1989 CERN–JINR SCHOOL OF PHYSICS

Egmond-aan-Zee
25 June–8 July 1989

PROCEEDINGS
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ABSTRACT

The CERN School of Physics is intended to give young experimental physicists an introduction to the theoretical aspects of recent advances in elementary particle physics. These Proceedings contain reports of lecture series on the following topics: gauge theories and applications, precision tests of the electroweak theory (in two parts), physics beyond the Standard Model, and the experimental programme at the IHEP accelerator.
PREFACE

The 1989 CERN-JINR School of Physics was held from 25 June to 8 July in the small seaside resort of Egmond-aan-Zee, in the Netherlands. It was attended by 104 students; 60 students came from CERN Member States, 38 students from JINR Member States, and six from other countries.

Our sincere thanks are due to the lecturers and discussion leaders for their active participation in the School and for making the scientific programme so stimulating. The high attendance at the lectures in spite of the warm and sunny weather testifies to the excellence of their work.

The School was organized in conjunction with the National Institute for Nuclear Physics and High Energy Physics (NIKHEF) Amsterdam, and we are indebted to the Institute for the strong support, both financial and material, that it provided for the School. Funds were also made available by the Netherlands Academy of Sciences and the Foundation for Fundamental Research on Matter (FOM). North Holland Publishing (Elsevier) generously offered the welcoming cocktail.

Our warmest thanks are extended to Professor W. Hoogland (NIKHEF/CERN) who, as Director of the School, ensured the smooth running of all the essential practical details of the day-to-day organization. Our particular thanks go also to Ms Rita van der Struy and her colleagues at NIKHEF for their willing help, including the loan of the library books and office equipment. Professor Hoogland was also ably assisted by his colleagues on the Organizing Committee. Lastly, the two Organizing Secretaries, Mrs. T. Donskova (JINR) and Miss S.M. Tracy (CERN), efficiently co-ordinated all the preparations for the School.

The School was held in the Hotel 't Zuiderduin, Egmond-aan-Zee, and our thanks go to the Managers, Mr and Mrs Groot and their staff for making us welcome and comfortable. Dinner by candlelight was an attractive setting for some lively talk; and the swimming pool and sauna proved a popular rendezvous.

A varied social programme was organized, which included visits to Haarlem, and to Enkhuizen and Amsterdam. A courageous few went to the Wierum nature reserve in Friesland, ably guided by their Dutch colleagues. The chess tournament with Mr. Kuyl, Dutch Chess Champion, provided a memorable evening's entertainment. Attended by members of the Organizing Committee and old friends of CERN, the School ended with a farewell barbecue, rescued from a thunderstorm with military precision, and followed by some lively singing and dancing.

W.O. Lock

on behalf of the CERN Organizing Committee
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Participants at the 1989 CERN-JINR School of Physics outside the Hotel "Zuiderduin, Egmond-an-Zee
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GAUGE THEORIES AND APPLICATIONS

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The purpose of these lectures is to give a practical introduction to field theory, in particular gauge theory, and the use of Feynman diagrams. Of course, it is really not possible to give you a self-contained and thorough treatment of these topics in just six lectures, but the hope is that, also with the help of the discussion sessions, you will get acquainted with the basic principles and ideas of gauge fields and acquire some experience in Feynman diagram calculations. The latter will be put to a test in Hollik’s lectures on precision tests of the electroweak theory later on in this school.

As you can see from the above table of contents, we will start with generic field theories and discuss the rules needed for the calculation of Feynman diagrams. After some applications we proceed to introduce simple abelian gauge theories and explain their properties. Then we consider the nonabelian version of these theories and discuss the ingredients that are necessary for the standard model.
1. The action

Field theories are usually defined in terms of a Lagrangian, or an action. The action, which has the dimension of Planck’s constant \( \hbar \), and the Lagrangian are well-known concepts in classical mechanics. For instance, for a point-particle subject to a conservative force

\[
F = -\frac{\partial V(r)}{\partial r},
\]

the Lagrangian is defined as the difference of the kinetic and the potential energy,

\[
L(r, \dot{r}) = \frac{1}{2}m \dot{r}^2 - V(r).
\]

Consider now some particle trajectory \( r(t) \), which does not necessarily satisfy the equation of motion, with fixed endpoints given by

\[
r_1 = r(t_1), \quad r_2 = r(t_2).
\]

The action corresponding to this trajectory is then defined as

\[
S[r(t)] = \int_{t_1}^{t_2} dt \, L(r(t), \dot{r}(t)).
\]

For each trajectory satisfying the boundary conditions (1.3) the action defines a number. According to Hamilton’s principle, the extremum of (1.4) (usually a minimum) is acquired for those trajectories that satisfy the equation of motion,

\[
m \ddot{r} = -\frac{\partial V(r)}{\partial r}.
\]

This is precisely Newton’s law.

Let us now generalize to a field theory. The system is now described in terms of fields, say, \( \phi(x) \), which are functions of the four-vector of space-time,

\[
x^\mu = (x_0, \mathbf{x}),
\]

where \( x_0 \equiv ct \) and \( c \) is the velocity of light (henceforth, we will use units such that \( c = 1 \)). Fields such as \( \phi \) may be used to describe the degrees of freedom of certain physical systems. For instance, they could describe the local displacement in a continuous medium like a violin string or the surface of a drum, or some force field such as an electric or a magnetic field.

Again one can define the action, which can now be written as the integral over space-time of the Lagrangian density (which is also commonly referred to as the Lagrangian),

\[
S[\phi(x)] = \int d^4x \, L(\phi(x), \partial_\mu \phi(x)),
\]

where the values of the fields at the boundary of the integration domain are fixed just as we did for the particle trajectory in (1.3). As indicated in (1.7), the Lagrangian is usually a function of the fields and their first-order derivatives. The action thus assigns a number to every field
configuration, and it is again possible to invoke Hamilton's principle and verify that the action has an extremum for fields that satisfy the classical equations of motion.

Let us now discuss a few examples of field theories that one encounters in particle physics. The simplest theory is that of a single scalar field $\phi(x)$. This field is called a scalar field because it transforms trivially under Lorentz transformations: $\phi'(x') = \phi(x)$, where $x'$ and $x$ are related by a Lorentz transformation. It can be used to describe spinless particles. A standard Lagrangian is

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \lambda \phi^3 - g \phi^4,$$

(1.8)

where

$$(\partial_\mu \phi)^2 = \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( \frac{\partial \phi}{\partial x_0} \right)^2.$$  

(1.9)

The quadratic part of (1.8) is called the Klein-Gordon Lagrangian, and the corresponding field equation the Klein-Gordon equation. Equation (1.9) shows that the time-derivatives appear with positive sign in the Lagrangian, just as in (1.2). Observe that we have introduced the Lorentz-invariant inner product of two four-vectors, defined by

$$x \cdot y = x_\mu y^\mu = x^\mu y_\mu = x \cdot y_0 = x_0 y_0.$$  

(1.10)

where four-vector indices are lowered (raised) with a metric $\eta_{\mu\nu}$ ($\eta^{\mu\nu}$), with $\mu, \nu = 0, 1, 2, 3$, which is a diagonal matrix with eigenvalues $(-, +, +, +)$. In the literature also a metric with opposite sign is used. Alternatively, we may use indices $\mu, \nu = 1, 2, 3, 4$ and define $x_4 \equiv i x_0$. In that case there is no difference between upper and lower indices and we do not need a metric in order to contract four-vectors. This convention is convenient when dealing with gamma matrices, to be introduced shortly.

Later we will consider the fields in the momentum representation, defined by the Fourier transform

$$\phi(x) = \int d^4k e^{ik \cdot x} \phi(k).$$

(1.11)

The inverse of this relation is

$$\phi(k) = (2\pi)^{-4} \int d^4x e^{-ik \cdot x} \phi(x).$$

(1.12)

For real fields, as in (1.8), the fields in the momentum representation satisfy the condition

$$\phi^*(k) = \phi(-k).$$

(1.13)

For complex fields there is no such condition. Complex fields are convenient if the theory is invariant under phase transformations. For instance, the Lagrangian

$$\mathcal{L} = -|\partial_\mu \phi|^2 - m^2 |\phi|^2 - g |\phi|^4$$

(1.14)

is invariant under

$$\phi \rightarrow \phi' = e^{i\xi} \phi,$$

(1.15)

with $\xi$ an arbitrary parameter. Of course, it is only a matter of convenience to use complex fields. One always has the option to decompose a complex field into its real and its imaginary
part, and to write the Lagrangian (1.14) in terms of two real fields. The invariance (1.15) then takes the form of a rotation between these two real fields. Observe the normalization factors in (1.14) which differ from those in (1.8).

In principle, it is only a small step to consider Lagrangians for fields that transform nontrivially under the Lorentz group. For instance, one has spinor fields (which transform as spinors under the Lorentz transformation), which describe the fermions. In spite of the fact that they are extremely important, we will largely ignore the spinor fields due to lack of time. However, in the tutorial sessions we will discuss several applications with fermions. Here we give a typical Lagrangian for spin-$\frac{1}{2}$ fermions interacting with a scalar and a pseudoscalar field, $\phi_s$ and $\phi_p$, respectively,

$$\mathcal{L} = -\overline{\psi} \gamma \psi - m \overline{\psi} \gamma \psi + G_s \phi_s \overline{\psi} \gamma \psi + iG_p \phi_p \overline{\psi} \gamma \gamma \psi,$$

(1.16)

Such couplings of the (pseudo)scalar fields to fermions are called Yukawa couplings. The quadratic terms in (1.16) constitute the Dirac Lagrangian. The corresponding field equation is the Dirac equation. The spinor fields have 4 independent components.* An important ingredient are the Dirac gamma matrices $\gamma^\mu$, which are $4 \times 4$ matrices satisfying the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} 1. \quad (\mu, \nu = 0, 1, 2, 3)$$

(1.17)

We should caution you that that exist many different conventions for gamma matrices and spinors in the literature. Adopting (1.17) as a starting point, it is convenient to define $\gamma^4 \equiv i\gamma^0$, so that all gamma matrices can be chosen hermitean,

$$(\gamma^\mu)^T = \gamma^\mu. \quad (\mu = 1, 2, 3, 4)$$

(1.18)

In this notation there is no difference between upper and lower four-vector indices. The conjugate spinor field $\overline{\psi}$ is then defined by

$$\overline{\psi} = \psi^\dagger \gamma_4, \quad \text{or, in components,} \quad \overline{\psi}_\alpha = \psi_\beta^\dagger (\gamma_4)_{\beta\alpha}.$$  

(1.19)

Another Lagrangian which is relevant is based on vector fields (i.e., fields that transform as vectors under Lorentz transformations). For the description of massive spin-1 particles one uses the Proca Lagrangian,

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{2} M^2 V_\mu^2.$$  

(1.20)

Modulo a total divergence, which we can drop because it contributes only a boundary term to the action (this follows from applying Gauss' law), this Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu V_\nu)^2 + \frac{1}{2} (\partial_\mu V^\nu)^2 - \frac{1}{2} M^2 V_\mu^2.$$  

(1.21)

The first and the last term in (1.21) are obvious generalization of the first two terms of (1.8). The second term is required, with precisely the coefficient $\frac{1}{2}$, in order that the Lagrangian describes

* The fact that in a four-dimensional space-time, spinors have also four components, should be regarded as a coincidence. In a $d$-dimensional space-time spinors have in general $2^d$ components.
pure spin-1 particles, and no additional spinless particles. From a simple counting argument one can already see that some care is required here. Massive particles with spin $s$ have in general $2s + 1$ independent polarizations. So a Lagrangian for spin-1 particles should give rise to 3 independent polarization states, whereas the field $V_\mu$ on which the Proca Lagrangian is based has 4 independent components. It is this discrepancy which forces us to include the second term in (1.21). We have implicitly assumed that $V_\mu$ is a real field, i.e. $V_\mu^* = V_\mu$, but it is perfectly possible to extend (1.20-21) to complex fields, analogous to what we did previously for scalar fields.

The $M \to 0$ limit of (1.20) describes massless spin-1 particles such as photons, and will play an important role in these lectures. The Lagrangian is called the Maxwell Lagrangian, and reads

$$\mathcal{L} = -\frac{i}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$
$$= -\frac{1}{4} (\partial_\mu A_\nu)^2 + \frac{1}{2} (\partial_\mu A_\mu)^2,$$  
(1.22)

where we have again suppressed a total divergence in the second line. Massless particles with spin have precisely 2 independent polarizations, irrespective of the value of the spin. We will discuss this in section 3. An important ingredient in the proof of this is the invariance of (1.22) under so-called gauge transformations,

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x),$$  
(1.23)

where $\xi(x)$ is an arbitrary function of $x$.

In relativistic quantum field theory it is convenient to use units such that the velocity of light in vacuum and Planck's constant are dimensionless and equal to unity: $c = \hbar = 1$. With this convention there is only one dimensional unit; for example, one may choose length, in which case mass parameters have dimension $[\text{length}]^{-1}$, or mass can be adopted as the basic unit so that length and time have dimension $[\text{mass}]^{-1}$. The action is then dimensionless, so that Lagrangians have dimension $[\text{mass}]^1$. It is then easy to see that scalar and vector fields have dimension $[\text{mass}]^1$, whereas spinor fields have dimension $[\text{mass}]^{3/2}$. Observe that all parameters that we have introduced in the above Lagrangians have positive mass dimension. This fact is important for the quantum mechanical properties of these theories. If quantum field theories have parameters with negative mass dimension, then the theory is not renormalizable. Usually this implies that the theory does not lead to sensible predictions. An example of such a theory is Einstein's theory of gravitation, general relativity, which is very successful as a classical field theory, but cannot be quantized consistently.

