DISPERSION OF STACKED PROTONS IN SYNCHROTRON PHASE

SPACE BY A MODULATED RADIO FREQUENCY VOLTAGE

by

E.W. Messerschmid

Geneva, August, 1972
1. Introduction

The passage of buckets created by a modulated radio frequency (r.f.) voltage causes a shift and a dilution of the stack density in longitudinal phase space. This effect can be treated by numerical computations using the computer programmes for r.f. accelerator studies \(^1\). From previous studies \(^2\),\(^3\) and from the analytical theory \(^4\) we know that the shift in momentum after one complete traversal of buckets corresponds to the averaged momentum spread of the buckets \(\delta p = A/2\pi\) \((A = \text{area of the moving buckets})\). As there exist only estimations from computations \(^5\) and a valuation only for particular phase angles (Ref. \(^2\) and E. Keil's computations of stacking efficiencies), H.G. Hereward suggested to study explicitly the scattering of a sample of stacked particles for a wide range of \(\Gamma = \sin \theta_0\).

2. Assumptions involved in the computer studies

The computer programmes for the investigation of the stacking process can be immediately applied to compute both shift and dispersion of particles in the stacking region. The first programme (ASTACK) traces the history of a number of particles as they circulate and change energy by traversing the r.f. cavities. The output of this programme consists of the energy \(E_{1k}\) and the phase for each particle \(k\) after predetermined intervals of time \(t_1\). It is analysed by the second programme BSTACK whose output describes the energy distribution \(V_1(E)\), mean \(\bar{E}_1\) and variance \(\sigma_1^2\) of a sample of particles \(N_P\) for the specified time steps \(t_1\):

\[
\bar{E}_1 = \frac{1}{N_P} \sum_{k=1}^{N_P} E_{1k}, \quad \sigma_1^2 = \frac{1}{N_P} \sum_{k=1}^{N_P} (E_{1k} - \bar{E}_1)^2.
\]

Thus one obtains a measure of the change of the energy spectrum as a function of the distance in energy between the particles and the current synchronous energy.

The programmes are based on the following assumptions:

\[\text{...}\]
1) The distributions obtained for a sample of particles initially within the reference channel after one cycle, depend only on the difference of the energy of that channel and the energy $E_i$ which the buckets were crossing).

2) According to reference 3) a uniform distribution in energy yields an almost uniform distribution in the Hamiltonian. Hence, all particles were put at the same phase and at equidistant intervals in energy.

3) We assume that there is no interaction between the particles in a stacked beam.

3. Choice of initial conditions

The scattering of particles is believed to depend only on the stable phase angle $\phi_s$. But in our numerical model it is not excluded that it depends on the actual parameters of a storage device too. For this reason, the machine parameters were matched as closely as possible to the ISR parameters. Furthermore, the results should not depend either on the energy which is sufficiently far above transition energy or on the number of cavities. Therefore, the computations were based on the simulation of acceleration by a single cavity beginning at the same energy $E_0$.

In order to reduce the computer time the ratio $A$ of the revolution frequency and the phase oscillation frequency of a particle in the proximity of the stable point has to be scaled by a suitably chosen scaling factor $\mu$. In spite of the scaling we obtain the same trajectories in the synchrotron phase space $(y,\phi)$ and hence the same results if we scale both the energy and revolution frequency by the same factor. This one verifies easily as $y$ is the canonical variable

$$y = \frac{1}{2\pi Q} \int_{E_0}^{E} \frac{dE}{f(E)}$$
where $Q$ is proportional to the area of the stationary bucket (see ref. 1).

The computer time to process a particle is proportional to the number $N$ of revolutions of the particle during one cycle. The required time for 60 particles performing $10^5$ revolutions is about 15 minutes on the CDC 6600 at CERN. Because the large variation of $\Gamma$ results in a drastic change of input parameters (among others $N$) we have chosen different matching conditions to have about the same accuracy for all computations. The total computer time was about $21$ hours. More details are attached in the Appendix.

