad (V.15)
\]
and is the required quasi-linear diffusion equation governing the slow evolution of the background distribution function.

The form of \( \hat{\mathcal{D}}(\mathbf{v}) \) given by Eq. (V.13) is not the usual one. The effect of this diffusion tensor can be appreciated by considering the behaviour of the function \( \Delta_{K}(x,\tau) \) and by letting the volume of the box become infinite. The summation over \( \mathbf{K} \) then becomes an integral in the usual way. The function \( \Delta_{K}(x,\tau) \) now always occurs under an integral sign, and we can see that for \( \tau \) large enough such that \( |\mathbf{K} \cdot \mathbf{v}_{t}| \gg 1, |\omega_{K}| \gg 1 \), and yet small enough so that \( |\gamma_{K}\tau| \ll 1 \), then \( \Delta_{K}(x,\tau) \) behaves like a \( \delta \)-function, i.e. it is negligible except in the vicinity of \( x = \mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K} = 0 \). Thus the area under the function is easily shown to be

\[
\int_{-\infty}^{\infty} dx \Delta_{K}(x,\tau) = \left[ \frac{\gamma}{|\gamma|} - \frac{\gamma}{|\gamma|} \exp \left\{ -(\gamma + |\gamma|)\tau \right\} + \exp \left\{ -(\gamma + |\gamma|)\tau \right\} \right] = 1 \text{ for } \gamma \lesssim 0. \quad (V.16)
\]

We may therefore replace \( \hat{\mathcal{D}}(\mathbf{v}) \) (V.13) by

\[
\overset{\ast}{\hat{\mathcal{D}}}(\mathbf{v}) = \frac{\pi e^{2}}{m_{e}^{2}} \int \frac{d^{3}K}{(2\pi)^{3}} k^{2} \left| \hat{E}_{K}(t) \right|^{2} \delta(\mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K}). \quad (V.17)
\]

To complete the picture, we make \( \hat{E}_{K}(t) \) self-consistent by substituting

\[
f_{K}(\mathbf{v},t) = \frac{e}{m_{e}} \frac{\partial f_{0}}{\partial \mathbf{v}} \hat{E}_{K}(t) \left[ 1 - \exp \left\{ -\frac{[\gamma_{K} + i(\mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K})]t}{\gamma_{K} + i(\mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K})} \right\} \right]
\]

into Poisson's equation

\[
\mathbf{\nabla} \cdot \hat{E}_{K}(t) = -\frac{e}{\varepsilon_{0}} \int d^{3}v f_{K}(\mathbf{v},t) \quad (V.18)
\]

which gives the dispersion relation

\[
e(\mathbf{K},\omega_{K} + i\gamma_{K}) = 0 \quad (V.19)
\]

where of course

\[
e(\mathbf{K},\omega_{K} + i\gamma_{K}) = 1 + \frac{e^{2}}{i\varepsilon_{0} m_{e} k^{2}} \int d^{3}v \mathbf{K} \cdot \frac{\partial f_{0}}{\partial \mathbf{v}} \left[ 1 - \exp \left\{ -\frac{[\gamma_{K} + i(\mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K})]t}{\gamma_{K} + i(\mathbf{K} \cdot \mathbf{v}_{t} - \omega_{K})} \right\} \right] \quad (V.20)
\]

We can write \( e = e_{1} + ie_{2} \) and using our previous assumption that \( |\gamma_{K}| \ll |\omega_{K}| \) we can solve \( e(\mathbf{K},\omega_{K} + i\gamma_{K}) = 0 \) for \( \omega_{K} \) and \( \gamma_{K} \) perturbatively. \( \omega_{c} \) is the solution of

\[
e_{1}(\mathbf{K},\omega_{K}) = 0 \quad (V.21)
\]
and \( \gamma_k \) is given by

\[
\gamma_k = -\varepsilon_2(k, \omega_k) / \partial \varepsilon_1 / \partial \omega_k
\]  \hspace{1cm} (V.22)

In the limit \( \tau = 0, \gamma \to 0 \) and \( \gamma \tau \to 0 \), \( \varepsilon_1 \) and \( \varepsilon_2 \) reduce to the usual Landau expressions.

The final equation needed to close this quasi-linear solution is the linear relation for the evolution of the electric field intensity

\[
\frac{\partial}{\partial t} |E_k(t)|^2 = 2\gamma_k |E_k(t)|^2 .
\]  \hspace{1cm} (V.23)

The self-consistent solution of Eqs. (V.15), (V.17), (V.20) to (V.23) gives the quasi-linear evolution of the system to a stationary (or saturated) state.

The (heuristic) derivation presented here requires a continuous spectrum of modes to be involved in the quasi-linear modification process in order for the replacement of the sum over \( k \) by an integral to be valid. Clearly, the mechanism would not apply to a discrete mode.

Finally, there are two other points which should be mentioned. The quasi-linear equation for the distribution function conserves wave energy and momentum. However, this equation, in contrast to Vlasov's equation, is no longer time reversible. Irreversibility was introduced both by the neglect of the homogeneous part of \( f_k \) and by the replacement of \( \delta_k(x) \) by \( \delta(x) \). Both of these approximations depend on the presence of a continuous spectrum of modes being present.

V.2 THREE-WAVE INTERACTIONS

In the quasi-linear theory of the last section a fundamental assumption was that the effect of mode coupling between all the fluctuations present in the system could be ignored. This involved the neglect of the second term on the right-hand side of Eq. (V.8), resulting in the linearization of this equation. The ultimate justification for such a procedure is that the time for the fluctuations to produce a change in the equilibrium must be shorter than the time scale for a redistribution of energy amongst the fluctuations.

In this section, we shall consider the opposite limit, in which the dominant effect is the non-linear interaction between the fluctuations, and the reaction of the fluctuations on the equilibrium is ignored. This procedure can be justified for small-amplitude fluctuations but where the frequencies and wave numbers satisfy certain resonant conditions which maximize the interaction. We shall confine ourselves to the simplest case of all -- namely a three-wave interaction in which the waves are coherent. The example we choose to analyse is that of stimulated Raman scattering in which an incident light wave (of finite amplitude) interacts with another light wave (propagating in the opposite direction) and a Langmuir wave. We shall obtain the equations describing this interaction and shall show that for an incident intensity above a definite threshold value the interaction is unstable, i.e. the light-wave propagating in the reverse direction and the Langmuir wave will be excited out of the background noise of the plasma.
Stimulated Raman scattering has been chosen as an illustration because the interaction can occur in one dimension and involves only the electrons. Three-wave interactions also include parametric instabilities which are important in plasma physics and many other fields. In parametric instability theory, the incident (or initial) finite amplitude wave (the source of the energy) is usually referred to as the pump wave. Let us now formulate the problem of stimulated Raman scattering.