Let us end this section by defining the field equations corresponding to a given Lagrangian. As we have already mentioned above, Hamilton's principle implies that the field configurations for which the action has an extremum, must satisfy the equations of motion. These field equations are the so-called Euler-Lagrange equations. For a general Lagrangian $\mathcal{L}$, defined in terms of fields $\phi$ and first-order derivatives of fields $\partial_\mu \phi$ only, these equations read

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$  
(1.24)

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\textbf{Problem 1:}  
Verify that the field equations corresponding to the Lagrangians (1.8), (1.14), (1.16) and (1.20-21) are given by

\begin{align*}
(\partial^2 - m^2)\phi &= 3\lambda \phi^3 + 4g \phi^3, \quad (1.25) \\
(\partial^2 - m^2)\psi &= 2g |\phi|^2 \phi, \quad (1.26a) \\
(\partial^2 - m^2)\phi^* &= 2g |\phi|^2 \phi^*, \quad (1.26b) \\
(\bar{\theta} + m)\psi &= G_s \phi_s \psi + iG_p \phi_p \gamma_5 \psi, \quad (1.27a) \\
\bar{\psi}(\bar{\theta} + m) &= G_s \phi_s \bar{\psi} + iG_p \phi_p \bar{\psi} \gamma_5, \quad (1.27b) \\
\partial^\nu (\partial_\nu V_\nu - \partial_\nu V_\mu) + M^2 V_\mu &= 0, \quad (1.28)
\end{align*}

where $\partial^2 \equiv \partial^\mu \partial_\mu$.

\textbf{Problem 2:}  
Add an extra source term $J^\mu A_\mu$ to the Lagrangian (1.22) and show that the corresponding field equations coincide with the inhomogeneous Maxwell equations (in the relativistic formulation)

$$\partial_\nu F^{\mu\nu} = J^\mu, \quad (1.29)$$

where the electromagnetic fields $F_{\mu\nu}$ are defined by $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

\textbf{Problem 3:}  
Consider plane wave solutions, i.e., solutions proportional to $\exp(ik \cdot x)$, for the free field equations found in problem 1. Find the conditions for the momentum $k_\mu$. Verify that the Proca Lagrangian gives rise to only three independent polarizations.

2. \textbf{Feynman rules}  
In the previous section we have presented field theories in terms of an action or a Lagrangian. Such theories can be studied as classical field theories, and this is often done in perturbation theory. Ultimately we are interested in the quantum mechanical scattering amplitudes for elementary particles. Those amplitudes can also be evaluated in perturbation theory, for which there exists a convenient graphical representation in terms of so-called Feynman diagrams. Some of these Feynman diagrams will correspond to the same contributions that one would find for a classical field theory. Such diagrams have the structure of tree diagrams. This in contradistinction with diagrams that contain closed loops. Their contributions do not follow from classical field theory, but can only be understood within the context of quantum field theory.

In these lectures there will be no time to give a detailed derivation of the Feynman diagrams. We shall just define the Feynman rules, which tell you how to evaluate the complicated mathematical expressions corresponding to a Feynman diagram. We refer to the literature for explicit derivations. The rules are presented in a number of steps:

- \textbf{The theory:} Begin with a field theory defined in terms of an action, which is expressed as an integral over space-time of a Lagrangian.
Fig. 1. Propagator line

• **Propagators:** Calculate the propagators of the theory, which follow from the terms in the action that are quadratic in the fields. The quadratic terms define a matrix in momentum space which is diagonal in the momentum variables. Suppose we take the Lagrangian (1.8) as an example. The action is

\[
S = \int d^4x \mathcal{L} = \int d^4x \left[ -\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + O(\phi^3) \right] \tag{2.1}
\]

Now express the action in terms of the Fourier transforms of the fields. The terms quadratic in the fields are then equal to

\[
S = -\frac{i}{2}(2\pi)^4 \int d^4k \, \phi^*(k) [k^2 + m^2] \phi(k), \tag{2.2}
\]

where we have made use of the fact that we are dealing with real fields (i.e. \( \phi^*(k) = \phi(-k) \)). Hence the elements of the diagonal matrix are just equal to \(-\frac{i}{2}(2\pi)^4[k^2 + m^2]\). For real fields the propagator is defined as a factor \(\frac{1}{2}i\) times the inverse of this matrix. For the case at hand we thus find

\[
\Delta(k) = \frac{1}{i(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon}. \tag{2.3}
\]

Its graphical representation is a line, with an arrow indicating the momentum flow, while the endpoints refer to two space-time points (see Fig. 1). The \(i\epsilon\)-term defines how to deal with the pole at \(k^2 = -m^2\); the limit \(\epsilon \downarrow 0\) should only be taken at the end of the calculations. This prescription for dealing with the propagator poles is crucial for the causality and the unitarity (i.e. probability conservation) of the resulting theory.

We have already pointed out that the normalization factors are different for complex fields. In that case the kinetic terms in the action are

\[
S = \int d^4x \mathcal{L} = \int d^4x \left[ -|\partial_\mu \phi|^2 - m^2 |\phi|^2 + O(|\phi|^4) \right], \tag{2.4}
\]

which, in terms of the Fourier transforms of the fields, leads to (we no longer have \(\phi^*(-k) = \phi(-k)\))

\[
S = -(2\pi)^4 \int d^4k \, \phi^*(k) [k^2 + m^2] \phi(k). \tag{2.5}
\]

The propagator is now defined by the inverse of \(-(2\pi)^4[k^2 + m^2]\) multiplied by a factor \(i\). This leads to the same diagram as for real fields, but now the arrow also indicates that the propagator
is oriented in the sense that the endpoints of the propagator lines refer to independent fields, namely \( \phi \) and \( \phi^* \); the standard convention is that incoming arrows refer to \( \phi \), and outgoing ones to \( \phi^* \). Of course, complex fields can always be regarded as a linear combination of two real fields, and by decomposing \( \phi = \frac{1}{2} \sqrt{2}(\phi_1 + i\phi_2) \) one makes contact with the description given for real fields.

- **Vertices:** The next step is to define the vertices of the graphs. We associate a vertex with \( n \) lines with every term in the Lagrangian that contains \( n \) field. A Lagrangian with a \( \phi^3 \)-term thus yields a vertex with three lines. Translational invariance ensures that the Fourier transform yields a delta-function in momentum space, thus guaranteeing energy-momentum conservation. Each vertex therefore has the structure

\[
\text{vertex} = i(2\pi)^4 \delta^4(\sum_j k_j) \times (\text{coefficient of } \phi^n \text{ in the Lagrangian}) ,
\]

where our conventions are such that the \( k_j \) denote incoming momenta associated with each of the fields. For example, the Feynman rules for the theory described by the Lagrangian (1.8) are summarized in Table 1.

If the vertices in the Lagrangian contain derivatives then each differentiation of the fields contributes a factor \( ik_j \) to the vertex where \( k_j \) is the incoming momentum of the \( j \)th line. These momentum factors are part of the coefficient indicated in the generic definition (2.6). Thus the terms \( g\phi^3 \) and \( g\phi(\partial_\mu\phi)^2 \) both correspond to three-point vertices, but yield different factors: \( i(2\pi)^4 g \delta^4(k_1 + k_2 + k_3) \) and \( i(2\pi)^4 g (-k_2 \cdot k_3) \delta^4(k_1 + k_2 + k_3) \), respectively. In the latter case, choosing the second and the third momentum as those corresponding to the differentiated fields, is arbitrary. A complete calculation must also include other possible line attachments, so that also factors \( (-k_1 \cdot k_3) \) and \( (-k_1 \cdot k_2) \) will contribute. The way in which these contributions must be summed will be discussed next, but it is rather obvious in this case that the total contribution of the second interaction becomes proportional to \( (k_1 \cdot k_2 + k_2 \cdot k_3 + k_3 \cdot k_1) \).

In the Feynman diagrams for complex fields the lines at the vertices carry an orientation; recall that fields correspond to lines with incoming arrows and their complex conjugates to lines.
with outgoing arrows. A formulation in terms of complex rather than real fields is useful if the theory is invariant under phase transformations, i.e.

\[ \phi \rightarrow \phi' = e^{i\alpha} \phi. \]

In that case every interaction must contain an equal number of fields \( \phi \) and their complex conjugates \( \phi^\ast \), so that each vertex has an equal number of incoming and outgoing lines. The lines coming from the vertices can now only be joined if their orientational arrows match (the orientation often corresponds to the flow of electric charge; obviously, charge will be conserved if the number of incoming and outgoing arrows is the same at each vertex).

- **Diagrams:** One now joins all the lines emanating from the vertices via propagators in order to form the various diagrams. The momentum flow through the various lines is determined by the momentum-conserving \( \delta \)-functions at the vertices, and for real fields one may readjust the arrows in order to reflect this fact. If the arrow denotes more than just the momentum assignment of the line, but also the orientation (e.g. charge flow) then the lines cannot always be joined, and the number of possible diagrams will be reduced.

- **Summing and combinatorics:** Finally one sums over all possible diagrams with the same configuration of external lines. In order to do so one must determine the combinatorial weight factor associated with each of the diagrams. In principle this weight factor counts the number of ways in which a diagram can be formed by connecting vertices to propagators and external lines, but diagrams that only differ in the position of the vertices are counted as identical (because we ultimately integrate over all vertex positions in space-time).

   There is only one exception to the above counting argument. If identical vertices occur in an indistinguishable way, i.e. not distinguished by their attachments to external lines, then one must avoid overcounting by dividing by \( n! \), where \( n \) is the number of such indistinguishable vertices.

   These rules also apply to diagrams with closed loops. However, in this case not all the momenta of the internal lines are fixed by momentum conservation, and one is left with one or more unrestricted momenta over which one should integrate. Likewise, we must also sum over all types of internal lines that are possible. Therefore, we have to sum over all components of vector and spinor fields that are possible.

   From these rules it is in principle straightforward to write down Feynman diagrams and their corresponding mathematical expressions. To get acquainted with their use, I recommend that you start from a simple Lagrangian, such as (1.8), and calculate some diagrams. The following problems contain a few suggestions.

*Problem 4:*

Consider the tree diagrams with four external lines that follow from the Lagrangian (1.8), and calculate the corresponding expressions. There are four Feynman diagrams, each giving rise to a characteristic dependence on the momenta associated with the external lines.
Problem 5:
Determine the three possible one-loop diagrams following from the Lagrangian (1.8) with two external lines. Write down the corresponding expressions and verify whether the momentum integrals are well defined.

3. Photons

We would now like to derive the Feynman rules for theories that involve massless spin-1 fields. In this section we refer to the particles described by these fields as photons. As it turns out there are a number of technical complications for theories with photons. Let us start by recalling the Lagrangian for massless spin-1 fields,

$$\mathcal{L} = -\frac{1}{4}(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})^2,$$

which is invariant under local gauge transformations

$$A_{\mu}(x) \rightarrow A_{\mu}(x) + \partial_{\mu} \xi(x).$$

This transformation is familiar from Maxwell's theory of electromagnetism where the vector potential is subject to the same transformations. The electromagnetic field strength is then equal to

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

The main consequence of an invariance under local gauge transformations is that the theory depends on a smaller number of fields. Correspondingly the number of plane wave solutions is also reduced in comparison to the massive case. To see this explicitly consider the field equation following from (3.1),

$$\partial^{\nu}(\partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}) = 0.$$ (3.4)

In order to examine plane wave solutions of this equation we take the Fourier transform of $A_{\mu}(x)$

$$A_{\mu}(k) = (2\pi)^{-4} \int d^4 x A_{\mu}(x) e^{-ikx}.$$ (3.5)

Under gauge transformations $A_{\mu}(k)$ changes by a vector proportional to $k_{\mu}$

$$A_{\mu}(k) \rightarrow A'_{\mu}(k) = A_{\mu}(k) + \xi(k) k_{\mu}.$$ (3.6)

The field equation (3.4) now takes the form

$$k^2 A_{\mu}(k) - k_{\mu} k^{\nu} A_{\nu}(k) = 0.$$ (3.7)

which is manifestly invariant under the transformation (3.6). Decomposing $A_{\mu}(k)$ into four independent vectors, $\varepsilon_{\mu}(k, \lambda), k_{\mu}$ and $\tilde{k}_{\mu}$, defined by

$$k_{\mu} \varepsilon_{\mu}(k, \lambda) = \varepsilon_0(k, \lambda) = 0, \quad (\lambda = 1, 2)$$

$$k_{\mu} = (k_0, k), \quad \tilde{k}_{\mu} = (-k_0, k),$$ (3.8)
we may write

$$A_\mu(k) = a^\lambda(k) \varepsilon_\mu(k, \lambda) + b(k) \bar{k}_\mu + c(k) k_\mu.$$  (3.9)

The field equation (3.7) then implies

$$k^2 a^\lambda(k) \varepsilon_\mu(k, \lambda) + b(k) \left[ k^2 \bar{k}_\mu - (k \cdot \bar{k}) k_\mu \right] = 0,$$  (3.10)

from which we infer for the coefficient functions (note that $k \cdot \bar{k}$ is positive)

$$k^2 a^\lambda(k) = 0 \quad \text{and} \quad b(k) = 0.$$  (3.11)

The field equation does not lead to any restriction on $c(k)$. This should not come as a surprise because $c(k)$ can be changed arbitrarily by a gauge transformation, whereas the field equation is gauge invariant. Consequently the field equation cannot fix the value of $c(k)$. By means of a gauge transformation we may adjust $c(k)$ to zero, which shows that $c(k)$ has no physical meaning. We thus find that there are only two independent plane wave solutions characterized by lightlike momenta ($k^2 = 0$) and transverse polarizations.

