4. Design of an Alternating-Gradient Synchrotron

Based on the Linear Theory.

by J.B. Adams

1. Introduction.

In this talk, the design of an alternating-gradient proton synchrotron, based on the linear theory outlined in the previous talks, will be considered in some detail. The machine design, which is presented, is not necessarily the final CERN proton synchrotron but rather the latest development in an evolutionary process that started several months ago and which will take a further several months to mature. However, the final design may not be very different from the one presented here and such items as costs, time schedules, and staff lists, based on this design will not need much alteration in the future.

The talk is divided into the following sections. The first describes the way in which the parameters of the machine, and the tolerances on those parameters, cause the protons to stray from the simple circular orbit and form a beam that has to be contained inside the useful region of the magnetic field. A preliminary machine design is then considered to show the way the cross-section of the proton beam depends on the magnetic field index, "n". The requirements that have to be satisfied by the periodic structure of the magnet are then stated and the performance of a structure that satisfies these requirements is presented in detail. This leads to an exact definition of the working point in the $n_1$, $n_2$ plane and the definition of the working diamond formed by the stopbands at the resonant and subresonant values of the periodic structure. Then, a description follows of the movement of the working point, inside the diamond, due to such factors as space charge, synchrotron oscillations and nonlinearities in the field. Finally, a table of parameters is presented with the tolerances that have been used in the calculations and some idea of the tolerances to be set on the nonlinearities of the magnetic field.
2. The Injected Beam Diameter.

The injection problem can be stated as follows. How best can a beam of protons from a linear accelerator be injected into a proton synchrotron and what will be the resulting beam diameter inside the proton synchrotron? The top diagram of Figure 1 shows a linac in which, for convenience, the motion of the protons has been drawn as sinusoidal. The linac beam is injected into a proton synchrotron in which the wavelength of oscillation is different from that in the linac. The envelope of the betatron oscillations in the proton synchrotron is seen to be sinusoidal with a large maximum diameter that has to be contained inside the useful magnetic field area. The lower diagram shows that, by adding a quarter-wave transformer, the beam can be correctly introduced into the proton synchrotron and a minimum beam diameter achieved.

From such simple ideas, it can be shown that the diameter of the beam inside the proton synchrotron, under correctly "matched" conditions, is related to the admittance of the linac and the admittance of the vacuum chamber of the synchrotron as, $d_s/d_{vc} = \sqrt{\frac{A_{LA}}{A_{vc}}}$, where $d_s$ is the beam diameter, $d_{vc}$ is the vacuum chamber height, $A_{LA}$ is the linac admittance, $A_{vc}$ is the admittance of a beam of particles filling the synchrotron vacuum chamber. The admittance is defined as the maximum cross-section area of the beam (or vacuum chamber fully filled by a beam) times the maximum angle of divergence of the beam at the point where the maximum cross-section occurs. A further useful relationship is that the diameter of the beam inside the synchrotron is inversely proportional to $n^{\frac{1}{2}}$. Therefore, increasing the value of "n" decreases the beam cross-section very slowly indeed.

3. The Closed Orbit.

If the magnet sectors in the synchrotron are not correctly aligned to a perfect circle, then the closed orbit, itself, is no longer a perfect circle. The linear theory gives the peak value of the
displacement, around the machine, of the closed orbit from the ideal orbit, averaged over all those machines that can be built within the specified tolerances. The variation in this average value, as the resonances are approached, is shown in Figure 2A. A typical closed orbit is shown underneath. The two horizontal lines represent the limits fixed by the mechanical tolerances. For example, if the ends of the magnet sectors are all measured from the centre of the machine, these lines represent the accuracy to which this length can be measured and all the sectors lie at random between the lines. The closed orbit looks very much like a betatron oscillation. As a resonance is approached, the amplitude of the oscillations increases and becomes infinite at the integral Q-value.

4. Oscillations About the Closed Orbit.

There is no guarantee that all the magnet sectors will have the correct n-value, or the same physical length. Variations in these values around the machine amplify betatron oscillations about the closed orbit. The linear theory predicts bands of instability of these oscillations near the main resonances, i.e., integral Q-values, and at the subresonances, i.e., half-integral Q-values. Between these unstable bands, called "stopbands", the betatron oscillation beats up and down between maximum and minimum values. Figure 2B shows the maximum and minimum values as a function of Q, under the assumption that injection occurs at a Q-value in the middle of the "safe" region between the stopbands (called the "working point") and then, for such reasons as magnet saturation, that "Q" varies adiabatically away from the working point. As the stopband edge is approached, the maximum amplitude of oscillation becomes infinitely large and the minimum amplitude goes to zero.