We shall consider the idealized model of an infinite, uniform plasma in which the plasma electrons are described as a warm fluid. The equations are the following:

\[
\frac{d\nabla e}{dt} + \gamma e \frac{kT_e}{m_e} \nabla T_e + \nabla \cdot \mathbf{v}_e \mathbf{e} = -\frac{e}{m_e} (\mathbf{E} + \mathbf{v}_e \mathbf{E}) \tag{V.24}
\]

\[
\frac{3n_e}{\delta t} + \mathbf{v} \cdot (n_e \mathbf{v}_e) = 0 \tag{V.25}
\]

\[
\mathbf{v} \times \mathbf{E} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\delta \mathbf{E}}{\delta t} \tag{V.26}
\]

\[
\mathbf{v} \times \mathbf{B} = -\frac{\delta \mathbf{E}}{\delta t}, \tag{V.27}
\]

where

\[
\frac{D}{\delta t} = \frac{\delta}{\delta t} + \mathbf{v} \cdot \nabla \mathbf{v},
\]

and

\[
\mathbf{j} = -n_e \mathbf{e} \mathbf{v}_e.
\]

Here \(\gamma_e\), \(k\), \(\mu_0\), and \(c\) are the ratio of specific heats for the electron fluid, Boltzmann's constant, the magnetic permeability of free space, and the velocity of light in vacuum respectively; \(T_e, n_e, \mathbf{v}_e,\) and \(e\) are the electron temperature (we use an isothermal model), the electron mass, the electron collision frequency, and the proton charge. The quantity \(\mathbf{v}_e\) may be viewed simply as the electron-ion collision frequency, or it may be interpreted as a term which simulates the effect of both electron Landau damping as well as collisional damping; \(\mathbf{v}_e, n_e, \mathbf{E},\) and \(\mathbf{B}\) represent the electron fluid velocity and density, and the electric and magnetic fields, respectively. The energy density of an electromagnetic wave in a plasma is given by \(\frac{1}{2} \varepsilon_0 |\mathbf{E}|^2\), where \(\varepsilon_0\) is the dielectric constant of a vacuum and \(\mathbf{E}\) is the electric field of the wave. The energy density of the plasma is \(n_e kT_e\). For all the instabilities with which we shall be concerned, the ratio of these two quantities \(\varepsilon_0 |\mathbf{E}|^2/2n_e kT_e\) is a small quantity for the threshold amplitude of the pump wave. This is a very fortunate circumstance since it enables us to calculate the quantities of interest using a perturbation analysis.
We assume that there are no steady electric fields present, and since we are considering an unmagnetized plasma the only magnetic fields present are the oscillating ones due to whatever electromagnetic waves are assumed. We now write Eqs. (V.3) to (V.6) in the form

\[
\frac{\partial \vec{v} e_1}{\partial t} + \gamma_e \frac{k_T e_1}{n_e m_e} \vec{v} n e_1 + v_e e_1 \vec{E}_1 + \frac{e}{m_e} \vec{v} e_1 \times \vec{B}_1 - (\vec{v} e_1 \cdot \vec{v}) \vec{v} e_1 + \gamma_e \frac{k_T e_n}{n_e n_1} \vec{v} e_1 = 0
\]

\[\tag{V.28}
\]

\[
\frac{\partial n e_1}{\partial t} + n_e \vec{v} \cdot \vec{v} e_1 = -\vec{v} \cdot (n_e \vec{v} e_1)
\]

\[\tag{V.29}
\]

\[
\vec{v} \times \vec{B}_1 - \frac{1}{c^2} \frac{\partial \vec{E}_1}{\partial t} + e v_n n_0 \vec{v} e_1 = -e v_n n_1 \vec{v} e_1
\]

\[\tag{V.30}
\]

\[
\vec{v} \times \vec{E}_1 + \frac{\partial \vec{B}_1}{\partial t} = 0
\]

\[\tag{V.31}
\]

where \( n_0 \) is the equilibrium electron density (in the absence of all wave fields) and the subscript 1 denotes a wave field. The non-linear terms on the right-hand sides of the above equations are now treated as small corrections which will provide coupling between the waves, which in the linear theory are independent of one another.

In order to describe the process

\[ T = T' + L, \]

[where \( T, T' \), and \( L \) represent the incident (pump), scattered (electromagnetic), and Langmuir waves], we choose a linearly polarized pump wave as follows:

\[
\vec{E}_{T0} = (0, E_{T0}, 0)
\]

\[
\vec{k}_{T0} = (0, 0, k_{T0})
\]

\[
\vec{p}_{T0} = (k_{T0}, 0, 0)
\]

and the frequency of the pump wave \( \omega_{T0} \) is given by

\[
\omega_{T0}^2 = \frac{e^2}{m_e} c^2 k_{T0}^2
\]

The pump wave is therefore taken as a travelling electromagnetic wave varying as

\[
\exp(i(k_{T0}x - \omega_{T0}t))
\]
with electric and magnetic fields as given above and an associated electron fluid velocity \( \mathbf{v}_{T0} = (0, v_{T0x}, 0) \), where \( v_{T0x} \) is given by the linearized equations and is

\[
v_{T0x} = \frac{eE_{T0x}}{i\omega_{T0}m_e} .
\] (V.32)

The velocity of the electrons in the field of the pump wave is often referred to as the "quiver velocity".

We must now obtain equations for the Langmuir and transverse electromagnetic wave perturbations. These perturbations will be coupled together because of the presence of the pump wave. The polarization of the transverse wave will be taken to be the same as the pump wave, and since we restrict the analysis to one dimension, the Langmuir wave propagates in the \( x \)-direction with its electric field vector also along this axis.