4. Scattering of a sample of particles

According to the analytic theory and previous computations the mean displacement in momentum $\delta p = \frac{\beta}{c} \delta \mathbf{E}$ ($\beta =$ particle velocity over velocity of light) corresponds to the momentum height $\delta p_b$ of the buckets when they pass through the sample of particles. Therefore, the number $\delta p / \delta p_b$ might serve as a confidence quantity and it is expected to be 1. Figure 1 shows for example ($\Gamma = 0.4, \lambda = 3$) the mean energy $\bar{E}_i$ as a function of the number $i$ of the crossed channel.

All measures of dispersion are to a large extent arbitrary, since the properties to be described by such parameters are too vaguely defined by means of a single number. For our purpose the mean square displacement in energy $\sigma^2$, and its square root (the r.m.s. displacement), have been chosen as they are convenient quantities to use as a measure of scattering 7). In Figure 2, with the same parameters as for the first figure, the energy variance $\sigma^2$ has been plotted against the number of the channels crossed. For this particular $\Gamma$ the variance increased by more than a factor of 100. This, of course, is due to the fact that the initial energy spread $\sigma_{in}^2$ of the particles was chosen to be smaller than the mean dispersion caused by the crossing buckets. We take it into account by the definition

$$\delta E_{rms} = \sqrt{\sigma^2 - \sigma_{in}^2}$$
where $\sigma_f^2$ is the final energy variance, which is practically constant, apart from statistical fluctuations due to the limited number of particles. Because of these, even for small fluctuations, $\sigma_f^2$ is averaged over the last few channels.

Equivalent to the equation above we can write:

$$\frac{\delta p_{rms}}{\delta p_s} = \frac{1}{\delta E_g} \sqrt{\sigma_f^2 - \sigma_{in}^2}$$

$$\frac{\delta p_{rms}}{\delta p_b} = \frac{1}{\delta E_b} \sqrt{\sigma_f^2 - \sigma_{in}^2}$$

The indices $s$ and $b$ correspond to the stationary and moving bucket respectively and are defined by $\delta p_b = a(\Gamma) \delta p_s$ (or $\delta E_b = a(\Gamma) \delta E_s$). $a(\Gamma)$ is the bucket area parameter and available in tabulated form.

5. Results

The results of the computations are attached in the table as well as the most essential matching conditions. Figure 3 shows the shift and associates the results plotted into the following diagrams with a confidence measure. Figure 4 is drawn for practical reasons though it is related to Figure 5 by merely the factor $a(\Gamma)$.

From Figure 3 we derive an increasing dispersion of mean energy displacement with increasing $\Gamma$. For $\Gamma > 0.82$ the shift $\delta p/\delta p_b$ is very different from 1. Figure 5 shows that the r.m.s. momentum displacement expressed in units of momentum width of a stationary bucket increases linearly with $\Gamma$. At $\Gamma > 0.82$ $\delta p_{rms}/\delta p_s$ falls off almost linearly. For particular $\Gamma$'s computations with different scaling factors yield practically no difference in the results. The same we find for computations at $\Gamma = 0.4$ where an accelerating voltage between 0.4 kV and 6.4 kV results in a relative deviation smaller than 1%.
6. **Conclusions**

The unexpected results for \( \Gamma > 0.82 \) in Figure 3 which have certainly influenced the results plotted in the Figures 4 and 5 respectively might be related to former experiences in the calculation of the stacking efficiency \(^9\). There it turned out to be impossible to make calculations for the high \( \Gamma \) value of 0.84 and \( \lambda = 6 \). The present analytical theory does not explain this behaviour. Among others by measurements in the ISR one could find out whether it comes from physical reasons "only" or from the scaling of the computer model to the real case for these large phase angles.

Restricting the discussions to values of \( \Gamma \) below the critical value mentioned above the linear increase of \( \delta p_{\text{rms}}/\delta p_s \) is obviously well approximated by the interrupted line in Figure 5:

\[
\frac{\delta p_{\text{rms}}}{\delta p_s} = \Gamma
\]

This states that Symon's rules \(^5\), \(^7\)

\[
\langle \delta p_{\text{rms}} \rangle_n = \sqrt{\langle \delta p^2 \rangle_n - \langle \delta p^2 \rangle_0} = 0.5 \sqrt{n} \delta p_s
\]

\( n \) being the number of bucket passages, is not in good agreement for \( n = 1 \) and values of \( \Gamma \) very different from 0.5.