The betatron oscillations about the closed orbit, around a typical machine, are shown in the lower diagram. If a beam from a linac is injected symmetrically about the closed orbit, then, after a revolution or two, the beam diameter will have increased and...
subsequently it beats up and down. At the working point, defined above, the worst injection condition giving the maximum peak amplitude of betatron oscillation occurs when the beam is injected at a minimum of the beating. The linear theory then gives the ratio of the maximum amplitude of betatron oscillation to the injected amplitude. This ratio is called $K$, the beating factor $^*$.

5. The Maximum Displacement of a Particle from the Ideal Orbit.

When the beam cross-section is calculated and, from this, the size of a useful magnetic field large enough to contain it, allowance must be made for coupling between the vertical and horizontal betatron oscillations. The shaded circle in Figure 3A shows the cross-section of the injected beam. The point distance from both axes, $\hat{y}_c$, is the maximum value of the closed orbit displacement from the ideal orbit. A particle, starting at the origin, oscillates about the closed orbit with an amplitude $\sqrt{2} \hat{y}_c$ times the beating factor, $K$, previously mentioned. Due to coupling, the energy in the radial direction is assumed to be completely transferred to the vertical direction. Thus, in the diagram, the oscillating vector rotates until it is vertical and the peak displacement is, then,

$$y = \hat{y}_c + K (\sqrt{2} \hat{y}_c + ds/2)$$

If the exact position of the closed orbit were known at injection and did not vary thereafter, then the best method of injection would be to arrange for the linac beam to be symmetrical about the closed orbit as shown in Figure 3B. However, the constancy of the closed orbit with time is not yet known, and detailed methods of establishing its position at a given point around the orbit have not been worked out. So, for the moment, the method of addition shown

* Note: This factor, $K$, is identical to the beating factor, $F$, mentioned in G. Lüders' lecture.
in Figure 3A is used to determine the maximum displacement of the particles from the ideal orbit. The peak value of the displacement of the closed orbit, that has been used, is a statistical value and has been averaged over all machines. In order to include, say, over 95% of all the machines that could be built inside the tolerances, we find that twice the RMS value of the peak displacement must be used. Another way of expressing this point is that there is no guarantee that CERN will build an RMS machine.

6. Machines with Different Field Indices, "n".

The results of calculations made on a simple preliminary machine, designed to meet the CERN specifications, are shown graphically in Figure 4. In the vertical direction, it is seen that the maximum displacement is very little affected by changing "n", but in the radial direction, because of synchrotron oscillations and frequency errors, the maximum displacement increases rapidly at low n-values. It should also be noted that variations in the remanent field, between the magnet blocks, become more serious as "n" is reduced. From these results, an n-value of 400 was chosen and an area of useful magnetic field, ± 4 cm high by ± 6 cm wide. The vacuum-chamber walls are shown dotted in the diagram.

7. The Periodic Structure of the Magnet.

There are various requirements that have to be met by whatever arrangement of magnet sectors is finally chosen for the synchrotron. They are:

a) The linear theory shows that it would be better to avoid making the radial and vertical betatron oscillation frequencies equal. Consideration of the pattern of the resonant lines on the stability diagram shows that the separation should be \( q_R - q_V = 1, 2, 3 \) etc.
b) To set up the stability diagram correctly, with reference to the actual n-value of the synchrotron magnets, and to compensate for subsequent changes in these n-values, a system of lenses must be included that can move the working diamond with respect to the working point in the simplest possible manner.

c) There must be an adequate number of field-free sectors of sufficient length, occurring around the orbit, to house the r-f accelerating gaps, the lenses, and the pickup electrodes for beam steering near the transition energy.

d) Lastly, the chosen arrangement of sectors should be as insensitive as possible to mechanical misalignments.