The fact that massless particles have fewer polarization states than massive ones, can also be understood as follows. For a massive spin-s particle one can always choose to work in the rest frame, where the four momentum of the particle remains unchanged under ordinary spatial rotations, so that its spin degrees of freedom transform according to a $(2s+1)$-dimensional representation of the rotation group $SO(3)$. In other words, there are $2s+1$ polarization states, which transform among themselves under rotations, and which can be distinguished in the standard way by specifying the value of the spin projected along a certain axis. However, for massless particles it is not possible to go to the rest frame and one is forced to restrict oneself to two-dimensional rotations around the direction of motion of the particle. These rotations constitute the group $SO(2)$ (actually, the group of transformations that leave the particle momentum $k_\mu = (\omega(k), k)$ invariant is somewhat larger, but the extra (noncompact) symmetries must act trivially on the particle states in order to avoid infinite-dimensional representations). The group $SO(2)$ has only one-dimensional complex representations. For spin s these representations just involve the states with spin (i.e. helicity) $\pm s$ in the direction of motion of the particle. Consequently massless particles have only two polarization states, irrespective of the value of their spin.

There is a further difficulty when one attempts to calculate Feynman diagrams for massless spin-1 particles, which is again related to the invariance under gauge transformations. To show this we rewrite (3.1) in the momentum representation,

$$S[A_\mu] = -\frac{i}{2} (2\pi)^4 \int d^4 k \ A^*_\mu(k) \left[ k^2 \eta^{\mu\nu} - k^\mu k^\nu \right] A_\nu(k).$$  (3.12)

According to the general prescription given in section 2, the propagator is proportional to the inverse of $(k^2 \eta_{\mu\nu} - k_\mu k_\nu)$. In this case, however, the inverse does not exist because this matrix has a zero eigenvalue, as we see from

$$(k^2 \eta_{\mu\nu} - k_\mu k_\nu) k^\nu = 0.$$  (3.13)
The presence of the zero eigenvalue is a direct consequence of the gauge invariance of the theory. Gauge invariance implies that the theory contains fewer degrees of freedom; this fact reflects itself in the presence of zero eigenvalues in the quadratic part of the Lagrangian. Indeed, the null vector associated with the zero eigenvalue is proportional to \( k_\mu \), which according to (3.6) characterizes gauge transformations in momentum space. Obviously, the degree of freedom that is absent in (3.12) should not reappear through the interactions. One can show that this is ensured provided that the photon couples to a conserved current.

The standard way to circumvent the singular propagator problem is to make use of a so-called gauge condition. A convenient procedure amounts to explicitly introducing the missing (gauge) degrees of freedom, which formally spoils the gauge invariance. However, the degrees of freedom are introduced only in order to make the propagator well-defined and they will not affect the interactions of the theory. Therefore, the effect of this procedure can still be separated from the true gauge invariant part of the theory, and the physical consequences remain unchanged. It is a rather subtle matter to prove that this is indeed the case. In this section we only present the prescription for defining the propagator. To do that one introduces a so-called "gauge-fixing" term to the Lagrangian. The most convenient choice is to add (3.1)

\[
\mathcal{L}_{\text{g.f.}} = -\frac{1}{2}(\lambda \partial_\mu A^\mu)^2, \tag{3.14}
\]

where \( \lambda \) is an arbitrary parameter. Because of this term the Fourier transform of the action corresponding to the combined Lagrangian becomes

\[
S[A_\mu] = -\frac{1}{2(2\pi)^4} \int d^4k \, A_\mu^*(k) \left[ k^2 \eta^{\mu\nu} - k^\mu k_\nu + \lambda^2 k^\mu k_\nu \right] A_\nu(k), \tag{3.15}
\]

so that for \( \lambda \neq 0 \) the propagator is equal to

\[
\Delta_{\mu\nu}(k) = \frac{1}{i(2\pi)^4} \left[ k^2 \eta_{\mu\nu} - (1 - \lambda^2)k_\mu k_\nu \right]^{-1} = \frac{1}{i(2\pi)^4} \frac{1}{k^2} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \tag{3.16}
\]

Clearly the propagator has more poles at \( k^2 = 0 \) than there are physical photons (characterized by transversal polarizations). However, one must realize that by making the above modification we have somewhat obscured the relation between propagator poles and physical particles. In order to extract the physical content of the theory one should only consider transversal polarizations. This requirement forms an essential ingredient of the proof that physical results do not depend on the parameter \( \lambda \).

Using the propagator (3.16) one can now construct Feynman diagrams and corresponding scattering and decay amplitudes for photons in the standard fashion. The \( \lambda \)-dependent \( k_\mu k_\nu \)-term of the propagator residue vanishes when contracting the invariant amplitude with transversal polarization vectors. In order to sum over photon polarizations one may use (for orthonormal polarization vectors)

\[
\sum_{\lambda=1,2} \epsilon_\mu(k, \lambda) \epsilon^*_\nu(k, \lambda) = \begin{cases} 
\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2} & \text{for } \mu, \nu = 1, 2, 3 \\
0 & \text{for } \mu \text{ and/or } \nu = 0
\end{cases} \tag{3.17}
\]
An alternative form is
\[ \sum_{\lambda=1,2} \varepsilon_\mu(k, \lambda) \varepsilon_\nu^*(k, \lambda) = \eta_{\mu\nu} \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{k \cdot \tilde{k}}, \]  
(3.18)

Obviously (3.17) and (3.18) are not manifestly Lorentz covariant, which is related to the fact that the transversality condition \( k \cdot \varepsilon(k, \lambda) = 0 \) is not Lorentz invariant. However, if the photons couple to conserved currents, such that the amplitude vanishes when contracted with the photon momentum,
\[ k^\mu \mathcal{M}_\mu = 0, \]
the noncovariant terms in (3.18) may be dropped. Consequently, when summing \( |\mathcal{M}_\mu \varepsilon^\mu(k, \lambda)|^2 \) over the transverse polarizations, one has
\[ \sum_{\lambda=1,2} |\mathcal{M}_\mu \varepsilon^\mu(k, \lambda)|^2 = \mathcal{M}^\mu \mathcal{M}_\mu, \]
(3.19)
which is manifestly Lorentz invariant. We will return to this aspect in section 6.

Problem 6:
To demonstrate that the gauge-fixing term only introduces an extra degree of freedom into the theory that does not interfere with the interactions, consider the Maxwell theory coupled to some conserved current. After addition of the gauge-fixing term (3.14), show that \( \theta \cdot A \) satisfies the free massless Klein-Gordon equation, so that the effect of the gauge-fixing term decouples from the rest of the theory.

Problem 7:
To prove the result (3.16) parametrize the propagator as \( \Delta_{\mu\nu}(k) = A(k) \eta_{\mu\nu} + B(k) k_\mu k_\nu \), and solve the equation
\[ [k^2 \eta^{\mu\nu} - k^\mu k^\nu + \lambda^2 k_\mu k_\nu] \Delta^{\ell\rho}(k) = \delta_\rho^\ell. \]
(3.20)

4. Annihilation of spinless particles by electromagnetic interaction
To get acquainted with the use of Feynman diagrams we will consider the annihilation reaction
\[ P^+ + P^- \rightarrow S^+ + S^- \]
(4.1)
mediated by a virtual photon in tree approximation. The particle \( P^\pm \) and \( S^\pm \) are hypothetical pointlike particles with no spin. A more relevant reaction is \( e^+ e^- \rightarrow \mu^+ \mu^- \), but considering the reaction (4.1) enables us to discuss the characteristic features of such a process without having to discuss some of the technical complications related to spin-\( \frac{1}{2} \) particles.

As a first step we consider the coupling of a complex spinless field \( \phi \) to photons. This is done by performing the so-called minimal substitution \( \partial_\mu \phi \rightarrow \partial_\mu \phi - ieA_\mu \phi \) in the free Klein-Gordon Lagrangian (cf. (1.14)), where \( \phi \) is a complex scalar field and \( \pm e \) is the electric charge of the particle associated with \( \phi \). Combining this with Maxwell’s Lagrangian gives
\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu - ieA_\mu)\phi|^2 - m^2 |\phi|^2 \]
\[ = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \]
\[ - ieA_\mu [\phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi] - e^2 A_\mu \phi^* \phi. \]
(4.2)
An important property of the Lagrangian (4.2) is its invariance under the combined gauge transformations

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \xi(x), \quad \phi(x) \rightarrow e^{ie\xi(x)}\phi(x).$$

(4.3)

This aspect will be extensively discussed in subsequent sections.

The propagators and vertices implied by (4.2) are shown in Table 2. The arrow on the pion line indicates the flow of (positive) charge rather than the momentum. We choose conventions such that an outgoing arrow on an external line indicates the emission of a positively charged particle absorption of its negatively-charged antiparticle. Combinatorial factors have not been included in the expressions for the vertices.

Assume that the particles \( P^\pm \) are associated with the field \( \phi \) introduced above. For the particle \( S^\pm \) we introduce a separate field \( \chi \), which has the same interactions with the photon (and thus the same electric charge) as \( \phi \), but a different mass denoted by \( M \). Observe that \( \chi \) is now subject to the same gauge transformation as \( \phi \).

Consider now the diagram shown in Fig. 2, which describes the reaction (4.1) in lowest order. Observe that \( p_1 \) and \( p_2 \) refer to the momenta of the incoming particles \( P^+ \) and \( P^- \), while \( q_1 \) and \( q_2 \) refer to the outgoing particles \( S^+ \) and \( S^- \), respectively. Extracting an overall factor of \( i(2\pi)^4 \) and a momentum-conserving \( \delta \)-function, the invariant amplitude is given by

$$\mathcal{M} = e^2(q_1 - q_2)^\mu \frac{1}{(p_1 + p_2)^2} \left( \delta^\nu_\mu - (1 - \lambda^{-2}) \frac{(q_1 + q_2)_\mu(p_1 + p_2)^\nu}{(p_1 + p_2)^2} \right) (p_1 - p_2)_\nu,$$

(4.4)

where we have set the photon momentum equal to \( k_\mu = (p_1 + p_2)_\mu = (q_1 + q_2)_\mu \). A first important observation is that the gauge-dependent part of the photon propagator vanishes when the pions are taken on the mass shell, because \( (p_1 + p_2) \cdot (p_1 - p_2) = p_1^2 - p_2^2 = 0 \) and \( (p_1 - p_2) \cdot (p_1 + p_2) = 0 \).
Fig. 2. Lowest-order Feynman diagram for the reaction (4.1)

\[ p_1^2 - p_2^2 = 0. \] This confirms that the physical consequences of the theory have not been affected by introducing the gauge-fixing term into the Lagrangian.

Introducing Mandelstam variables

\[
s = -(p_1 + p_2)^2, \quad t = -(p_1 - q_1)^2, \quad u = -(p_1 - q_2)^2,
\] (4.5)

which satisfy

\[
s + t + u = 2m^2 + 2M^2,
\] (4.6)

the amplitude can be written in a simple form

\[
\mathcal{M} = e^2 \frac{u - t}{s}.
\] (4.7)

In the centre-of-mass frame \( t \) and \( u \) are expressed in terms of \( s \) and the scattering angle \( \theta \) between \( p_1 \) and \( q_1 \):

\[
t = -\frac{1}{2} s + m^2 + M^2 + \frac{1}{2} \sqrt{(s - 4m^2)(s - 4M^2)} \cos \theta,
\]
\[
u = -\frac{1}{2} s + m^2 + M^2 - \frac{1}{2} \sqrt{(s - 4m^2)(s - 4M^2)} \cos \theta,
\] (4.8)

We now use the general formula for the differential cross section for a quasi-elastic scattering reaction \( 1 + 2 \rightarrow 3 + 4 \),

\[
\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2} \frac{1}{s} \sqrt{\lambda(s, m_1^2, m_4^2)} \sqrt{\lambda(s, m_1^2, m_2^2)} |\mathcal{M}|^2,
\] (4.9)

where \( m_1 \sim m_4 \) denote the masses of the particles 1-4, and the function \( \lambda \) is defined by

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.
\] (4.10)

Application of the above formulae gives rise to

\[
\frac{d\sigma}{d\Omega_{CM}} = \frac{\alpha^2}{4s} \sqrt{\frac{s - 4M^2}{s - 4m^2}} \left( 1 - \frac{4m^2}{s} \right) \left( 1 - \frac{4M^2}{s} \right) \cos^2 \theta.
\] (4.11)

Here, \( \alpha \) denotes the fine structure constant \( \alpha = e^2/4\pi \). Integration over the angles gives the total cross section

\[
\sigma = \frac{\pi \alpha^2}{3s} \sqrt{\frac{s - 4M^2}{s - 4m^2}} \left( 1 - \frac{4(m^2 + M^2)}{s} + \frac{16m^2M^2}{s^2} \right).
\] (4.12)
For $s \gg m^2, M^2$, these results become

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{\alpha^2}{4s} \cos^2 \theta.$$  \hfill (4.13)

and

$$\sigma = \frac{\pi \alpha^2}{3s}. \hfill (4.14)$$

For comparison we give the corresponding expression for $e^+e^- \rightarrow \mu^+\mu^-$,

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{\alpha^2}{4s} \sqrt{\frac{s - 4M^2}{s - 4m^2}} \left\{ 1 + \frac{4(m^2 + M^2)}{s} + \left( 1 - \frac{4m^2}{s} \right) \left( 1 - \frac{4M^2}{s} \right) \cos^2 \theta \right\}, \hfill (4.15)$$

where we have averaged over the (incoming) electron and summed over the (outgoing) muon spins. Observe that the $\cos \theta$ dependent terms coincide with (4.11). After integration over the angles, one obtains the total cross-section

$$\sigma = \frac{4\pi \alpha^2}{3s} \sqrt{\frac{s - 4M^2}{s - 4m^2}} \left\{ 1 + \frac{2(m^2 + M^2)}{s} + \frac{4m^2M^2}{s^2} \right\}, \hfill (4.16)$$

where $m$ is the electron mass and $M$ the muon mass. When $E \gg M, m$, as is usually the case, one finds the well-known results

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta), \hfill (4.17)$$

and

$$\sigma = \frac{4\pi \alpha^2}{3s}. \hfill (4.18)$$

5. **Gauge theory of $U(1)$**

In the previous section we have considered scalar electrodynamics, the theory of photons coupled to charged spinless fields. This is one of the simplest examples of an interacting field theory based on gauge invariance. Such theories are called gauge theories. In this section we will present the main ingredients of these theories. For simplicity we will first consider the case of abelian gauge transformations, i.e., gauge transformations that commute. In later sections we shall also discuss theories based on nonabelian gauge groups.