First, let us obtain the equation for the transverse perturbation. Taking the \( y \)-components of Eqs. (V.28) and (V.30) and the \( z \)-component of Eq. (V.31), and assuming the perturbation varies as \( \exp i(kx - \omega t) \), we obtain

\[
(\omega^2 - \omega_p^2 - c^2k^2)E_{1y} = -i\nu e \frac{\omega_p^2}{\omega} E_{1y} - \omega_p^2 \nu \frac{v_{1x}}{\varepsilon_0} \frac{\partial v_{1y}}{\partial x} + \frac{e\nu}{\varepsilon_0} \frac{\partial v_{1y}}{\partial x} + i \frac{\omega}{\varepsilon_0} \varepsilon_1 v_{1y} .
\] (V.33)

The first term on the right-hand side is a linear damping term, and the remaining terms are due to the non-linear coupling. We shall treat the damping and coupling terms as small perturbations to the linear waves whose dispersion relation, in the absence of the pump wave, is given by the left-hand side of the equation. We shall now denote the electric field of the transverse wave (the scattered light) by \( E_{1y} = E_{T1} \) to distinguish it from the incident electric field, \( E_{T0} \). The dependence of \( E_{T1} \) will be written as

\[
E_{T1}(x,t) = \text{Re} \left\{ \mathcal{E}_{T1}(t) \exp \left[ i(k_{T1}x - \omega_{T1}t) \right] \right\} ,
\] (V.34)

where

\[
\omega_{T1}^2 = \omega_p^2 + c^2k_{T1}^2 ,
\]

and the amplitude \( \mathcal{E}_{T1}(t) \) is assumed to vary slowly in time in comparison with the rapidly varying linear phase. Notice that we must now be careful to take the real parts of all complex amplitudes since we now have products of such quantities.

We shall also express all transverse quantities in terms of the electric field. Since we are performing a perturbation analysis, we can relate \( \mathbf{B}_T \) and \( \mathbf{v}_T \) to \( E_T \) by means of the linear equations. We then obtain

\[
\mathbf{B}_T = \frac{k_T}{\omega_T} E_T ,
\]

\[
\mathbf{v}_T = -\frac{ieE_T}{\omega_T m_e} .
\] (V.35)

\[
\mathbf{B}_T \quad \text{and} \quad \mathbf{v}_T \quad \text{are now written instead of} \quad \mathbf{B}_{1z} \quad \text{and} \quad v_{1y} , \quad \text{but they may represent either the incident or the scattered light. The coupling terms in Eq. (V.35) must consist of the product of a}
\]
transverse field and a Langmuir field. It is clear that the Langmuir fields are $v_{1x}$ and $n_{e_1}$. Since we shall also represent the Langmuir wave amplitude by its electric field, which we denote by $E_{1x} = E_L$, we express $v_{1x} = v_L$ and $n_{e_1} = n_L$ in terms of $E_L$, again by means of the linear equations

$$v_L = -\frac{i\omega_{Te}\alpha eL}{en_e}$$  \hspace{1cm} (V.37)

$$n_L = -\frac{ik_c^L \alpha E_L}{e}$$  \hspace{1cm} (V.38)

We can now obtain a non-linear differential equation for the scattered light wave by expanding Eq. (V.33) about the linear solution and identifying $\omega$ with $i\omega/\omega$:

$$\exp\left[i(k_{T1}x - \omega_{T1}t)\right] i2\omega_{T1} \frac{\partial \varepsilon_{T1}}{\partial t} = -iv_L \frac{\omega^2_P e}{\omega_{T1}} E_{T1} + \frac{i}{2} \frac{\omega_{T1} L_e^*}{\omega_{10}^m} E_{T0}^* E_{T0}.$$  \hspace{1cm} (V.39)

In order to obtain the final form of this equation, we put

$$E_{T0} = \text{Re} \left\{ \varepsilon_{T0} \exp\left[i(k_{T0}x - \omega_{T0}t)\right] \right\}$$

and express $E_L$ as the product of a slowly varying amplitude and the linear phase

$$E_L(x,t) = \text{Re} \left\{ \varepsilon_L(t) \exp\left[i(k_Lx - \omega_Lt)\right] \right\}.$$  \hspace{1cm} (V.40)

Finally we make explicit use of the frequency and wave-number matching conditions

$$\omega_{T0} = \omega_{T1} + \omega_L$$  \hspace{1cm} (V.41)

$$k_{T0} = k_{T1} + k_L$$  \hspace{1cm} (V.42)

in order to be able to eliminate the phase factors. The resulting equation for $\varepsilon_{T1}$ is

$$\frac{\partial \varepsilon_{T1}}{\partial t} + \frac{v_L}{2} \frac{\omega^2_P e}{\omega_{T1}^2} \varepsilon_{T1} = \frac{1}{4} \omega_{T0}^m \varepsilon_L^* \varepsilon_{T0}^* e^{-i\psi t},$$  \hspace{1cm} (V.43)

where we have imposed perfect $k$-matching but allowed for a small frequency mismatch by permitting $\psi = \omega_{T0} - \omega_{T1} - \omega_L$ to be small but possibly non-zero; $\psi = 0$ corresponds to perfect frequency matching.

The equation for $E_L$ is obtained in a similar way. Assuming an exp $i(kx - \omega t)$ dependence as before, and taking the $x$-component of Eqs. (V.28) and (V.50) together with Eq. (V.29), we obtain

$$(\omega^2 - \omega^2_P e - \gamma e k^2 v^2_e) E_L = -i v_L \omega E_L + \omega^2_P v \gamma B_{12}.$$  \hspace{1cm} (V.44)
The left-hand side gives the linear dispersion relation for a Langmuir wave, and the right-hand side consists of the linear damping and the non-linear coupling. The only coupling term in this case comes from the $\vec{v} \times \vec{B}$ force in the equation of motion, since this is the only term which consists of a product of two transverse fields. Expanding Eq. (V.44) about the linear solution, using Eqs. (V.35) and (V.36), and imposing the matching conditions, we obtain the equation for the amplitude $\varepsilon_L(t)$:

$$\frac{\partial \varepsilon_L}{\partial t} + \frac{v_e}{2} \varepsilon_L = \frac{1}{4} c_{\phi L} \frac{e}{m_e} \frac{k_L}{\omega_L} \varepsilon_T \varepsilon^*_{T1} e^{-i \omega t}.$$  (V.45)