A structure which satisfies these conditions is as follows. The focusing and defocusing sectors are each divided into two parts and straight sections placed between the parts so that the structure is $S - \frac{1}{2}F - \frac{1}{2}D - S - \frac{1}{2}D - \frac{1}{2}F - S - \frac{1}{2}F - \frac{1}{2}D$ etc, where F and D are the focus and defocus sectors, respectively, and S are the straight sectors in which the correcting lenses and r-f gaps are located. The difference in radial and vertical betatron frequencies is obtained by making the focusing sectors 2.6% larger than the defocusing sectors. Each half focusing sector is rigidly clamped to a half defocusing sector to form a unit. This system is less sensitive to mechanical misalignments than one with whole focusing and defocusing sectors as separate units. In Figure 5, the small diagram shows the structure and the position of the correcting lenses $L_1$ and $L_2$. The stability diagram is drawn with the lenses turned off but thicker lines have been drawn at $Q = 9.5$ and 19 cycles, corresponding to the stopbands introduced when the lenses are energised. It is the variation of the width of these stopbands, with lens excitation, that moves the whole pattern of lines on the diagram and so moves the working diamond with respect to the working point.
The working diamond is contained in the little black diamond marked on the diagram. Figure 6 shows this region of the diagram greatly magnified. The working point, drawn as a large dot, is at $n_1 = r_2 = 392$. The betatron frequencies at this point are $Q_v = 6.75$ and $Q_R = 7.75$. The working diamond surrounds this working point. The unstable lines due to twist distortion are shown to bisect those diamonds where the two frequencies differ by odd multiples of $\frac{1}{2}$.

8. Effect of the Lenses.

The steering power of the lenses is shown in Figure 7, where the displacement of the working diamond is shown for four combinations of a given lens power. If both lenses are energised in the same sense, the diamond is displaced sideways along the $\Delta n_1 = -\Delta n_2$ line. If the lenses are energised in opposition, the diamond moves along the $\Delta n_1 = \Delta n_2$ line. The shift of the diamond is not symmetrical along the $\Delta n_1 = \Delta n_2$ line for the same lens power, since there is a product term, in $L_1 L_2$, which remains unaltered as the lenses are reversed. Apart from the initial setting up, the lenses enable the working diamond to follow the working point if, for example, "n" changes as the magnet saturates. The amount of control shown in Fig. 7 is obtained by lenses 50 cm long, with a maximum gradient of 360 gauss/cm, that is, a maximum field at the edge of the vacuum chamber of 2160 gauss.


Figure 8 shows the physical dimensions of the structure, just discussed. The top strip is a scaled plan drawing of part of the periphery of the machine. There are 57 complete magnet periods consisting of $(S-\frac{1}{2}F.\frac{1}{2}D-S-\frac{1}{2}D.\frac{1}{2}F)$ sectors. In each main period, consisting of three magnet periods, there are two lenses: one between the focusing sectors and one between the defocusing sectors. Therefore, there are 38 lenses in all, in two sets of 19. In each main period there are two r-f accelerating gaps, i.e., a total of 38 gaps in all. It is planned
that all these gaps shall work in phase, on the 38th harmonic of the orbital frequency. The lower diagrams, in Fig. 8, show the magnet cross-section and a form of four-pole lens.


One must now consider whether the working point is a simple point, or something more complicated. The two effects now mentioned were brought to my notice by R. Q. Twiss.

The synchrotron oscillations are oscillations in momentum of the particles about the equilibrium momentum. It can be shown that, to particles with a momentum different from the equilibrium momentum, the magnetic field presents an \( n \)-value different from that applied to the equilibrium particle. The modification to \( n \) is,

\[
\Delta (p/p) n z = 0 \quad \text{i.e.} \quad \Delta n = n_0 \Delta p/p
\]

Both the focusing and defocusing magnets have their \( n \)-values changed so that the working point is, in fact, a line with extension in the \( \Delta n_1 = \Delta n_2 \) direction. This effect is illustrated in Figure 9A, where the correct extension is drawn for the machine under consideration.

When a beam of particles is injected into the synchrotron, there will be space-charge forces that tend to spread the beam. This is equivalent to an increase in the defocusing forces in the defocusing sectors, and a decrease in the focusing forces in the focusing sectors. Both the radial and vertical periodicities are modified in the same manner by the space-charge effect so that \( \mu_r \) and \( \mu_v \) vary together and in the same direction as the beam intensity is increased. The net effect on the working point is to move it along the \( n_1 = n_2 \) line.