As a first example let us construct a field theory which is invariant under local phase transformations. Our starting point is the free Dirac Lagrangian

$$\mathcal{L}_\psi = -\bar{\psi}\gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi, \hfill (5.1)$$

which is obviously invariant under rigid phase transformations, i.e., phase transformations which are the same at each point in space-time. This is so because
\[ \psi \rightarrow \psi' = e^{i\eta \xi} \psi, \]  

implies

\[ \overline{\psi} \rightarrow \overline{\psi}' = e^{-i\eta \xi} \overline{\psi}. \]

Here we have introduced a parameter \( \eta \) that measures the strength of the phase transformations, because eventually we want to simultaneously consider fields transforming with different strengths. Phase transformations generate the group of \( 1 \times 1 \) unitary matrices called \( U(1) \).

Let us now consider local phase transformations, and verify whether (5.1) remains invariant. Of course, the local aspect of the transformation is not important for the invariance of the mass term, since the variation of that term only involves the transformation of fields taken at the same point in space-time. But as soon as we compare fields at different points in space-time the local character of the transformation is crucial. A derivative, which depends on the variation of the fields in an infinitesimally small neighbourhood, will be subject to transformations at neighbouring space-time points. To see the effect of this, let us evaluate the effect of a local transformation on \( \partial_\mu \psi \):

\[ \partial_\mu \psi(x) \rightarrow (\partial_\mu \psi(x))' = \partial_\mu (e^{i\eta \xi(x)} \psi(x)) \]

\[ = e^{i\eta \xi(x)} (\partial_\mu \psi(x)) + i\eta \partial_\mu \xi(x) \psi(x). \]  

(5.4)

Clearly \( \partial_\mu \psi \) does not have the same transformation rule as \( \psi \) itself. There is an extra term induced by the transformations at neighbouring space-time points which is proportional to the derivative of the transformation parameter. This term is responsible for the lack of invariance of the Lagrangian (5.1).

In order to make (5.1) invariant under local phase transformations, one may consider the addition of new terms whose variation will compensate for the \( \partial_\mu \xi \) term in (5.4). As a first step one could attempt to construct a modified derivative \( D_\mu \) transforming according to

\[ D_\mu \psi(x) \rightarrow (D_\mu \psi(x))' = e^{i\eta \xi(x)} (D_\mu \psi(x)). \]  

(5.5)

If such a derivative exists we can then simply replace the ordinary derivative \( \partial_\mu \) in the Lagrangian (5.1) by \( D_\mu \) and preserve invariance under local phase transformations.

Since the transformation of \( D_\mu \psi \) is entirely determined by the transformation parameter at the same space-time coordinate as \( \psi \), \( D_\mu \) is called a covariant derivative. To appreciate this definition one should realize that a local phase transformation may be regarded as a product of independent phase transformations each acting at a separate space-time point. It is possible that local quantities transform only under the gauge transformation taken at the same space-time point. Such quantities are then said to transform covariantly. For instance, according to this nomenclature, the field \( \psi \) transforms in a covariant fashion under local phase transformations, whereas the transformation behaviour of ordinary derivatives ( cf.(5.4)), although correctly representing the action of the full local group, is clearly noncovariant. It is obviously convenient to have local quantities that transform covariantly, and this is an extra motivation for introducing the covariant derivative.
Let us now turn to an explicit construction of the covariant derivative. Comparing (5.4) and (5.5) we note that the modified derivative $D_\mu$ must contain a quantity whose transformation can compensate for the second term in (5.4). If we define

$$D_\mu \psi(x) = (\partial_\mu - iqA_\mu(x))\psi(x), \quad (5.6)$$

we obtain

$$D_\mu \psi \to (D_\mu \psi)' = (\partial_\mu \psi)' - iq(A_\mu \psi)'$$
$$= e^{i\xi}(\partial_\mu \psi + iq\partial_\mu \xi \psi - iqA'_\mu \psi). \quad (5.7)$$

Comparing this to (5.5) shows that the new quantity $A_\mu$ must have the following transformation rule

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \xi. \quad (5.8)$$

Hence the requirement of local gauge invariance has led us to introduce a new field $A_\mu$, whose transformation is given by (5.8). This new field is called a gauge field. Note that the gauge field does not transform in a covariant fashion.

Introducing the covariant derivative (5.6) into the Lagrangian (5.1) shows that the theory is no longer free, but describes interactions of the fermions with the gauge field

$$\mathcal{L}_\psi = -\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi,$$
$$= -\bar{\psi} \not{\partial} \psi - m\bar{\psi} \psi + iqA_\mu \bar{\psi} \gamma^\mu \psi. \quad (5.9)$$

Usually one assumes that $A_\mu$ describes some new and independent degrees of freedom of the system, although this can sometimes be avoided. But in any case it is clear that the requirement of local gauge invariance leads to interacting field theories of a particular structure.

Covariant derivatives play an important role in theories with local gauge invariance, so we discuss them here in more detail. First we note that $D_\mu$ consists of two terms which are both related to an infinitesimal transformation. The derivative $\partial_\mu$ generates an infinitesimal displacement of the coordinates, $x^\mu \to x^\mu + a^\mu$, whereas the second term $-iqA_\mu \psi$ represents the variation under an infinitesimal gauge transformation $\delta \psi = iq\xi \psi$, with parameter $\xi = -A_\mu$. The combination of an infinitesimal transformation over a distance $a^\mu$ and a field-dependent gauge transformation with parameter $\xi = -a^\mu A_\mu$ is sometimes called a covariant translation. Under such a translation a field transforms as

$$\delta \psi = a^\mu D_\mu \psi. \quad (5.10)$$

The observation that $D_\mu$ corresponds to an infinitesimal variation shows that covariant derivatives must satisfy the Leibnitz rule, just as ordinary derivatives do

$$D_\mu(\psi_1 \psi_2) = (D_\mu \psi_1)\psi_2 + \psi_1(D_\mu \psi_2). \quad (5.11)$$

To appreciate this result one should realize that the precise form of the covariant derivative is tied to the transformation character of the quantity on which it acts. For instance, if $\psi_1$ and $\psi_2$
transform under local phase transformations with strength $q_1$ and $q_2$, respectively, then we have

$$D_\mu(\psi_1 \psi_2) = (\partial_\mu - iq_1 A_\mu)(\psi_1 \psi_2),$$

$$D_\mu \psi_1 = (\partial_\mu - iq_1 A_\mu)\psi_1,$$

$$D_\mu \psi_2 = (\partial_\mu - iq_2 A_\mu)\psi_2.$$  \hspace{1cm} (5.12)

With these definitions it is straightforward to verify the validity of (5.11).

Of course, repeated application of covariant derivatives will always yield covariant quantities. This fact may be used to construct a new covariant object which depends only on the gauge fields. Namely, we apply the antisymmetric product of two derivatives on $\psi$

$$[D_\mu, D_\nu] \psi = D_\mu(D_\nu \psi) - D_\nu(D_\mu \psi).$$  \hspace{1cm} (5.13)

Writing explicitly

$$D_\mu(D_\nu \psi) = \partial_\mu \partial_\nu \psi - iq A_\mu \partial_\nu \psi - iq(\partial_\mu A_\nu)\psi - iq A_\nu \partial_\mu \psi - q^2 A_\mu A_\nu \psi,$$  \hspace{1cm} (5.14)

one easily establishes

$$[D_\mu, D_\nu] \psi = -iq F_{\mu\nu} \psi,$$  \hspace{1cm} (5.15)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (5.16)

However, since $\psi$ transforms covariantly and the left-hand side of (5.15) is covariant, we may conclude that $F_{\mu\nu}$ is itself a covariant object. In fact, application of the gauge transformation (5.8) shows that $F_{\mu\nu}$ is even gauge invariant,

$$\delta F_{\mu\nu} = \partial_\mu \partial_\nu \xi - \partial_\nu \partial_\mu \xi = 0.$$  \hspace{1cm} (5.17)

but, as we will see later, this is a coincidence related to the fact that $U(1)$ transformations are abelian.

The result (5.15) is called the Ricci identity. It specifies that the commutator of two covariant derivatives is an infinitesimal gauge transformation with parameter $\xi = -F_{\mu\nu}$, where $F_{\mu\nu}$ is called the field strength. This field strength $F_{\mu\nu}$ is sometimes called the curvature tensor. The reason for this nomenclature is not difficult to see: if the left-hand side of (5.15) were zero then two successive infinitesimal covariant translations, one in the $\mu$ and the other in the $\nu$ direction, would lead to the same result when applied in the opposite order. According to the Ricci identity this is not the case for finite $F_{\mu\nu}$. One encounters the same situation when considering translations on a curved surface, which do not commute for finite curvature. As the tensor $F_{\mu\nu}$ on the right-hand side of (5.15) measures the lack of commutativity, its effect is analogous to that of curvature.

We can use covariant derivatives to obtain yet another important identity. Consider the double commutators of covariant derivatives $[D_\mu, [D_\nu, D_\rho]]$. According to the Jacobi identity the cyclic combination vanishes identically,
\[ [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0, \]  

(5.18)

as can be verified by writing all the terms. To see the consequence of (5.18) let us write the first term acting explicitly on \( \psi(x) \),

\[ [D_\mu, [D_\nu, D_\rho]]\psi = D_\mu([D_\nu, D_\rho]\psi) - [D_\nu, D_\rho]D_\mu\psi \]

\[ = -iq(D_\mu F_{\nu\rho})\psi, \]  

(5.19)

where we used (5.11) and (5.15). Therefore the Jacobi identity implies

\[ D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0. \]  

(5.20)

In this case the field strength is invariant under gauge transformations so we may replace covariant by ordinary derivatives and obtain

\[ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \]  

(5.21)

This result is called the Bianchi identity; it implies that \( F_{\mu\nu} \) can be expressed in terms of a vector field, precisely in accord with (5.16). The identity (5.21) is well known in electrodynamics as the homogeneous Maxwell equations.

The field strength tensor can now be used to write a gauge invariant Lagrangian for the gauge field itself

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2. \]  

(5.22)

This Lagrangian can now be combined with (5.9),

\[ \mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi \]

\[ = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi} \Theta \psi - m \bar{\psi} \psi + iqA_\mu \bar{\psi} \gamma^\mu \psi, \]  

(5.23)

so that we have obtained an interacting theory of a vector field and a fermion field invariant under the combined local gauge transformations (5.2) and (5.8). It is not difficult to see that this theory coincides with electrodynamics: the gauge field \( A_\mu \) is just the vector potential (subject to its familiar gauge transformation), which couples to the fermion field via the minimal substitution, and \( F_{\mu\nu} \) is the electromagnetic field strength.

To derive the field equations corresponding to (5.23) is straightforward. They read

\[ \partial^\nu F_{\nu\mu} = J_\mu, \]  

(5.24)

\[ (\Theta + m)\psi = iqA \psi, \]  

(5.25)

\[ \bar{\psi}(\bar{\Theta} - m) = -iq\bar{\psi}A, \]  

(5.26)

where the right-hand side of (5.24) is equal to

\[ J_\mu = iq\bar{\psi} \gamma_\mu \psi. \]  

(5.27)

Clearly (5.24) corresponds to the inhomogeneous Maxwell equation (1.29), while, as was mentioned above, the Bianchi identity (5.21) coincides with the homogeneous Maxwell equations.
The field equations (5.25) and (5.26) describe the dynamics of the charged fermion, and are strictly speaking not part of Maxwell’s equations.