Equations (V.43) and (V.45) are non-linear coupled equations for the slowly varying amplitudes $\varepsilon_{T1}$ and $\varepsilon_L$. These equations can be linearized by assuming that $\varepsilon_{T0} \gg \varepsilon_{T1}$ and $\varepsilon_{T0} \gg \varepsilon_{L1}$, so that the pump amplitude can be taken as constant. Using $\varepsilon_{T1}$ and $\varepsilon_L e^{-i \omega t}$ as the amplitudes, the coupled equations then have constant coefficients, and a solution proportional to exp ($-i \omega t$) can be assumed. The dispersion relation for the coupled waves is then

$$(\Omega + i \gamma_T) (\Omega + i \gamma_L - \psi) + c_{\phi L} c_{\phi_1} |\varepsilon_T|^2 = 0,$$  (V.46)

where

$$\gamma_L \equiv \frac{v_e}{2}, \quad \gamma_T \equiv \frac{v_e}{2} \frac{\omega e}{\omega T1}, \quad c_{\phi L} \equiv \frac{e}{m_e} \frac{k_L}{\omega L}, \quad \text{and} \quad c_{\phi_1} \equiv \frac{e}{m_e} \frac{\omega}{\omega T0}.$$  

Putting $c_{\phi L} c_{\phi_1} |\varepsilon_T|^2 \equiv K$, we obtain the instability threshold from Eq. (V.46):

$$K = \gamma_T \gamma_L + \frac{\psi^2 \gamma_T \gamma_L}{(\gamma_T + \gamma_L)^2}.$$  (V.47)

There are two contributions to the threshold field, the first from the natural damping rates of the excited waves, and the second due to the frequency mismatch. The minimum threshold clearly occurs for $\psi = 0$. For $\psi = 0$ and the pump well above threshold the growth rate of the instability is given by

$$\gamma = (c_{\phi L} c_{\phi_1})^{1/2} |\varepsilon_T|^2.$$  (V.48)

In terms of the plasma parameters, the minimum threshold can be written:

$$\frac{v_{e_T}^2}{v_{e_L}^2} = \frac{4}{k_L^2 \omega T1 \omega L},$$  (V.49)

where

$$\lambda^2_{Te} = v_{e_T}^2 / \omega_{e_T}^2.$$
Let us now derive some conservation relations for the waves taking part in the Raman scattering process. In order to do this, we need an equation for the pump wave which is no longer assumed to be a constant amplitude wave. The equation for \( \varepsilon_{T0}(t) \) can be obtained from Eq. (V.33) and is

\[
\frac{3 \varepsilon_{T0}}{\partial t} + \frac{\nu_e}{2 \omega_{T0}^2} \varepsilon_{T0}^2 - \frac{1}{4} \frac{e k_L}{\omega_{T1}^2 m_e} \varepsilon_L \varepsilon_{T1} e^{i \nu t}.
\]  

Equations (V.43), (V.45), and (V.50) now form a closed set of coupled non-linear equations for the three waves involved in the stimulated Raman scattering problem. Let us now normalize the amplitudes of these waves so that the amplitude squared represents the total wave energy density of the mode. With the aid of the linearized fluid equations, it is straightforward to show that the total wave energy densities for transverse and Langmuir waves in a plasma are given by

\[
\rho_{E_T} = \left( \frac{1}{4} n_0 m_e v_T^2 + \frac{1}{4} e \varepsilon_T^* \varepsilon_T + \frac{1}{4} \mu_0 n_{T1}^* \varepsilon_{T1} \right) \omega_{T1}, k_T
\]  

\[
\rho_{E_L} = \left( \frac{1}{4} n_0 m_e v_L^2 + \frac{1}{4} \gamma_e \frac{k_T^2}{n_e} n_{L1}^* n_L + \frac{1}{4} e \varepsilon_L^* \varepsilon_L \right) \omega_{L1}, k_L.
\]  

Expressing all transverse variables in terms of \( E_T \) with the aid of Eqs. (V.35) and (V.36), and all Langmuir variables in terms of \( E_L \) using Eqs. (V.37) and (V.38), we obtain

\[
\rho_{E_T} = \frac{1}{2} \varepsilon_0 |E_T|^2
\]  

and

\[
\rho_{E_L} = \frac{1}{2} \varepsilon_0 |E_L|^2 \frac{\omega_L^2}{\omega_{Pe}^2}.
\]  

We may now introduce the normalized amplitudes

\[
a_{T0,1} = \left( \frac{\varepsilon_0}{2} \right)^{\frac{1}{2}} \varepsilon_{T0,1}
\]  

\[
a_L = \left( \frac{\varepsilon_0}{2} \right)^{\frac{1}{2}} \frac{\omega_L}{\omega_{Pe}} \varepsilon_L.
\]  

and, neglecting the damping terms and assuming perfect frequency matching, the equations for the interacting waves are

\[
\frac{3 a_{T0}}{\partial t} = -i \omega_{T0} a_{T1} a_L
\]  

(V.57)
\[ \frac{3 \alpha_{T1}}{\delta t} = \Gamma_{TT1} \alpha_{T0} a_{T0}^* \]  

(V.58)

\[ \frac{3 \alpha_{L}}{\delta t} = \Gamma_{LT1} \alpha_{L} a_{T0}^* \]  

(V.59)

where

\[ \Gamma = \left( \frac{2}{\epsilon_e} \right)^{\frac{1}{2}} \frac{e k_{pe} \omega_{pe}}{m_e \omega_{T1} \omega_{T0} \omega_{L}} \]  

and \(|a_{T0}|^2, |a_{T1}|^2, |a_{L}|^2\) correspond to the total wave energy densities of the modes.

With the aid of Eqs. (V.57) to (V.59) it is a simple matter to show that

\[ \frac{3}{\delta t} (|a_{T0}|^2 + |a_{T1}|^2 + |a_{L}|^2) = 0 \]  

(V.60)

which corresponds to the conservation of the total wave energy density for the interacting waves. However, a more significant result concerns the wave action density \(|a_{n}|^2/\omega_{n}\). Calculating the rate of change of this quantity from Eqs. (V.57) to (V.59), we obtain the celebrated Manley-Rowe relations (first discussed in the field of electronics):

\[ - \frac{1}{\omega_{T0}} \frac{3}{\delta t} |a_{T0}|^2 = \frac{1}{\omega_{T1}} \frac{3}{\delta t} |a_{T1}|^2 = \frac{1}{\omega_{L}} \frac{3}{\delta t} |a_{L}|^2 \]  

(V.61)

The significance of the Manley-Rowe relations is that they give the proportion of pump energy converted to each parametrically excited wave. When the Raman back-scattering instability occurs in the tenuous outer corona of a laser-produced plasma where \(\omega_{T1} \gg \omega_{L}\), most of the incident laser energy going into the excited waves is carried back out of the plasma by the back-scattered electromagnetic wave, i.e. the ratio of the energy in the back-scattered wave to that in the Langmuir wave is \(\omega_{T1}/\omega_{L} \gg 1\).