In Ref. \(^2\), Reilly gives the average displacement and the r.m.s. scatter for \( \Gamma = 0.55 \) and for small differences in energy of sample position relative to the bucket centre of only few times the bucket width. For the same parameters we derive similar results, but it is to realise that within these small differences we have high fluctuations and still an increase of dispersion with increasing synchronous energy (see Figure 2).

Furthermore, E. Keil's former computations yield in our notation the following numbers
which confirm the r.m.s. scattering in units of a stationary bucket to
be well approximated by the sin of the stable phase angle. Finally,
the computations showed that different scaling factors yield the same
results within the statistical fluctuations.

7. Acknowledgements

I would like to thank H.G. Hereward for useful comments and
suggestions and E. Keil for helpful support he granted, not only for
these studies.

References

1) E. KEIL and A.G. RUGGIERO, "Computer programmes for RF accelerator
studies", ISR-TH/68-37.

2) D. REILLY, "Computer studies of beam stacking effects", MURA 477.

3) E. KEIL and A. NAKACH, "Beam stacking at high values of $\Gamma = \sin \phi_s$",
CERN 66-9, ISR Division, 1966.

Accelerators (Geneva) 1, p. 44 (1956).

5) SYMON's rule, quoted in
W. SCHNELL, "An estimate of the efficiency of phase-displacement
acceleration in the ISR", ISR-RF/68-18.

6) D.A. SWENSON, "A study on the Beam Stacking process", Int. Conf.
7) H.G. HEREWARD, Private communication.

8) I. GUMOWSKI, "CPS RF bucket, width, height and area", MPS/Int. RF 67-1.

APPENDIX

I. Fixed parameters or parameters at convenience are:

harmonic number \( n = 30 \), number of channels \( N_h = 160 \),
60 particles are equally distributed in channel 80 or in
\( \lambda \cdot y_c / \zeta \) for \( \lambda \neq \zeta \) in the table.
\( \lambda = A / (2 \pi \Delta y_c) = 3 \) with \( A \) : bucket area
\( \Delta y_c \) : height of a channel,

Parameters which depend only on \( \Gamma \):

\( a, y_{\max} = \sqrt{2} y \) from Ref. \( \delta \) and
\( a_1 = \pi \Gamma (1 - \Gamma^2)^{1/2} \left( \Delta \alpha \right)^{-1} \)
\( a_2 = 32 a_1 a_2 (\Gamma \cdot \pi^3)^{-1} \)

and \( E_0 = 26.544 \text{ GeV}, \ V = 1.6 \text{ kV}. \)

II. Given by I : \( \Delta = f_{\text{rev}}/f_p \), \( f_{\text{rev}} \) : revolution frequency
\( f_p \) : phase oscillation frequency

\( \mu = \frac{\Delta}{\Delta_c} \), \( \Delta_c = (f_{\text{rev}})_{\text{comp}}/f_p \).

Choice of \( \mu \):

i. to reduce computer time for small \( \Gamma \)'s we start computations
   at the first channel, i.e., \( E_0 - E_1 = 0 \). From 2.25) and
   2.26) in Ref. \( \delta \) then follows

\[ \zeta = \frac{a_2 N_h}{2 a_1 \lambda^2} \quad \text{and} \quad \Delta_c = \frac{N \cdot \Gamma \cdot \lambda}{2 a_2 \cdot N_h} \]

with number of revolutions \( N = 5,000/\Gamma \), thus proportional
to the actual case. The results are indicated by circles
in the figures.

ii. \( \mu = \Delta/1,000 \) ; dots in the figures

iii. deviations from parameters above, which are indicated in the
    last column of the table and by a triangle in the figures.
## TABLE