It is easy to repeat the above construction for other fields. For example, a complex scalar field \( \phi \) may transform under local phase transformations according to

\[
\phi(x) \rightarrow \phi'(x) = e^{i\xi(x)} \phi(x).
\]  

(5.28)

As before, the requirement of local gauge invariance forces one to replace the ordinary derivative by a covariant derivative,

\[
D_\mu \phi = (\partial_\mu - iqA_\mu)\phi,
\]  

(5.29)

so that one obtains a gauge invariant version of the Klein-Gordon Lagrangian

\[
\mathcal{L}_\phi = -|D_\mu \phi|^2 - m^2|\phi|^2
= -|\partial_\mu \phi|^2 - m^2|\phi|^2 - iqA_\mu \phi^* \overrightarrow{\partial}_\mu \phi - q^2 A_\mu^2|\phi|^2.
\]  

(5.30)

This is the Lagrangian for scalar electrodynamics, which we have been using in section 4. Observe that the effect of the covariant derivative coincides with that of the minimal substitution procedure. Unlike in spinor electrodynamics, which is defined by the Lagrangian (5.9), there are interaction terms in (5.30) that are quadratic in \( A_\mu \). Therefore the corresponding expression for the current (5.28) now depends also on the gauge field \( A_\mu \); it reads

\[
J_\mu = iq(D_\mu \phi^*)\phi - \phi^*(D_\mu \phi),
\]  

(5.31)

and appears on the right-hand side of the Maxwell equation (5.24). The field equation of the scalar field can be written as

\[
D^\mu D_\mu \phi - m^2 \phi = 0.
\]  

(5.32)

The parameter \( q \) that we have been using to indicate the relative magnitude of the change of phase caused by the gauge transformation, also determines the strength of the interaction with the gauge field \( A_\mu \). Hence in electromagnetism the particles described by the various fields carry electric charges \( \pm q \). We shall give a more precise definition of the electric charge in section 7.

6. Current conservation

The four-vector current \( J_\mu \) that appears on the right-hand side of the inhomogeneous Maxwell equation must satisfy an obvious restriction. To see this contract (5.24) with \( \partial^\mu \) and use the fact that \( F_{\mu\nu} \) is antisymmetric in \( \mu \) and \( \nu \). It then follows that \( J_\mu \) must be conserved, i.e.

\[
\partial_\mu J^\mu = 0.
\]  

(6.1)

Another way to derive the same result is to make use of the matter field equations. For instance, for a fermion field one has

\[
\partial_\mu J^\mu = \partial_\mu (iq\overline{\psi}\gamma^\mu \psi)
= iq \overline{\psi}(\slashed{D}\psi) + iq(\overline{\psi} \slashed{D})\psi
= 0,
\]  

(6.2)
by virtue of (5.25) and (5.26). Similarly one can show that the current (5.31) is conserved by virtue of the field equation (5.32) and its complex conjugate.

The fact that photons must couple to a conserved current has direct consequences for invariant amplitudes that involve external photon lines. Such an amplitude with one photon and several other incoming and outgoing lines takes the form

$$\mathcal{M}(k, \cdots) = \epsilon_\mu(k) M^\mu(k, \cdots), \quad (6.3)$$

where \( k \) and \( \epsilon(k) \) denote the momentum and polarization vector of the photon. Current conservation now implies that

$$k_\mu M^\mu(k, \cdots) = 0, \quad (6.4)$$

provided that all external lines other than that of the photon are on the mass shell. The latter condition can be understood as a consequence of the fact that we had to impose the matter field equation in (6.2).

Actually, the mass-shell condition for the external lines can be somewhat relaxed: it is sufficient to require that only the external lines associated with charged particles are on the mass shell. Therefore it is possible to exploit (6.4) for amplitudes with several off-shell photons, but we should already caution the reader that this result does not hold for nonabelian gauge theories. If all external lines are taken off shell one obtains relations between Green's functions, which have a more complicated structure than (6.4). Such identities are called Ward identities. In the context of quantum electrodynamics these identities are usually called Ward-Takahashi identities.

The fact that \( A_\mu \) couples to a conserved current is essential in order to establish that interactions of the massless spin-1 particle associated with \( A_\mu \) are Lorentz invariant. We have already observed this in one particular example in section 3. To explain this aspect in more detail, we recall that massless particles have fewer polarization states than massive ones. The spin of massless spin-\( s \) particles must be parallel or anti-parallel to the direction of motion of the particle, so that the helicity is equal to \( \pm s \). Consequently massless particles have only two polarization states, irrespective of the value of their spin. Physical photons have therefore helicity \( \pm 1 \), and are described by transverse polarization vectors \( \epsilon(k) \) that satisfy the condition

$$k \cdot \epsilon(k) = 0, \quad \epsilon_0(k) = 0, \quad (6.5)$$

corresponding to two linearly independent polarization vectors.

The second condition in (6.5) is obviously not Lorentz invariant, and one may question whether the interactions of massless spin-1 particles will be relativistically invariant. To make this more precise, consider a physical process involving a photon, for which the invariant amplitude takes the form

$$\mathcal{M} = \epsilon_\mu(k) M^\mu(k, \cdots), \quad (6.6)$$

where \( \epsilon(k) \) is the photon polarization vector, \( k_\mu \) the photon momentum (\( k^2 = 0 \)) and the dots indicate the other particle momenta that are relevant. Obviously, (6.6) has a Lorentz invariant
form, but since \( \varepsilon(k) \) will not remain transverse after a Lorentz transformation, the amplitude will in general no longer coincide with the expression (6.6) when calculated directly in the new frame.

To examine this question in more detail let us first derive how a transverse polarization vector transforms under Lorentz transformations. What we intend to prove is that \( \varepsilon_\mu(k) \), satisfying (6.5) with \( k^2 = 0 \), transforms into a linear combination of a transverse vector \( \varepsilon'(k') \) and the transformed photon momentum \( k' \), where \( \varepsilon'(k') \) is transverse with respect to the new momentum \( k' \). More precisely,

\[
\varepsilon_\mu(k) \rightarrow \varepsilon'_\mu(k') + \alpha k'_\mu, \tag{6.7}
\]

with \( \alpha \) some unknown coefficient which depends on the Lorentz transformation. To derive (6.7) it is sufficient to note that the condition \( k \cdot \varepsilon(k) = 0 \) is Lorentz invariant, so that the right-hand side of (6.7) should vanish when contracted with \( k' \); therefore it follows that this vector can be decomposed into a transverse vector satisfying (6.5) (but now in the new frame) and the momentum \( k' \). What remains to be shown is that the transverse vector \( \varepsilon'(k') \) has the same normalization as \( \varepsilon(k) \). This is indeed the case, since

\[
\varepsilon^\mu(k) \varepsilon_\mu(k) = (\varepsilon'^\mu(k') + \alpha k'^\mu)(\varepsilon'_\mu(k') + \alpha k'_\mu)
= \varepsilon'^\mu(k') \varepsilon'_\mu(k'), \tag{6.8}
\]

where we have used (6.5) and \( k'^2 = 0 \). Using (6.7) one easily establishes that the amplitude (6.3) transforms under Lorentz transformations as

\[
\varepsilon_\mu(k) \mathcal{M}^\mu(k, \cdots) \rightarrow \varepsilon'_\mu(k') \mathcal{M}'^\mu(k', \cdots) + \alpha k'_\mu \mathcal{M}'^\mu(k', \cdots). \tag{6.9}
\]

The first term on the right-hand side corresponds precisely to the amplitude that one would calculate in the new frame. Therefore relativistic invariance is ensured provided that the photons couple to a conserved amplitude.

7. Conserved charges

In classical field theory current conservation implies that the charge associated with the current is locally conserved. For scattering and decay reactions of elementary particles it seems obvious that charge conservation should imply that the total charge of the incoming particles is equal to the total charge of the outgoing particles. It is the purpose of this section to prove that this is indeed the case and to establish that the charge of a particle can be defined in terms of the invariant amplitude for a particle to emit or absorb a zero-frequency photon. To elucidate this definition consider the amplitude for the absorption of a virtual photon with momentum \( k \) by a spinless particle. The corresponding diagram is shown in Fig. 3. The momenta of the incoming and outgoing particles are denoted by \( p \) and \( p' \), respectively, so that \( k = p' - p \). Both \( p \) and \( p' \) refer to physical particles of the same mass, so that \( p^2 = p'^2 = -m^2 \). The invariant amplitude can generally be decomposed into two terms

\[
\mathcal{M}_\mu(p', p) = F_1(k^2)(p'_\mu + p_\mu) + iF_2(k^2)k_\mu \tag{7.1}
\]
where $F_1$ and $F_2$ are called form factors. Current conservation implies that $k \mu \mathcal{M}^\mu(p', p)$ should vanish, so

$$F_1(k^2)(p'^2 - p^2) + iF_2(k^2)k^2 = 0. \quad (7.2)$$

Since the incoming and outgoing particles have the same mass, $F_1$ drops out from (7.2) and we are left with

$$F_2(k^2) = 0. \quad (7.3)$$

In order to obtain (7.3) it is essential that we assume current conservation for off-shell photons. The simplification resulting from this assumption will not imply a loss of generality in what follows, as we will mainly be dealing with physical photons. The function $F_1(k^2)$ is called the charge form factor, and as, we have been alluding to above, its value at $k^2 = 0$ defines the electric charge of the particle in question. For a pointlike particle this is easily verified, and one finds $F_2(k^2) = 0$ and $F_1(k^2) = e$, where $e$ is the coupling constant in the Lagrangian that measures the strength of the photon coupling. Experimentally it is not possible to measure the probability for absorbing or emitting a zero-frequency photon, so that the charge of a particle is not measured in this way. It is more feasible to use low-energy Compton scattering (also called Thomson scattering) for this purpose. Another process is the elastic scattering of a particle by a Coulomb field.

We will now show that, with the above definition of charge, one has charge conservation in any possible elementary particle reaction. The derivation starts from considering a process in which a soft photon is being emitted or absorbed; for instance

$$A \rightarrow B + \gamma \quad (7.4)$$

where $A$ and $B$ denote an arbitrary configuration of incoming and outgoing particles. It is possible to divide the amplitude for this process into two terms

$$\mathcal{M}(A \rightarrow B + \gamma) = \mathcal{M}^B(A \rightarrow B + \gamma) + \mathcal{M}^R(A \rightarrow B + \gamma), \quad (7.5)$$

where $\mathcal{M}^B$ consists of all the Born approximation diagrams in which the photon is attached to one of the external lines, as shown in Fig. 4, and $\mathcal{M}^R$ represents the remainder. The Born approximation diagrams have the form
\[ M^B = \varepsilon_\mu(k) \left\{ \sum_i M(A[i] \rightarrow B) \frac{(2p_i - k)^\mu}{(p_i - k)^2 + m_i^2} F_i(k^2) \\
+ \sum_j \frac{(2p_j + k)^\mu}{(p_j + k)^2 + m_j^2} F_j(k^2) M(A \rightarrow B[j]) \right\}, \quad (7.6) \]

where \( M(A[i] \rightarrow B) \) and \( M(A \rightarrow B[j]) \) denote the invariant amplitude for the process \( A \rightarrow B \) in which one of the external lines is shifted from its mass shell by an amount \( k \). The index \( i \) labels the off-shell incoming line with momentum \( p_i - k \) and \( p_i^2 = -m_i^2 \), whereas \( j \) labels the outgoing line with momentum \( p_j + k \) and \( p_j^2 = -m_j^2 \). The reason why we consider the Born approximation diagrams of Fig. 4 separately is that they become singular when the photon momentum \( k \) tends to zero because the propagator of the virtual particle diverges in that limit. Therefore we are entitled to restrict ourselves to the charge form factors \( F_i(k^2) \) or \( F_j(k^2) \) as they are measured for real particles, since the deviation from their on-shell value leads to terms in which the propagator pole cancels, and which are therefore regular if \( k \) approaches zero. Those terms are thus contained in the second part of (7.5) which is assumed to exhibit no singularities in the soft-photon limit.

We now use current conservation on the full amplitude (7.5), i.e., we require that the amplitude vanishes when the photon polarization vector \( \varepsilon_\mu(k) \) is replaced by \( k_\mu \). Contracting the momentum factors in the Born approximation amplitude with \( k_\mu \) leads to the following factors

\[ k_\mu \frac{(2p_i - k)^\mu}{(p_i - k)^2 + m_i^2} = -\frac{(p_i - k)^2 + p_i^2}{(p_i - q)^2 + m_i^2} = -1, \quad (7.7) \]

\[ k_\mu \frac{(2p_j + k)^\mu}{(p_j + k)^2 + m_j^2} = \frac{(p_j + k)^2 - p_j^2}{(p_j + k)^2 + m_j^2} = +1. \quad (7.8) \]

Using the fact that the remaining diagrams in (7.5) are regular for vanishing \( k \), we thus find

\[ -\sum_i M(A[i] \rightarrow B) F_i(k^2) + \sum_j F_j(k^2) M(A \rightarrow B[j]) = O(k), \quad (7.9) \]

or, in the soft-photon limit

\[ (\sum_j F_j(0) - \sum_i F_i(0)) M(A \rightarrow B) = 0, \quad (7.10) \]

where \( M(A \rightarrow B) \) is now the full on-shell amplitude for the process \( A \rightarrow B \). The implication of (7.10) should be obvious. In every possible process the sum of the charges of the incoming

\[ \gamma \]

\[ B \{ \]
particles should equal the sum of the charges of the outgoing particles. This result justifies the definition of electric charge as the charge form factor taken at zero momentum transfer.