The situation is even more dramatic for the stimulated Brillouin instability, since the frequency of the ion acoustic wave is always a very small fraction of the frequency of the back-scattered wave. Thus, according to the simple theory of parametric instability, 99% or more of the incident light may be back-scattered owing to the stimulated Brillouin instability!

The Manley-Rowe relations can also be interpreted in terms of pump and excited-wave "quanta". Each pump quantum produces two excited-wave quanta, or vice versa. Equation (V.61) and the perfect k-matching condition also lead to the conservation of total wave momentum density. These conservation relations, although derived for a specific instability, are characteristic of any three-wave interaction.

### V.3 STABILITY OF A LARGE-AMPLITUDE OSCILLATION

In the previous section on the resonant interaction of three coherent waves, the analysis was restricted to small-amplitude waves such that the energy density of the waves was small compared with the total energy density of the plasma. Despite this limitation, many
results of physical significance can be derived with the aid of this model, since the
threshold amplitude for a finite-amplitude wave (e.g. a laser pump) to excite other waves
in a plasma is consistent with the condition $\epsilon E_z E_0/n_e k T_e \ll 1$, where $E_0$ is the electric field
of the finite (but small) amplitude pump. However, it is of interest to analyse a case
where the restriction on the smallness of the amplitude of the initial wave is removed. In
this last section on non-linear effects we shall obtain a solution of the dynamical equations
for the plasma (the equations of ideal MHD) which is valid for any amplitude. We
shall then analyse the stability of this large-amplitude wave.

Let us begin the analysis of this problem by writing down the equations of ideal MHD:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla (p c_s^2) + \frac{1}{\mu_0} \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} - \mathbf{J}^{\text{ext}} \times \mathbf{B} \quad (V.62)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (V.63)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{J} \times (\mathbf{v} \times \mathbf{B}) \quad (V.64)$$

where $c_s$ is the velocity of sound in the plasma, $\rho$ the mass density, $\mathbf{v}$ the fluid velocity
and $\mathbf{B}$ the magnetic field; $\mu_0$ is the magnetic permeability and $\mathbf{J}^{\text{ext}}$ represents an external
source of current, the physical significance of which will be discussed later. Note that
the MHD equations have been reduced to three by substituting $\mathbf{J} = (1/\mu_0) \nabla \times \mathbf{B}$ from Maxwell's
equations into the equation of motion, by assuming the isothermal equation of state $p = c_s^2 \rho$
(where $p$ is the pressure) and by substituting the equation for infinite conductivity
$\mathbf{E} = -\nabla \times \mathbf{B}$ into the Maxwell equation $-\partial \mathbf{B}/\partial t = \nabla \times \mathbf{E}$.

The large-amplitude solution we are going to construct is a circularly polarized field.
We therefore introduce the variables which are most suitable for describing such a field.
Thus, we define

$$\mathbf{v}_z = v_x \pm i v_y \quad (V.65)$$

$$\mathbf{B}_z = B_x \pm i B_y \quad (V.66)$$

$$\mathbf{J}^{\text{ext}}_z = J^{\text{ext}}_x \pm i J^{\text{ext}}_y \quad (V.67)$$

and look for a one-dimensional solution of Eqs. (V.62)-(V.64) such that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 .$$

In terms of the variables defined in Eqs. (V.65)-(V.67), Eqs. (V.62)-(V.64) can be written

$$\rho \left( \frac{\partial \mathbf{v}_z}{\partial t} + \mathbf{v}_z \frac{\partial \mathbf{v}_z}{\partial z} \right) v_z = \frac{B_z}{\mu_0} \frac{\partial B_z}{\partial z} \pm i \mathbf{J}^{\text{ext}}_z \quad (V.68)$$

$$\rho \left( \frac{\partial \mathbf{v}_z}{\partial t} + \mathbf{v}_z \frac{\partial \mathbf{v}_z}{\partial z} \right) v_z = -\frac{1}{2 \mu_0} \left( B_+ - \frac{\partial B_+}{\partial z} + B_- \frac{\partial B_-}{\partial z} \right) + \frac{i}{2} \left( J^{\text{ext}}_{B_+} - J^{\text{ext}}_{B_-} \right) \quad (V.69)$$

$$\frac{\partial B_z}{\partial t} = \frac{n}{\mu_0} (v_x B_z) - \frac{1}{\mu_0} \left( v_z B_z \right) \quad (V.70)$$

$$\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial z} \left( \rho v_z \right) . \quad (V.71)$$
The required large-amplitude solution of Eqs. (V.68)-(V.71) is easily obtained by assuming

\[ B_{z0} = \pm iB_{z0} e^{\pm i(\omega_0 t - k_0 z)} \]  \hspace{1cm} (V.72)

\[ j_{ext} = \pm iJ_0 e^{\pm i(\omega_0 t - k_0 z)} \]  \hspace{1cm} (V.73)

\[ v_{z0} = \pm iv_{z0} e^{\pm i(\omega_0 t - k_0 z)} \]  \hspace{1cm} (V.74)

and that \( v_z \), \( \rho \), and \( B_{z0} \) are uniform and constant. Substituting Eqs. (V.72)-(V.74) into Eqs. (V.68)-(V.71), it is easily verified that the large-amplitude wave is given by

\[ \tilde{B}_0(z,t) = \tilde{B}_{z0} + B_{z0} \left[ -\dot{X} \sin (\omega_0 t - k_0 z) + \tilde{Y} \cos (\omega_0 t - k_0 z) \right] \]  \hspace{1cm} (V.75)

\[ j_{ext} = J_0 \left[ -\dot{X} \sin (\omega_0 t - k_0 z) + \tilde{Y} \cos (\omega_0 t - k_0 z) \right] \]  \hspace{1cm} (V.76)

and

\[ \dot{v}_0(z,t) = \hat{\nu} \hat{v}_d + v_{z0} \left[ -\dot{X} \sin (\omega_0 t - k_0 z) + \tilde{Y} \cos (\omega_0 t - k_0 z) \right]. \]  \hspace{1cm} (V.77)