*Note: This effect was incorrectly presented during the talk as a sideways movement of the working point in the diamond i.e. along the \( \Delta n_1 = -\Delta n_2 \) line.
For an injected current of 3 milliamp and an injection energy of 50 Mev, the working point moves to the end of the diamond, as is shown in Figure 9A. If the injection energy is reduced to 25 Mev, the working point moves outside the diamond. The lenses can be used, first, to offset the diamond over the working point and, then, to bring the diamond back to the n = 392 point when the working point returns as the particles gain energy.

It will be noticed that a smaller diamond has been drawn inside the working diamond. This smaller diamond allows the n-value of the different magnet sectors to have a random variation in "n" between the limits ± 1% of n. In this case, over 95% of the possible machines built to these tolerances, will have very little increase in betatron amplitudes over the area of the smaller diamond.


Gas scattering increase the beam cross-section from the time of injection to a time when the particles have about twice the injection energy. After this time, the damping due to the rising magnetic field gives a gradual reduction of this increase. The peak value of the amplitude due to gas scattering is covered by various papers by Courant and Blachman. The peak value must be compared with the amplitude of betatron oscillation at twice the injection energy, where the peak occurs. Then the correct damping factor must be applied to see whether there is a net increase in the beam cross-section at any energy. Figure 9B shows that, for an air pressure of $10^{-5}$ mm Hg, and an injection energy of 50 Mev, gas-scattering effects can be ignored.

12. Table of Parameters.

Figure 10 shows the displacement of the particles away from the ideal orbit as a function of the field index, "n". As was found for the preliminary machine, there is no advantage to be gained in having "n" higher than 400. For values of "n" lower than 400, the
synchrotron oscillations require a wider magnetic field and vacuum chamber.

The frequency error at injection gives only a trivial increase in the radial displacement, for an easy tolerance of 1 in $10^3$. As the transition energy is approached, however, the tolerance becomes very tight to keep the beam within 1 cm of the ideal orbit. For example, at 200 Mev away from the transition energy, the frequency error for a 1-cm movement is of the order of a few parts in $10^6$. It is in this region that the pickup electrodes must detect shifts in the mean position of the beam and circuits must apply these as frequency corrections to the r-f accelerating system.

Other parameters that have been assumed and used in this talk are collected together in the table given in Fig. 10. The synchrotron oscillation frequency is rather high. A particle, executing a synchrotron oscillation, takes 12.4 revolutions around the machine to complete one cycle. Thus, there is the possibility that the working diamond might have a fine structure of other unstable lines across the "safe" area. In this case, there would be about six such lines in number, arising from perturbations that occur every twelve revolutions.

In the column dealing with injection parameters, the magnetic field at injection is seen to be 119 gauss even for the relatively high injection energy of 50 Mev. This is a further justification for assuming a high injection energy. In the Cosmotron, at injection, the field is 300 gauss.


The small signal effect of a nonlinear magnetic field has been discussed by M.C.N. Hine. A particle, whose closed orbit is displaced away from the ideal orbit during a synchrotron oscillation in a magnetic field quadratically dependent on the displacement, runs through a magnet structure in which "$n" is linearly dependent on displacement but of opposite sign in the focusing and defocusing sectors. The working point consequently moves off sideways along the $\Delta n_1 = -\Delta n_2$ line.
As the particle is executing a synchrotron oscillation, the working point is also moving up and down the $\Delta n_1 = \Delta n_2$ line, as already described. The resultant motion is the vectorial sum of these two motions. A particle, undergoing synchrotron oscillations in a magnet field that is cubically dependent on displacement, runs through a magnet structure in which "n" is quadratically dependent on displacement. But, in this case, "n" decreases with displacement in both the focusing and defocusing sectors, thus, the working point moves off along the $\Delta n_1 = \Delta n_2$ line and the motion just adds to the normal synchrotron oscillation sweep.

These effects are illustrated in Figure 11. The assumptions contained in this figure must be emphasized. The spread of the working point has been drawn for a change in "n" of $\frac{1}{2} \%$ at the maximum radius of the synchrotron oscillation swing. It can be seen that, with this change in "n", the working point is conveniently contained inside the working diamond; this is the reason for the choice of $\frac{1}{2} \%$. Errors in the applied radio frequency also give a radial displacement of the equilibrium orbit and, consequently, have a similar effect on the working point but, for simplicity, they have been ignored in the diagram.