8. Nonabelian gauge fields

In the previous chapter we have introduced theories with local gauge invariance. In order to demonstrate the essential ingredients we stayed primarily within the context of theories such as electrodynamics that are invariant under local phase transformations. However, the same framework can be applied to theories that are invariant under more complicated gauge transformations. One distinctive feature of the latter is that they depend on several parameters and also that they are not always commuting. Groups of noncommuting transformations are called nonabelian. This in contradistinction with phase transformations, which depend on a single parameter $\xi$ and are obviously commuting (and therefore called abelian). The fact that the gauge transformations depend on several parameters forces us to introduce several independent gauge fields. Also the matter fields must have a certain multiplicity in order that the gauge transformations can act on them. In more mathematical terms, the fields should transform according to representations of the gauge group. Each representation consists of a set of fields which transform among themselves, just as the components of a three-dimensions vector transform among themselves under the group of rotations. The coupling of the gauge fields to matter will involve certain matrices, which will appear in the expressions for the charges. Charges can be defined along the same lines as in section 7. If the gauge transformations do not commute, these matrices will not commute either. This requires that the nonabelian gauge fields exhibit selfinteractions (in other words, they are not neutral as the photon), so that the Lagrangian for the gauge fields will be considerably more complicated than the Lagrangian (5.14). One way to discover the need for selfinteractions of nonabelian gauge fields follows from analyzing Feynman diagrams with several external lines associated with nonabelian gauge fields. One can then establish that the corresponding amplitudes are only conserved if the gauge fields have direct interactions with themselves. However, we will proceed differently and start in the same vein as in section 5, assuming invariance under nonabelian transformations.

The relevant gauge groups consist of transformations, usually represented by matrices, that can be parametrized in an analytic fashion in terms of a finite number of parameters. Such groups are called Lie groups. The number of independent parameters defines the dimension of the group.* For instance, phase transformations constitute the group $U(1)$, which is clearly of dimension one. As mentioned above, this group is abelian because phase transformations commute. Two important classes of nonabelian groups are the groups $SO(N)$ of real rotations in $N$ dimensions ($N > 2$), and the groups $SU(N)$ of $N \times N$ unitary matrices with unit determinant ($N > 1$). As we shall see in a moment the dimension of these groups is $\frac{1}{2} N(N - 1)$ and $N^2 - 1$, respectively.

* Mathematically, a set of transformations forms a group if the product of every two transformations, the identity and the inverse of each transformation is contained in the set, and if the product of transformations is associative. It is usually rather obvious that the complete set of transformations that leave a theory invariant, forms a group.
Let us generally consider fields that transform according to a representation of a certain Lie group \( G \). This means that, for every element of the group \( G \), we have a matrix \( U \); these matrices \( U \) satisfy the same multiplication rules as the corresponding elements of \( G \). Under a group transformation the fields rotate as follows

\[
\psi(x) \rightarrow \psi'(x) = U \psi(x),
\]

(8.1)

where \( \psi \) denotes an array of different fields written as a column vector. More explicitly, we may write

\[
\psi_i(x) \rightarrow \psi'_i(x) = U_{ij} \psi_j(x).
\]

(8.1')

For most groups the matrices \( U \) can generally be written in exponential form

\[
U = \exp(\xi^a t_a),
\]

(8.2)

where the matrices \( t_a \) are called the generators of the group defined in the representation appropriate to \( \psi \), and the \( \xi^a \) constitute a set of real parameters in terms of which the group elements can be described. The number of generators, which is obviously equal to the number of independent parameters \( \xi^a \) and therefore to dimension of the group, is unrelated to the dimension of the matrices \( U \) and \( t_a \). It is usually straightforward to determine the generators for a given group. For example, the generators of the \( SO(N) \) group must consist of the \( N \times N \) real and antisymmetric matrices, in order that (8.2) defines an orthogonal matrix: \( U^T = U^{-1} \). As there are \( \frac{1}{2} N(N-1) \) independent real and antisymmetric matrices the dimension of the \( SO(N) \) group is equal to \( \frac{1}{2} N(N-1) \). For the \( SU(N) \) group, the defining relation \( U^\dagger = U^{-1} \) requires the generators \( t_a \) to be antihermitean \( N \times N \) matrices. Furthermore, to have a matrix (8.2) with unit determinant it is necessary that these antihermitean matrices \( t_a \) are traceless. There are \( N^2 - 1 \) independent antihermitean traceless matrices so that the dimension of \( SU(N) \) is equal to \( N^2 - 1 \). To verify these properties it is usually sufficient to consider infinitesimal transformations, where the parameters \( \xi^a \) are small, so that \( U = 1 + \xi^a t_a + O(\xi^2) \).

Because the matrices \( U \) defined in (8.2) constitute a representation of the group, products of these matrices must be of the same exponential form. This leads to an important condition on the matrices \( t_a \), which can already be derived by considering a product of two infinitesimal transformations: the matrices \( t_a \) generate a group representation if and only if their commutators can be decomposed into the same set of generators. These commutation relations define the Lie algebra \( g \) corresponding to the Lie group \( G \),

\[
[t_a, t_b] = f_{ab}^c t_c,
\]

(8.3)

where the proportionality constants \( f_{ab}^c \) are called the structure constants, because they define the multiplication properties of the Lie group. As we shall see the Lie algebra relation (8.3) plays a central role in what follows.

Let us now follow the same approach as in section 5 and consider the extension of the group \( G \) to a group of local gauge transformations. This means that the parameters of \( G \) will become functions of the space-time coordinates \( x^\mu \). As long as one considers variations of the field at a
single point in space-time this extension is trivial, but the local character of the transformations becomes important when comparing changes at different space-time points. In particular this is relevant when considering the effect of local transformations on derivatives of the fields, i.e.,

\[ \psi(x) \rightarrow \psi'(x) = U(x) \psi(x), \]  
(8.4)

\[ \partial_\mu \psi(x) \rightarrow (\partial_\mu \psi(x))' = U(x) \partial_\mu \psi(x) + (\partial_\mu U(x)) \psi(x). \]  
(8.5)

Just as in section 5 local quantities such as the vector \( \psi \), which transform according to a representation of the group \( G \) at the same space-time point, are called \textit{covariant}. Due to the presence of the second term on the right-hand side of (8.5), \( \partial_\mu \psi \) does not transform covariantly. Although the action of the space-time dependent extension of \( G \) is still correctly realized by (8.5) this type of behaviour under symmetry variations is difficult to work with. Therefore one attempts to replace \( \partial_\mu \) by a so-called \textit{covariant derivative} \( D_\mu \), which constitutes a covariant quantity when applied on \( \psi \),

\[ D_\mu \psi(x) \rightarrow (D_\mu \psi(x))' = U(x) D_\mu \psi(x). \]  
(8.6)

The construction of a covariant derivative has been discussed in the previous chapter for abelian transformations, where it was noted that a covariant derivative can be viewed as the result of a particular combination of an infinitesimal displacement generated by the ordinary derivative and a field-dependent infinitesimal gauge transformation. Such an infinitesimal displacement was called a \textit{covariant translation}. Its form suggests an immediate generalization to the covariant derivative for an arbitrary group. Namely, we take the linear combination of an ordinary derivative and an infinitesimal gauge transformation, where the parameters of the latter \textit{define} the nonabelian gauge fields. Hence

\[ D_\mu \psi \equiv \partial_\mu \psi - W_\mu \psi, \]  
(8.7)

where \( W_\mu \) is a matrix of the type generated by an infinitesimal gauge transformation. This means that \( W_\mu \) takes values in the Lie-algebra corresponding to the group \( G \), i.e., \( W_\mu \) can be decomposed into the generators \( t_a \),

\[ W_\mu = W_\mu^a t_a. \]  
(8.8)

Indeed \( W_\mu \) has the characteristic feature of a gauge field, as it can carry information regarding the group from one space-time point to another.

Let us now examine the consequences of (8.7). Combining (8.5) and (8.6) shows that \( W_\mu \psi \) must transform under gauge transformations as

\[
(W_\mu \psi)' = \partial_\mu \psi' - (D_\mu \psi)'
= U(\partial_\mu \psi)' + (\partial_\mu U)\psi - U(D_\mu \psi)
= \{UW_\mu U^{-1} + (\partial_\mu U)U^{-1}\} \psi'.
\]  
(8.9)

This implies the following transformation rule for \( W_\mu \),

28
\[ W_\mu \rightarrow W'_\mu = UW_\mu U^{-1} + (\partial_\mu U)U^{-1}. \]  

(8.10)

Clearly the gauge fields do not transform covariantly. The first term in (8.10) indicates that the gauge fields \( W_\mu^a \) transform according to the so-called adjoint representation of the group; the second noncovariant term is a modification that is characteristic for gauge fields. It is easy to evaluate (8.10) for infinitesimal transformations by using (8.3) and we find

\[ W_\mu^a \rightarrow (W'_\mu)^a = W_\mu^a + f_{bc}^{\ a} \xi^b W_\mu^c + \partial_\mu \xi^a + O(\xi^2). \]  

(8.11)

This result differs from the transformation law of abelian gauge fields by the presence of the term \( f_{bc}^{\ a} \xi^b W_\mu^c \).

We have already made use of the observation that the \( D_\mu \) can be viewed as the generators of covariant translations, which consist of infinitesimal space-time translations combined with infinitesimal field-dependent gauge transformations in order to restore the covariant character of the translated quantity. Since both these infinitesimal transformations satisfy \( \delta(\phi\psi) = (\delta\phi)\psi + \phi(\delta\psi) \), we have Leibnitz' rule for covariant derivatives,

\[ D_\mu(\phi\psi) = (D_\mu\phi)\psi + \phi(D_\mu\phi). \]  

(8.12)

Note that the covariant derivative always depends on the representation of the fields on which it acts through the choice of the generators \( t_a \). Hence each of the three terms in (8.12) may contain a different representation for the generators (see the simple abelian example in (5.12)).

Unlike ordinary differentiations, two covariant differentiations do not necessarily commute. It is easy to see that the commutator of two covariant derivatives \( D_\mu \) and \( D_\nu \), which is obviously a covariant quantity, is given by

\[ [D_\mu, D_\nu]\psi = D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) \]
\[ = -(\partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu])\psi. \]  

(8.13)

This result leads to the definition of a covariant antisymmetric tensor \( G_{\mu\nu} \),

\[ G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu], \]  

(8.14)

which is called the field strength. As \( \psi \) and \( D_\mu D_\nu\psi \) transform identically under the gauge transformations the field strength must transform covariantly according to

\[ G_{\mu\nu} \rightarrow G'_{\mu\nu} = UG_{\mu\nu}U^{-1}. \]  

(8.15)

Because \( W_\mu \) is Lie-algebra valued and the quadratic term in (8.14) is a commutator, the field strength is also Lie-algebra valued, i.e., \( G_{\mu\nu} \) can also be decomposed in terms of the group generators \( t_a \),

\[ G_{\mu\nu} = G_{\mu\nu}^a t_a, \]  

(8.16)

with
\[ G_{\mu \nu}^a = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - f_{bc}^{\;a} W^b_\mu W^c_\nu. \]  
(8.17)

Note that (8.17) differs from the abelian field strength derived in section 5 by the presence of the term quadratic in the gauge fields.

Under an infinitesimal transformation \( G_{\mu \nu} \) transforms as

\[ G_{\mu \nu} \to G'_{\mu \nu} = G_{\mu \nu} + [\xi^a t_a, G_{\mu \nu}], \]
(8.18)
or, equivalently, as

\[ G_{\mu \nu}^a \to (G_{\mu \nu}^a)' = G_{\mu \nu}^a + f_{bc}^{\;a} \xi^b G_{\mu \nu}^c. \]
(8.19)

For abelian groups the structure constants vanish (i.e., the adjoint representation is trivial for an abelian group), so that the field strengths are invariant in that case.

The result (8.13) can now be expressed in a representation independent form

\[ [D_\mu, D_\nu] = -G_{\mu \nu}, \]
(8.20)
implying that the commutator of two covariant derivatives is equal to an infinitesimal gauge transformation with \(-G_{\mu \nu}^a\) as parameters. This is the Ricci identity. Precisely as for the abelian case we may apply further covariant derivatives to (8.20). In particular we consider

\[ [D_\mu[D_\nu, D_\rho]] + [D_\nu[D_\rho, D_\mu]] + [D_\rho[D_\mu, D_\nu]] \]

which vanishes identically because of the Jacobi identity. Inserting (8.20) we obtain the result

\[ D_\mu G_{\nu \rho} + D_\nu G_{\rho \mu} + D_\rho G_{\mu \nu} = 0, \]
(8.21)

where, according to (8.18), the covariant derivative of \( G_{\mu \nu} \) equals

\[ D_\mu G_{\nu \rho} = \partial_\mu G_{\nu \rho} - [W_\mu, G_{\nu \rho}], \]
(8.22)
or, in components,

\[ D_\mu G^a_{\nu \rho} = \partial_\mu G^a_{\nu \rho} - f_{bc}^{\;a} W^b_\mu G^c_{\nu \rho}. \]
(8.23)

The relation (8.21) is called the Bianchi identity; in the abelian case the Bianchi identity corresponds to the homogeneous Maxwell equations.