The self-consistency of this solution is completed by the following relationships between \( B_{z0} \), \( J_0 \), and \( v_{z0} \):

\[ J_0 = \frac{k_0}{v_0} B_{z0} \left[ 1 - \frac{(\omega_0 - k_d v_d)^2}{k_0^2 c_A^2} \right] \]  \hspace{1cm} (V.78)

\[ v_{z0} = \frac{\omega_0 - k_d v_d}{k_d B_{z0}} B_{z0}, \]  \hspace{1cm} (V.79)

where \( c_A \) is the Alfvén velocity \( B_0/(\rho_0 \mu_0)^{1/2} \) and we have included the uniform flow \( v_d \) along the \( z \)-axis as an additional source of energy in the plasma.

Before considering the stability of this oscillating field of arbitrary amplitude, let us say a word about the external current source \( J_0 \). In the special case when the large-amplitude wave is a natural mode of the plasma (i.e. an Alfvén wave where \( \omega_0 = k_0 c_A \) for \( v_d = 0 \) or \( \omega_0 = k_0 v_d \mp k_0 c_A \) for \( v_d \neq 0 \)), the external current \( J_0 = 0 \) and the large-amplitude magnetic field is excited entirely by currents flowing in the plasma itself. In all other cases an external current \( J_0 \) is required to excite the large-amplitude magnetic field \( B_{z0} \).

In practice, these external currents would be carried by some arrangement of coils. However, in order to keep the analysis as simple as possible and to enable a one-dimensional model to be used, we adopt the convenient idealization of an external current source which is distributed throughout the entire plasma. Thus we imagine that the plasma is threaded by a set of infinitely thin wires carrying the external current \( j_{ext} \) of Eq. (V.76). The presence of these wires does not impede the resulting plasma flows but acts solely as a source of the resulting magnetic and velocity fields \( \vec{B}_{z0}(z,t) \) and \( \vec{v}_{z0}(z,t) \).

Let us now consider the stability properties of the large-amplitude wave described by Eqs. (V.75)-(V.79). We suppose that a perturbation of the density \( \rho_1 \) occurs which varies as \( \exp i(kz - \omega t) \). This density perturbation will now beat with the large-amplitude
magnetic fields $B_{\pm 0} \equiv B_{X0} \pm iB_{Y0}$ to generate sidebands $B_{\pm 1} \equiv iB_{y1}$ which vary as $\exp \{i[(k + k_0)x - (\omega + \omega_0)t]\}$. The solutions for the sideband fields can be written as

$$B_{\pm 1} = \alpha_{\pm} \rho_1 B_{\mp 0} ,$$

(V.80)

where $\alpha_{\pm}$ are coefficients depending on the various frequencies and wave numbers. Now, the sideband fields can also beat with the large-amplitude field to regenerate the original density perturbation. Thus

$$\rho_1 = \beta_{-} B_{\mp 0} \rho_{+1} + \beta_{+} B_{\mp 0} \rho_{-1} ,$$

(V.81)

where $\beta_{\pm}$ are again coefficients similar to $\alpha_{\pm}$ above. It is easily seen that the combination of large-amplitude and sideband fields shown on the right-hand side of Eq. (V.81) will act as sources for the density perturbation $\rho_1$. Substituting the expressions for $B_{\pm 1}$ in terms of $\rho_1$ from Eq. (V.80) into Eq. (V.81) yields the dispersion relation for perturbations about the large-amplitude state.

We shall now show how to calculate the coefficients $\alpha_{\pm}$ and $\beta_{\pm}$, thus obtaining the required dispersion relation. The procedure is to linearize Eqs. (V.68)-(V.71) in which the fields are written (in the usual way) as the sum of the initial state (i.e. the large-amplitude wave) plus a small perturbation. Products of a perturbed variable with a field of the large-amplitude wave will be retained, whereas products of perturbed quantities will be neglected. The linearized equations for the transverse perturbations can be written,

$$\left( \frac{\partial}{\partial t} + v_d \frac{\partial}{\partial z} \right) v_{\pm 1} = \frac{B_{Z0}}{\rho_0 u_0} \frac{\partial B_{\pm 1}}{\partial z} - \frac{\partial_1}{\rho_0} \left( \frac{\partial}{\partial t} + v_d \frac{\partial}{\partial z} \right) v_{\mp 0} + v_{z1} \frac{\partial v_{\pm 0}}{\partial z}$$

(V.82)

$$\left( \frac{\partial}{\partial t} + v_d \frac{\partial}{\partial z} \right) B_{\pm 1} = \frac{B_{Z0}}{\rho_0} \frac{\partial v_{\pm 1}}{\partial z} - \frac{\partial_1}{\rho_0} (v_{z1} B_{\pm 0})$$

(V.83)

where all perturbed quantities are denoted by a subscript 1. The variable $v_{\pm 1}$ stands for

$$v_{\pm 1} \equiv v_{x1} \pm i v_{y1}$$

with a similar expression for $B_{\pm 1}$. Note that if we take an equilibrium state in which there is no large-amplitude wave and no plasma flow (i.e. $v_{\pm 0} = B_{\pm 0} = 0$ and $v_d = 0$), Eqs. (V.82) and (V.83) reduce to

$$\frac{\partial^2 B_{\pm 1}}{\partial t^2} = \frac{B_{Z0}^2}{\rho_0 u_0} \frac{\partial^2 B_{\pm 1}}{\partial z^2} ,$$

(V.84)

which describes the propagation of Alfvén waves with phase and group velocity given by $c_A = B_{Z0}/(\rho_0 u_0)^{1/2}$.