Whereas it can be argued that frequency errors can be limited by correcting devices, only an increase in "n" can reduce the synchrotron-oscillation amplitude. Therefore, the latter effect has been presented in the diagram since it is unavoidable. The synchrotron oscillation amplitudes are those at injection and, at this time, the frequency errors should be small. At higher energies, damping reduces the synchrotron amplitudes but frequency errors become more important, especially at the transition energy.

At what stage the nonlinearities in the magnetic field are more important, at injection when eddy-current and remanent field effects are predominant, or at high energies when the magnet saturates is, as yet, unknown. The field tolerances on the nonlinearities, calculated for the maximum permissible radial excursion of a phase-stable particle at injection are as follows:
a) Quadratic fields \( \frac{\Delta B}{B} = 0.004\% \text{ at } 0.7 \text{ cms } = 0.14\% \text{ at } 4.0 \text{ cms} \)

\( \frac{\Delta n}{n} = 0.25\% \text{ at } 0.7 \text{ cms } = 1.4\% \text{ at } 4.0 \text{ cms} \)

b) Cubic fields \( \frac{\Delta B}{B} = 0.007\% \text{ at } 0.7 \text{ cms } = 1.46\% \text{ at } 4.0 \text{ cms} \)

\( \frac{\Delta n}{n} = 1.0\% \text{ at } 0.7 \text{ cms} \)

These tolerances are obviously very tight. During the injection period, "n" must be kept constant over ± 1 cm aperture to within about ± \( \frac{1}{4}\% \). During the rest of the accelerating period, frequency errors must not shift the mean position of the closed orbit more than ± 0.5 cms away from the ideal orbit and, over this region, "n" must be held constant to within the same tolerance of ± \( \frac{1}{4}\% \).

14. Conclusion.

In the introduction to this talk, the remark was made that the machine, to be described, was the latest product in an evolutionary process. What do we learn from this machine design and how would we alter it to ease the tolerances? It is probable that "n" could be further reduced and the increased width of the magnet aperture offset against the better pole geometry. It is unlikely that the magnet weight would be affected by this change. As more detailed information on the behaviour of the oscillating particle becomes available, and methods of injecting the beam on the closed orbit are discovered and perfected, it will be possible to reduce the displacement of the particles away from the ideal circle and, thus, materially reduce the magnet aperture. Since this reduces the weight of iron and the cost of the most expensive piece of equipment on the project, it is clearly the most important subject to work on, in the future.
### ORBIT PARAMETERS

<table>
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<tr>
<th></th>
<th>TOLERANCE</th>
<th>$n = 100$</th>
<th>$n = 392$</th>
<th>$n = 900$</th>
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<tr>
<td>BEAM RADIUS</td>
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<td>0.80</td>
<td>0.57</td>
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<td>CLOSED ORBIT DISPLACEMENT</td>
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<td>2.83</td>
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<tr>
<td>SYN: OSC: : AMP: @ INJECTION</td>
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<td>0.68</td>
<td>0.30</td>
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<tr>
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<td>$n = 100$</td>
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<td>MAX: RADIAL DISPLACEMENT</td>
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<td>3.42</td>
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### GENERAL PARAMETERS

- FINAL KINETIC ENERGY: $30.10^9$ eV
- RADIUS OF SECTORS: 3600 cm
- MEAN RADIUS: $1200$ cm
- MAX FIELD, $B_\text{max}$: 12,000 gauss
- RISE OF FIELD $dB/dt$: 12 Kgauss/sec

### INJECTION PARAMETERS

- KINETIC ENERGY: $50.10^6$ eV
- MAGNETIC FIELD, $B_0$: 119 gauss
- LENGTH OF PULSE: $7.5\times10^{-6}$ sec
- BEAM CURRENT: 3 mA

### RADIO FREQUENCY

- FREQ: RANGE MIN: $0.13$, MAX: $4.94$ MHz
- HARMONIC NO: 38
- VOLTS PER TURN: $1.57 \times 10^5$ V
- VOLTS PER GAP: $4.12 \times 10^6$ V

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**Vacuum Chamber**

**Figure 10**

**Quadratic Field Error**

**Figure 11**

**Cubic Field Error**