9. **Gauge invariant Lagrangians for spin-0 and spin-\( \frac{1}{2} \) fields**

By making use of the covariant derivatives constructed in the previous section it is rather straightforward to construct gauge invariant Lagrangians for spin-0 and spin-\( \frac{1}{2} \) fields. To demonstrate this, consider a set of \( N \) spinor fields \( \psi_i \) transforming under transformations \( U \) belonging to a certain group \( G \) according to \( (i, j = 1, \ldots, N) \)

\[ \psi_i \to \psi'_i = U_{ij} \psi_j, \]
(9.1)
or, suppressing indices \(i, j\), and writing \(\psi\) as an \(N\)-dimensional column vector,

\[
\psi \rightarrow \psi' = U\psi. \tag{9.2}
\]

Conjugate spinors \(\bar{\psi}\), then transform as

\[
\bar{\psi}_i \rightarrow \bar{\psi}'_i = U^*_{ij} \bar{\psi}_j, \tag{9.3}
\]

or, regarding \(\bar{\psi}\) as a row vector and again suppressing indices, as

\[
\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} U^\dagger. \tag{9.4}
\]

Obviously, if \(U\) is unitary, i.e., if \(U^\dagger = U^{-1}\), the massive Dirac Lagrangian,

\[
\mathcal{L} = -\bar{\psi}_i \partial_i \psi_i - m \bar{\psi}_i \psi_i, \tag{9.5}
\]

is invariant under \(G\). This Lagrangian thus describes \(N\) spin-\(\frac{1}{2}\) (anti)particles of equal mass \(m\).

We now require that the Lagrangian be invariant under \textit{local} \(G\) transformations. To achieve this we simply replace the ordinary derivative in (9.5) by a covariant derivative (here and henceforth we will suppress indices \(i, j\), etc.),

\[
\mathcal{L} = -\bar{\psi} \partial \psi - m \bar{\psi} \psi
= -\bar{\psi} \partial \psi - m \bar{\psi} \psi + \bar{\psi} \gamma^\mu W_\mu \psi, \tag{9.6}
\]

where \(W_\mu = W_\mu^a t_a\) is the Lie-algebra valued gauge field introduced in the previous section. Hence the gauge field interactions are given by

\[
\mathcal{L}_{\text{int}} = W_\mu^a \bar{\psi} \gamma^\mu t_a \psi, \tag{9.7}
\]

where \(t_a\) are the parameters of the gauge group \(G\) in the representation appropriate to \(\psi\). Observe that the matrices \(t_a\) are antihermitean in order that the gauge transformations be unitary. The reader will have noticed that there is no obvious coupling constant in (9.7), but we shall see in the next section how this coupling constant can be extracted from the fields \(W_\mu^a\). The matrices \((t_a)_{ij}\) can be regarded as nonabelian charges (up to a proportionality factor \(i\)). The commutation relation for these charges is a consequence of the Lie algebra relation (9.3) which is necessary and sufficient in order that the nonabelian gauge transformations form a group.

To illustrate the above construction, let us explicitly construct a gauge invariant Lagrangian for fermions transforming as doublets under the group \(SU(2)\). This group consists of all \(2 \times 2\) unitary matrices with unit determinant. Such matrices can be written in exponentiated form

\[
U(\xi) = \exp(\xi^a t_a), \quad (a = 1, 2, 3) \tag{9.8}
\]

where the three generators of \(SU(2)\) are expressed in terms of the isotopic spin matrices \(\tau_a\),

\[
t_a = \frac{1}{2} i \tau_a, \tag{9.9}
\]
which coincide with the Pauli matrices used in the context of ordinary spin,

\[ \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(9.10)

As can be explicitly verified the generators \( t_a \) satisfy the commutation relations

\[ [t_a, t_b] = -\epsilon_{abc} t_c, \]  

(9.11)

ensuring that the matrices (9.8) form a group.

Historically the first construction of a nonabelian gauge field theory was based on \( SU(2) \) and was motivated by the existence of the approximate isospin invariance in Nature. According to the notion of isospin (or isobaric spin) invariance the proton and the neutron can be regarded as an isospin doublet. Therefore one introduces a doublet field

\[ \psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \]  

(9.12)

analogous to the \( s = \frac{1}{2} \) doublet of ordinary spin. Conservation of isospin is just the requirement of invariance under isospin rotations

\[ \psi \rightarrow \psi' = U \psi, \]  

(9.13)

where \( U \) is an \( SU(2) \) matrix as defined in (9.8). If isospin invariance would be an exact symmetry then it is a matter of convention which component of \( \psi \) would correspond to the proton and which one to the neutron. If one insists on being able to define this convention at any space-time point separately, then one is led to the construction of a gauge field theory based on local isospin transformations (this is the heuristic argument that motivated Yang and Mills to attempt the construction of the gauge theory of \( SU(2) \)).

Starting from (9.13) it is straightforward to construct the covariant derivative on \( \psi \),

\[ D_\mu \psi = \begin{pmatrix} \partial_\mu \psi_p \\ \partial_\mu \psi_n \end{pmatrix} - \frac{1}{2} i W^a_\mu \tau_a \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} = \begin{pmatrix} \partial_\mu \psi_p \\ \partial_\mu \psi_n \end{pmatrix} - \frac{1}{2} i \begin{pmatrix} W^3_\mu & W^1_\mu - i W^2_\mu \\ W^1_\mu + i W^2_\mu & -W^3_\mu \end{pmatrix} \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \]  

(9.14)

A locally \( SU(2) \) invariant Lagrangian is then obtained by replacing the ordinary derivative by a covariant one in the Lagrangian of a degenerate doublet of spin-\( \frac{1}{2} \) fields,

\[ \mathcal{L} = -\bar{\psi} \partial_\mu \psi - m \bar{\psi} \psi = -\bar{\psi} \partial_\mu \psi - m \bar{\psi} \psi + \frac{1}{2} i W^a_\mu \bar{\psi} \gamma^\mu \tau_a \psi. \]  

(9.15)

The field strength tensors follow straightforwardly from the \( SU(2) \) structure constants exhibited in (9.11)

\[ G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon_{abc} W^b_\mu W^c_\nu, \]  

(9.16)
while under infinitesimal $SU(2)$ transformations the gauge fields transform according to

$$W_\mu^a \to (W_\mu^a)' = W_\mu^a + \epsilon_{abc} W_\mu^b \xi^c + \partial_\mu \xi^a. \quad (9.17)$$

However, contrary to initial expectations, local $SU(2)$ transformations have no role to play in the strong interactions. Instead these forces are governed by an $SU(3)$ gauge theory called quantum chromodynamics because one has introduced the term colour for the degrees of freedom transforming under $SU(3)$. The corresponding gauge fields are called gluon fields because they are assumed to bind the elementary hadronic constituents, called quarks, into hadrons. The quarks transform as triplets under $SU(3)$ and can thus be viewed as a straightforward extension of the $SU(2)$ doublet (9.12). Theories based on $SU(2)$ gauge transformations are relevant for the weak interactions.

Gauge invariant Lagrangians with spin-0 fields are constructed in the same way. For instance, consider an array of complex scalar fields transforming under transformations $U$ as in (9.1). If we regard $\phi$ as a column vector and the complex conjugate fields as a row vector $\phi^*$, we may write

$$\phi \to \phi' = U\phi$$
$$\phi^* \to (\phi^*)' = \phi^* U^\dagger. \quad (9.18)$$

Covariant derivatives read

$$D_\mu \phi = \partial_\mu \phi - W_\mu^a t_a \phi,$$
$$D_\mu \phi^* = \partial_\mu \phi^* - \phi^* t_a^\dagger W_\mu^a. \quad (9.19)$$

Provided that the transformation matrices $U$ in (9.18) are unitary (so that $t_a^\dagger = -t_a$) the following Lagrangian is gauge invariant

$$\mathcal{L} = -|D_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4 \quad (9.20)$$

where we have used a complex inner product $|\phi|^2 \equiv \phi^*_i \phi_i$. Substituting (9.19) leads to

$$\mathcal{L} = - (\partial_\mu \phi^* + \phi^* t_a W_\mu^a)(\partial_\mu \phi - W_\mu^b t_b \phi) - m^2 |\phi|^2 - \lambda |\phi|^4$$
$$= - |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4$$
$$- W_\mu^a (\phi^* t_a \partial_\mu \phi - (\partial_\mu \phi^*) t_a \phi) + W_\mu^a W_\mu^b (\phi^* t_a t_b \phi), \quad (9.21)$$

where in the gauge field interaction terms $\phi^*$ and $\phi$ are written as row and column vectors. This result once more exhibits the role played by the generators $t_a$ as matrix generalizations of the charge. Using (9.9) it is easy to give the corresponding Lagrangian invariant under $SU(2)$. In that case (9.21) reads

$$\mathcal{L} = - |\partial_\mu \phi|^2 - m^2 |\phi|^2 - \lambda |\phi|^4$$
$$- \frac{1}{2} i W_\mu^a \left( \phi^* \tau_a \partial_\mu \phi \right) - \frac{1}{4} (W_\mu^a)^2 |\phi|^2, \quad (9.22)$$

where we have used $\tau_a \tau_b + \tau_b \tau_a = 1 \delta_{a b}$. 

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10. The gauge field Lagrangian

In the preceding section we discussed how to construct locally invariant Lagrangians for matter fields. Starting from a Lagrangian that is invariant under the corresponding rigid transformations, one replaces ordinary derivatives by covariant ones. Until that point the gauge fields are not yet treated as new dynamical degrees of freedom. For that purpose one must also specify a Lagrangian for the gauge fields, which must be separately locally gauge invariant. A transparent construction of such invariants make use of the field strength tensor $G_{\mu\nu}$. Let us recall that $G_{\mu\nu}$ transforms according to (cf. (8.15)),

$$G_{\mu\nu} \rightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{-1},$$  \hspace{1cm} (10.1)

so that for any product of these tensors we have

$$G_{\mu\nu} G_{\rho\sigma} \cdots G_{\lambda\tau} \rightarrow G'_{\mu\nu} G'_{\rho\sigma} \cdots G'_{\lambda\tau} = U(G_{\mu\nu} G_{\rho\sigma} \cdots G_{\lambda\tau}) U^{-1}.$$  \hspace{1cm} (10.2)

Consequently the trace of arbitrary products of the form (10.2) is gauge invariant, i.e.,

$$\text{Tr}(G_{\mu\nu} \cdots G_{\lambda\tau}) \rightarrow \text{Tr}(U G_{\mu\nu} \cdots G_{\lambda\tau} U^{-1}) = \text{Tr}(G_{\mu\nu} \cdots G_{\lambda\tau}).$$  \hspace{1cm} (10.3)

by virtue of the cyclicity of the trace operation. The simplest Lorentz invariant and parity conserving Lagrangian can therefore be expressed as a quadratic form in $G_{\mu\nu}$,

$$\mathcal{L}_W = \frac{1}{4g^2} \text{Tr}(G_{\mu\nu} G^{\mu\nu}),$$  \hspace{1cm} (10.4)

where we have introduced $(4g^2)^{-1}$ as an arbitrary normalization constant. Notice that two alternative forms $G_{\mu\mu}$ and $G_{\mu\nu} G_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ are excluded; the first one vanishes by antisymmetry of $G_{\mu\nu}$, and the second one is not parity conserving (in fact one can show that the second term is equal to a total divergence).

After rescaling $W_\mu$ to $gW_\mu$, the gauge field Lagrangian (10.4) acquires the form

$$\mathcal{L}_W = \text{Tr}(t_a t_b) \left\{ \frac{1}{4}(\partial_\mu W_\mu^a - \partial_\nu W_\nu^a)(\partial_\mu W_\mu^b - \partial_\nu W_\nu^b) - g f_{cd}^{\alpha} W_\mu^c W_\nu^d \partial_\mu W_\nu^\alpha + \frac{1}{4} g^2 f_{cd}^{\alpha} f_{ef}^{\beta} W_\mu^c W_\nu^d W_\sigma^e W_\nu^\beta \right\},$$  \hspace{1cm} (10.5)

where one may distinguish a kinetic term, which resembles the abelian Lagrangian (5.12), and $W^3$- and $W^4$-interaction terms, which depend on the structure constants of the gauge group.

For the Lie groups that we will be interested in, we have $t_a^\dagger = -t_a$ and the generators can be defined such that $\text{Tr}(t_a t_b)$ becomes equal to $-\delta_{ab}$. In that case the Lagrangian reads

$$\mathcal{L}_W = -\frac{1}{4}(\partial_\mu W_\mu^a - \partial_\nu W_\nu^a)^2 + g f_{abc} W_\mu^a W_\nu^b \partial_\mu W_\nu^c$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade} W_\mu^a W_\nu^d W_\sigma^e W_\nu^\gamma.$$  \hspace{1cm} (10.6)

In this particular case there is no need to distinguish between upper and lower indices $a, b, \ldots$ and $f_{abc} \equiv f_{bac}$ is totally antisymmetric.