We may now obtain the solutions for $B_{\pm 1}$ from Eqs. (V.82) and (V.83) corresponding to the expressions given by Eq. (V.80). To do this, we note that $\rho_1$, and hence $v_{\pm 1}$ with which it is linearly connected, are assumed to vary as $\exp i(kz - \omega t)$. The variation of $B_{\pm 0}$ and $v_{\pm 0}$ is given by Eqs. (V.72) and (V.74), and, using the fact that $v_{z1}$ and $B_{z1}$ are driven by the products $\rho_1 v_{z0}$, $v_{z1} v_{z0}$, and $v_{z1} B_{z0}$, the variation of $B_{z1}$ and $v_{z1}$ is given by

$$\exp \left[ -i(\omega + \omega_0)t + i(k + k_0)z \right] .$$
The solution for $B_{z_1}$ is now obtained as follows. Equation (V.83) is first operated on by $[\partial/\partial t + v_d (\partial/\partial z)]$, thus eliminating $v_{z_1}$ in favour of $B_{z_1}$ with the aid of Eq. (V.82). The explicit dependence of all the variables on $z$ and $t$ is then substituted into the equation for $B_{z_1}$, and Eq. (V.79) is used to express $v_{z_0}$ in terms of $B_{z_0}$. Finally, $v_{z_1}$ is expressed in terms of $\rho_1$ with the aid of the linearized version of Eq. (V.71):

$$\left(\frac{\partial}{\partial t} + v_d \frac{\partial}{\partial z}\right) \rho_1 = -\rho_0 \frac{\partial v_{z_1}}{\partial z},$$

(V.85)

giving

$$v_{z_1} = \frac{\omega - kv_c}{k_0 c_A} \rho_1 .$$

(V.86)

One then obtains Eq. (V.80):

$$B_{z_1} = \alpha_z \rho_1 B_{z_0},$$

where

$$\alpha_z = \frac{1}{\rho_0 k_0} \frac{(k + k_0)(\omega_0 - k_0 v_d)}{[(\omega + \omega_0) - (k + k_0)v_d]^2 - (k + k_0)^2 c_A^2]} \times \left\{1 - \frac{k_0}{k} \frac{\omega - kv_d}{\omega_0 - k_0 v_d} \frac{\omega + \omega_0 - (k + k_0)v_d}{(\omega_0 - k_0 v_d)^2} \left[\frac{\omega + \omega_0 - (k + k_0)v_d}{k_0 c_A^2}\right]^{\frac{1}{2}}\right\}. $$

(V.87)

In order to obtain the coefficients $\beta_x$, we must use the linearized equation for $v_{z_1}$. This is obtained from Eq. (V.69) and can be written

$$\rho_0 \left(\frac{\partial}{\partial t} + v_d \frac{\partial}{\partial z}\right)v_{z_1} + c_s^2 \frac{\partial \rho_1}{\partial z} = -\frac{1}{2\mu_0} \left(\beta_{-0} \frac{\partial B_{+1}}{\partial z} + B_{-1} \frac{\partial B_{+0}}{\partial z} + B_{+0} \frac{\partial B_{-1}}{\partial z} + B_{+1} \frac{\partial B_{-0}}{\partial z}\right)$$

$$+ \frac{1}{2} (J^\text{ext}_{B_{-1}} - J^\text{ext}_{B_{+1}}).$$

(V.88)

Again, it is easily verified, using the known variations of $B_{z_0}, B_{z_1},$ and $J^\text{ext}_x$, that the terms on the right-hand side of Eq. (V.88) will drive $\rho_1$ and $v_{z_1}$ at the frequency $\omega$ and wave number $k$ originally assumed. Substituting the exp $i(kz - \omega t)$ dependence into Eq. (V.88), we obtain

$$-\rho_0 (\omega - kv_d)v_{z_1} + c_s^2 \frac{\partial \rho_1}{\partial z} = \frac{1}{2\mu_0} \left\{(k - k_0)B_{-1}B_{+1} - k_0 B_{+0} B_{-1}\right.$$ 

$$+ (k + k_0)B_{+0} B_{-1} + k_0 B_{-0} B_{+1}\left\} = \frac{1}{2} (J^\text{ext}_{B_{-1}} - J^\text{ext}_{B_{+1}}).$$

(V.89)

Using Eq. (V.86) to eliminate $v_{z_1}$ in terms of $\rho_1$, we find Eq. (V.81):

$$\rho_1 = \beta_{-0} B_{+1} + \beta_{+0} B_{-1},$$

where

$$\beta_{-0} = \frac{k}{2\mu_0 c_s^2} \left[ k - k_0 + \frac{(\omega_0 - k_0 v_d)^2}{k_0 c_A^2} \right].$$

(V.90)
\[ \beta_+ = \frac{k}{2\nu_0 D_z} \left[ k + k_0 - \frac{(\omega_s - k_0 v_d)^2}{k_s c_s^2} \right] \]  \hspace{1cm} (V.91)

and

\[ D_z \equiv (\omega - kv_d)^2 - k^2 c_s^2 . \]

The dispersion relation can now be obtained by substituting the expressions for \( B_{\perp 1} \) in terms of \( \rho_1 \) from Eq. (V.80) into Eq. (V.81). The required dispersion relation is

\[ 1 = (\alpha_\perp \beta_- + \alpha_\parallel \beta_+) B_{\perp 1}^2 . \]  \hspace{1cm} (V.92)

Equation (V.92) is a sixth-order equation in \( \omega \). Within the limitations of the simplified one-dimensional MHD model it is a very general dispersion relation valid for arbitrary amplitude \( B_{\parallel 1} \) and any value of \( \omega/\omega_s \). As illustrations we shall consider two special cases of Eq. (V.92) for which approximate analytic solutions can be obtained. First, however, we shall write the dispersion relation in a less formal and physically more revealing way.

Substituting the expressions for \( \alpha_\perp \) and \( \beta_- \) into Eq. (V.92), and after a certain amount of manipulation, we obtain the following form:

\[ \frac{\omega_d^2}{k^2 c_s^2} - \frac{B_{\perp 0}^2}{B_{\parallel 0}^2} \left[ k^2 c_s^2 + k_0^2 c_s^2 - \omega_s^2 \right] = \frac{B_{\perp 0}^2}{2B_{\parallel 0}^2} c_s^2 \left\{ \frac{(k + k_0)^2 (k + k_0 - \omega_s^2/k_s c_s^2)}{(\omega_d + \omega_s)^2 - (k + k_0)^2 c_s^2} \right. \]

\[ + \frac{(k - k_0)^2 (k - k_0 + \omega_s^2/k_s c_s^2)}{(\omega_d - \omega_s)^2 - (k - k_0)^2 c_s^2} \} \]  \hspace{1cm} (V.93)

where \( \omega_d \equiv \omega - kv_d \) and \( \omega_s \equiv \omega_s - k_0 v_d \).