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$\Delta_c$</th>
<th>$\zeta$</th>
<th>$\lambda$</th>
<th>$\delta p/\delta Pb$</th>
<th>$\delta p_{rms}/\delta Pb$</th>
<th>$\delta p_{rms}/\delta ps$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.05</td>
<td>130.17</td>
<td>145.02</td>
<td>3</td>
<td>.986</td>
<td>.083</td>
<td>.073</td>
</tr>
<tr>
<td>.10</td>
<td>144.23</td>
<td>59.29</td>
<td>3</td>
<td>.985</td>
<td>.155</td>
<td>.124</td>
</tr>
<tr>
<td>.10</td>
<td>174.67</td>
<td>2.36</td>
<td>1</td>
<td>1.045</td>
<td>.153</td>
<td>.123</td>
</tr>
<tr>
<td>.20</td>
<td>176.71</td>
<td>20.05</td>
<td>3</td>
<td>.998</td>
<td>.364</td>
<td>.241</td>
</tr>
<tr>
<td>.20</td>
<td>1000</td>
<td>21.0</td>
<td>21</td>
<td>.972</td>
<td>.419</td>
<td>.277</td>
</tr>
<tr>
<td>.30</td>
<td>219.77</td>
<td>8.88</td>
<td>3</td>
<td>1.033</td>
<td>.545</td>
<td>.294</td>
</tr>
<tr>
<td>.40</td>
<td>280.59</td>
<td>4.25</td>
<td>3</td>
<td>1.055</td>
<td>.928</td>
<td>.400</td>
</tr>
<tr>
<td>.40</td>
<td>1000</td>
<td>5.0</td>
<td>5</td>
<td>1.080</td>
<td>.863</td>
<td>.372</td>
</tr>
<tr>
<td>.40</td>
<td>1000</td>
<td>5.0</td>
<td>5</td>
<td>1.075</td>
<td>.872</td>
<td>.375</td>
</tr>
<tr>
<td>.50</td>
<td>373.00</td>
<td>2.04</td>
<td>3</td>
<td>1.079</td>
<td>1.525</td>
<td>.508</td>
</tr>
<tr>
<td>.50</td>
<td>1000</td>
<td>3.0</td>
<td>3</td>
<td>1.075</td>
<td>1.439</td>
<td>.479</td>
</tr>
<tr>
<td>.50</td>
<td>178.87</td>
<td>13.0</td>
<td>13</td>
<td>.989</td>
<td>1.473</td>
<td>.491</td>
</tr>
<tr>
<td>.55</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>1.010</td>
<td>2.037</td>
<td>.588</td>
</tr>
<tr>
<td>.60</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>1.078</td>
<td>2.276</td>
<td>.560</td>
</tr>
<tr>
<td>.65</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>0.910</td>
<td>3.337</td>
<td>.687</td>
</tr>
<tr>
<td>.70</td>
<td>2.0</td>
<td>3</td>
<td>3</td>
<td>1.155</td>
<td>3.729</td>
<td>.626</td>
</tr>
<tr>
<td>.70</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>.835</td>
<td>4.531</td>
<td>.760</td>
</tr>
<tr>
<td>.75</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>.941</td>
<td>5.759</td>
<td>.762</td>
</tr>
<tr>
<td>.80</td>
<td>2.0</td>
<td>3</td>
<td>3</td>
<td>.819</td>
<td>7.884</td>
<td>.781</td>
</tr>
<tr>
<td>.80</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>.827</td>
<td>7.838</td>
<td>.777</td>
</tr>
<tr>
<td>.82</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>1.103</td>
<td>8.710</td>
<td>.753</td>
</tr>
<tr>
<td>.85</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>1.827</td>
<td>9.533</td>
<td>.654</td>
</tr>
<tr>
<td>.90</td>
<td>2.0</td>
<td>2</td>
<td>2</td>
<td>2.514</td>
<td>13.557</td>
<td>.554</td>
</tr>
<tr>
<td>.90</td>
<td>1.0</td>
<td>3</td>
<td>3</td>
<td>3.498</td>
<td>13.447</td>
<td>.549</td>
</tr>
<tr>
<td>.95</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>13.640</td>
<td>23.055</td>
<td>.395</td>
</tr>
</tbody>
</table>

$\lambda = 15$

$V = 6.4$ kV

$E_0 = 22$ GeV

$\mu = 100$
Figure 1  Mean energy of the particle distribution versus the number of the channels crossed by the moving buckets.

Figure 2  Variance of the energy of the particle distribution versus the number of the channels crossed by the moving buckets.
Figure 3 Displacement of the sample in units of expected mean displacement

Figure 4 R.m.s. scattering in units of expected mean displacement
Figure 5 R.m.s. scattering in units of a stationary bucket