We shall now discuss two special cases.

a) \( \omega_s = k_s c_s, \quad v_d = 0 \)

This corresponds to the situation where the driving (or pump) wave (i.e., the large-amplitude initial state) is a natural mode of the plasma. If we also assume that \( \omega << k_s c_s \), Eq. (V.93) reduces to the simple form

\[ \omega^2 = k^2 c_s^2 - \frac{B_{\perp 0}^2}{B_{\parallel 0}^2} k^2 c_s^2 - \frac{k_0^2}{(k^2 - 4k_0^2)} . \]  \hspace{1cm} (V.94)

This shows that the initial density perturbation will be unstable for perturbed wave numbers \( k >> k_0 \) provided the second term on the right-hand side exceeds the first. The larger the ratio \( k/2k_0 \), the larger the ratio \( B_{\perp 0}/B_{\parallel 0} \) required for instability. However, \( B_{\perp 0}/B_{\parallel 0} \) can evidently be as small as one pleases as the resonance \( k = 2k_0 \) is approached.

This resonance can be described by returning to the full dispersion relation, Eq. (V.93) (with \( v_d = 0 \) and \( \omega_s = k_s c_s \)), and using the fact that one of the denominators on the right-hand side is resonant. We may therefore neglect the non-resonant term and write the dispersion relation as

\[ \omega^2 = k^2 c_s^2 \approx \frac{B_{\perp 0}^2}{2B_{\parallel 0}^2} c_s^2 \frac{k^2 (k - k_0)^2}{(\omega - \omega_0)^2 - (k - k_0)^2 c_s^2} . \]  \hspace{1cm} (V.95)
Now the left-hand side describes ion acoustic perturbations \( \omega = \pm k c_s^2 \). On the other hand, zeros of the denominator on the right-hand side refer to Alfvén perturbations \( \omega = \omega_s = \pm (k - k_o) c_A \) shifted in frequency and wave number by the presence of the pump wave. We will have a resonance when these natural modes coincide. The condition for this is

\[
k c_s^2 = \omega_s - (k - k_o) c_A,
\]

which gives

\[
k = \frac{2 k_o}{[1 + (c_s/c_A)]}.
\]

For most laboratory plasmas \( c_s \ll c_A \), so that the resonance condition is \( k = 2 k_o \). The solution of Eq. (V.95) at the resonance is obtained by writing it in the form

\[
(\omega - k c_s)[\omega - \omega_s + (k - k_o) c_A] = \frac{k^2}{2 B_o^2} \frac{k^2(k - k_o)^2}{c_A^2} \frac{1}{[\omega - \omega_s - (k - k_o) c_A]}(\omega + k c_s^2).
\]

This exhibits the coupling between the two natural modes of the plasma due to the presence of the pump. To demonstrate instability we must assume that \( B_{ho}/B_{z0} \) is small enough for us to write a perturbation solution

\[
\omega \approx k c_s^2 + \delta \omega,
\]

where we of course also make use of the resonance conditions given above. Substituting Eq. (V.99) into the left-hand side of Eq. (V.98), and \( k = 2 k_o \) and \( \omega = k c_s^2 \) on the right-hand side, we finally obtain the perturbed solution

\[
(\delta \omega)^2 = -\frac{k^2}{2 B_o^2} \frac{c_A^2}{c_s^2} k_o^2 c_s^2,
\]

which shows that a very small value of \( B_{ho}/B_{z0} \) will be sufficient to drive an acoustic wave and an Alfvén wave unstable simultaneously.

b) \( \omega_s = 0, v_d \neq 0 \)

This corresponds to an initial state which is not a natural mode of the plasma and which must be driven by external currents. By analogy with the first case we again look for resonant conditions corresponding to a three-wave interaction. The resonant condition can again be obtained by requiring coincidence of the ion acoustic and transverse wave roots. This gives

\[
k v_d - k c_s = k v_d - k v_d - (k - k_o) c_A
\]

or

\[
k = k_o \frac{[1 - (v_d/c_A)]}{[1 - (c_s/c_A)]}.
\]

For \( v_d \sim c_s \), this reduces to \( k \approx k_o \). For this case, the most unstable wavelength is the one which just matches the periodicity of the external large-amplitude driving field, a condition which is also found for the MHD stability of magnetically confined plasmas.
We may obtain the unstable solution corresponding to the resonance condition given by Eq. (V.101) by means of a perturbation solution similar to the previous case. Thus, we assume

\[ \omega = kv_d - kc_s + \delta \omega , \]  

(V.103)

and with the aid of Eq. (V.101) and retaining only the resonant term in Eq. (V.93) we find

\[ (\delta \omega)^2 = -\frac{B_{\perp 0}^2}{8\pi^2 \epsilon_0} \frac{(v_d - c_s)}{c_s} \frac{k^2}{c_s^2} \frac{(v_d - c_s)c_A - v_A^2}{c_A^2} . \]

(V.104)

The solution given by Eq. (V.104) is only valid for \( \delta \omega \ll k(v_d - c_s) \). However, apart from demonstrating instability it also shows that a necessary condition for instability in this case is that

\[ v_d > c_s . \]

In summary, we have obtained a dispersion relation, Eq. (V.93), which is valid for arbitrarily large values of \( B_{\perp 0} \) and for any value of \( \omega \). The only limitations on \( B_{\perp 0} \) and \( \omega \) are that the associated velocity \( v_{\perp 0} \) should be non-relativistic and \( \omega < \omega_{ci} \), where \( \omega_{ci} \) is the ion cyclotron frequency. In fact, it is interesting to note that the above analysis can be extended to include relativistic dynamics. The general solutions of Eq. (V.93) can only be obtained numerically. However, we have demonstrated two unstable solutions, one for the case when the pump is a natural Alfvén wave, and the other when the pump is a static magnetic field and the plasma is drifting. This case is analogous to the interaction occurring in one of the operating modes of the free electron laser. The drifting plasma is equivalent to the electron beam, the periodic magnetic field to the magnetic "wiggler" field, and the excited Alfvén wave to the excited electromagnetic field.

One final remark concerning the distinction between cases (a) and (b) concerns the effect of damping. The introduction of dissipation into the system gives rise to a finite threshold for instability. It is interesting to note that such an analysis shows that case (b) rather than case (a) has the lowest threshold for instability.

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