One may combine the gauge field Lagrangian (10.6) with a gauge invariant Lagrangian for the matter fields, and derive the corresponding Euler-Lagrange equations from Hamilton's
principle. To be specific let us choose the gauge invariant Lagrangian (9.6) for fermions and combine it with (10.6),

\[ \mathcal{L} = \mathcal{L}_W + \mathcal{L}_\psi \]
\[ = \frac{1}{4g^2} \text{Tr}(G_{\mu\nu}G^{\mu\nu}) - \overline{\psi} \gamma^\mu \psi - m \overline{\psi} \psi \]

(10.7)

The field equations for the fermions are obviously a covariant version of the Dirac equation, i.e.,

\[ (\slashed{D} + m)\psi = 0, \quad \overline{\psi} \gamma^\mu (\slashed{D} - m) = 0, \]

(10.8)

or explicitly (using \( t_a^\dagger = -t_a \))

\[ (\slashed{D} + m)\psi = W^a t_a \psi, \]
\[ \overline{\psi} (\gamma^\mu (\slashed{D} + m) = -\overline{\psi} t_a \gamma^\mu W^a, \]

(10.9)

(10.10)

where we have not yet extracted the gauge coupling constant \( g \) from \( W_\mu \). These equations are the nonabelian generalizations of (5.12) and (5.13).

To derive the field equations for the gauge fields requires more work and we give the result without proof,

\[ \frac{1}{g^2} D^\nu G^a_{\mu\nu} = J^a_\mu, \]

(10.11)

which is obviously a nonabelian extension of (5.24). The explicit form of this equation is rather complicated,

\[ \frac{1}{g^2} \left\{ \partial^\nu (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) + f_{bc}^a (W^{b\nu} \partial_\mu W_c^\nu + W_\mu^b \partial_\nu W^{c\nu} - 2W^{b\nu} \partial_\nu W_\mu^c) \right. \]
\[ \left. + f_{de}^a f_{bc}^c W_\mu^d W_\mu^b W_\mu^c \right\} = J^a_\mu, \]

(10.12)

where

\[ J^a_\mu = \overline{\psi} \gamma_\mu t_a \psi. \]

(10.13)

To examine whether \( J^a_\mu \) is conserved, we apply a covariant derivative \( D_\mu \) to (10.11). On the left-hand side this leads to

\[ \frac{1}{g^2} D_\mu D_\nu G^{a\mu\nu} = \frac{1}{2} [D_\mu, D_\nu] G^{a\mu\nu} \]
\[ = \frac{1}{2} G^{b\mu} f_{bc}^a G^{c\nu}, \]

(10.14)

where we have used the Ricci identity (10.28). Because \( f_{bc}^a \) is antisymmetric in \( b \) and \( c \), (10.14) vanishes. Therefore the current satisfies a covariant divergence equation

\[ D_\mu J^{a\mu} = 0, \]

(10.15)

or, explicitly,

\[ \partial_\mu J^{a\mu} - f_{bc}^a W_\mu^b J^{c\mu} = 0. \]

(10.16)

This result implies that gauge fields can only couple consistently to currents that are covariantly constant. According to (10.15) the charges associated with the current are not quite conserved.
The reason is obviously that the gauge fields are not neutral. Their contributions must be included in order to define charges that are conserved.

11. Spontaneously broken symmetries

It is possible for a theory to be exactly invariant under a continuous symmetry, while its ground-state solution does not exhibit this symmetry. In itself it is not surprising that a symmetric theory can give rise to nonsymmetric states; take, for instance, the hydrogen atom which is described by a hamiltonian that is rotationally invariant, while its eigenstates with nonvanishing angular momentum are not inert under rotations. Nevertheless its ground state has zero angular momentum, and is thus rotationally symmetric.

An example of a rotationally invariant system which is realized in such a way that the ground state is not symmetric, is the ferromagnet. Nonferromagnetic materials have a rotationally symmetric ground state in which the atomic spins are randomly oriented. Therefore the gross magnetization is zero. However, in a ferromagnet the spin-spin interactions are such that in the state of lowest energy all spins are aligned. This gives rise to a finite magnetization which breaks the manifest rotational symmetry; thus the rotational symmetry is realized in a spontaneously broken way. This does not mean that rotational symmetry has no consequences anymore, but the most obvious implications of having a symmetric theory are absent. One important aspect of a spontaneously broken realization is that the ground state must be infinitely degenerate. From a nonsymmetric ground state one may construct an infinite number of states by applying the symmetry transformations on the ground state. All these different states must have the same energy as the original one, because of the symmetry of the theory; the hamiltonian of the system still commutes with all symmetry transformations. Indeed for the ferromagnet with all spins aligned in a given direction, one obtains an infinite set of ground states by rotations of the magnet.

We now discuss these phenomena in the context of a field-theoretic model based on a complex spinless field $\phi$:

$$\mathcal{L} = -|\partial_\mu \phi|^2 - V(|\phi|).$$

This Lagrangian is invariant under constant phase transformations of $\phi$

$$\phi(x) \rightarrow \phi'(x) = e^{i\xi} \phi(x).$$

Such $U(1)$ transformations can also be represented as two-dimensional rotations of the real and imaginary parts of $\phi$. Hence the groups $U(1)$ and $O(2)$ are equivalent.

In theories such as (11.1) the fields are usually expanded about some constant value for which the potential (and thus the energy) has an absolute minimum. This value characterizes the ground state of the system, in the same way as the magnetization of the ferromagnet specifies its ground state. Of course, the actual value changes when quantum corrections are included, but such effects will not concern us here. The field $\phi$ will have fluctuations about this classical value corresponding to dynamical degrees of freedom. Such degrees of freedom can be associated with particles; the constant field value which represents the field configuration with minimal energy is called the vacuum expectation value. The nature of the fluctuations about this value
is determined by the Lagrangian. Expanding the field about its vacuum expectation value \( v \), we obtain a Lagrangian of the Klein-Gordon type for the two field components contained in \( \phi \), which describe particles with a mass determined by the second derivative of \( V(|\phi|) \) at \( \phi = v \). Rather than working this out in detail, we give a systematic description of the various possibilities.

The first possibility is that the potential acquires its minimum at \( \phi = 0 \). In that case expanding the Lagrangian about \( \phi = 0 \) gives rise to

\[
\mathcal{L} = -|\partial_\mu \phi|^2 - \mu^2 |\phi|^2 + \ldots
\]  \hspace{1cm} (11.3)

This shows that the excitations described by \( \phi \) correspond to particles with mass \( \mu \); note that \( \mu^2 \) must be positive because we have expanded the potential about a minimum. We may thus distinguish two kinds of particles, corresponding to the real and imaginary part of \( \phi \). Both have the same mass, which can be understood on the basis of the symmetry (11.2) which rotates the real and imaginary parts of the field. Such a symmetric realization of the theory is called the Wigner-Weyl mode.

We now consider the case where the minimum is acquired for a non-zero field value. In that case one immediately realizes that there must be an infinite set of minima because of the symmetry (11.2). In the plane of real and imaginary components of \( \phi \) these minima are located on a circle (see Fig. 5), and each of them represents a possible ground state. This situation describes a spontaneously broken realization of the symmetry (11.2), because the fact that we are forced to consider the theory for nonvanishing vacuum expectation value means that the symmetry is no longer manifest. However, further inspection shows that the symmetry still has an important implication. Because of the degeneracy there is one direction in which the potential remains constant when expanding about the minimum. Consequently, one of the excitations about the ground state value of \( \phi \) is massless. This is in accordance with the Goldstone theorem, which states that to every generator of the symmetry group that is spontaneously broken,
there corresponds a massless particle. This particle is called the Goldstone particle, and the spontaneously broken realization is called the Goldstone mode. One recognizes that the massless degrees of freedom are related to the symmetry that is broken, since it is this symmetry that causes the degeneracy of the potential. Because the symmetry in this case is generated by a scalar parameter, which shifts the phase of $\phi$, the particle is a scalar particle. But more complicated examples of spontaneously broken symmetries are possible.

We now consider the Goldstone mode in somewhat more detail. Since we are expanding the field about some nonvanishing value, it is convenient to make a decomposition

$$
\phi(x) = \frac{1}{\sqrt{2}} \rho(x) e^{i\xi(x)}.
$$

(11.4)

This leads to

$$
\partial_\mu \phi = \frac{1}{\sqrt{2}} \left( \partial_\mu \rho + i \rho \partial_\mu \xi \right),
$$

(11.5)

which is inserted into the Lagrangian (11.1)

$$
\mathcal{L} = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{1}{2} \rho^2 (\partial_\mu \xi)^2 - V(\rho/\sqrt{2})
$$

(11.6)

Clearly the radial degrees of freedom describe a particle with a mass given by

$$
\mu^2 = \frac{\partial^2}{\partial \rho^2} V(\rho/\sqrt{2}) \bigg|_{\rho=\mu} = \frac{1}{2} V''(\rho/\sqrt{2}).
$$

(11.7)

But the angular degrees of freedom related to $\theta$ do not have a mass, and we find a standard kinetic term for a massless scalar field with some additional derivative interactions. This confirms the result of our heuristic considerations, and is in agreement with Goldstone's theorem.

12. The Brout-Englert-Higgs mechanism

The existence of two possible realizations of a symmetry naturally raises the question whether a similar phenomenon exists for local gauge symmetries. We will see that this is indeed the case; there exists a second realization of theories with local gauge invariance, which causes the generation of a mass term for the gauge fields. To analyse this in detail we extend the model of the previous section by introducing an abelian gauge field $A_\mu$ and by requiring invariance under local $U(1)$ transformations. The combined transformation rules thus read

$$
\phi(x) \rightarrow \phi'(x) = e^{iq\xi(x)} \phi(x),
$$

$$
A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x).
$$

(12.1)

Following the rules of the previous sections, it is easy to write down a Lagrangian invariant under these transformations. We add a kinetic term for the gauge field to (12.1), and replace the derivatives of $\phi$ by covariant derivatives

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(A) - |D_\mu \phi|^2 - V(|\phi|),
$$

$$
F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu,
$$

$$
D_\mu \phi = \partial_\mu \phi - iq A_\mu \phi.
$$

(12.2)
We assume that the potential acquires an absolute minimum for nonvanishing field values; therefore we adopt the decomposition (11.4). In this parametrization the phase transformation is expressed by

$$\theta(x) \rightarrow \theta'(x) = \theta(x) + q \xi(x),$$

(12.3)

and the covariant derivative takes the form

$$D_\mu \phi = \frac{1}{\sqrt{2}} e^{i\theta} (\partial_\mu \rho - iq\rho(A_\mu - q^{-1}\partial_\mu \theta)).$$

(12.4)

We now define a new field $B_\mu$ by

$$B_\mu = A_\mu - q^{-1}\partial_\mu \theta,$$

(12.5)

which is inert under the gauge transformations. The covariant derivative can then be written as

$$D_\mu \phi = \frac{1}{\sqrt{2}} e^{i\theta} (\partial_\mu \rho - iq\rho B_\mu).$$

(12.6)

Since the relation between $B_\mu$ and $A_\mu$ takes the form of a (field-dependent) gauge transformation, we can simply replace the field strength $F_{\mu\nu}(A)$ by the corresponding tensor $F_{\mu\nu}(B)$. Therefore the Lagrangian can be expressed entirely in terms of the fields $\rho$ and $B_\mu$, which are both gauge invariant,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2(B) - \frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{2} q^2 \rho^2 B_\mu^2 - V(\rho/\sqrt{2}).$$

(12.7)

If we now expand the field $\rho$ about its vacuum expectation value $v$, we find that the Lagrangian (12.7) describes a massive spin-1 field $B_\mu$, with a mass given by

$$M_B = |qv|.$$  

(12.8)

The massless field that corresponds to the Goldstone particle in the model of the previous section has simply disappeared, while the massive spinless field remains.

At this point we realize that we could have derived (12.7) directly, by exploiting the gauge invariance in order to put $\theta(x) = 0$ from the beginning. This amounts to choosing a gauge condition $\phi = \rho/\sqrt{2}$, which is called the unitary gauge. The advantage of this gauge is that the physical content of the model is immediately clear. However, this gauge is extremely inconvenient for calculating quantum expectations, because it leads to many more ultraviolet divergences than the so-called renormalizable gauge conditions.

It is important to realize that the degeneracy of the ground state that was present in the case without local gauge invariance, has disappeared. The degeneracy is related to the fictitious degrees of freedom that are affected by the gauge transformations; these degrees of freedom have no physical content. Hence the phase of $\phi$ becomes irrelevant and only the radial degree of freedom, which is gauge invariant, has physical significance. The same remark applies to the vacuum expectation of $\phi$. Actually the term vacuum expectation value is somewhat misleading in this context. Since the physical states are gauge invariant it is not possible to have nonvanishing expectation values for quantities that are not gauge invariant. Strictly speaking the only relevant quantity is the expectation value of $|\phi|$, while the phase of $\phi$ is not relevant. This represents
a crucial difference with the situation described in the previous section where the phase of \( \phi \) does represent a physical degree of freedom. In that case the physical states are not required to be invariant under the symmetry, which leads to the connection between a nonzero expectation value and the infinite degeneracy of the ground state.

Hence we have discovered that in the spontaneously broken mode the gauge-dependent degrees of freedom, which effectively reside in the phase \( \theta \), decouple from the theory. The reason for this decoupling is rather obvious, since a gauge invariant theory does not depend on gauge degrees of freedom. Unlike in the realization where the potential acquires a minimum at \( \phi = 0 \) and the gauge field remains massless, the decoupling takes place in a purely algebraic manner without the necessity of making nonlocal field redefinitions. The number of field components has not been changed in this way. Previously we had a complex field and a gauge field representing \( 2 + 3 = 5 \) degrees of freedom; in this realization we have only one spinless field and a vector field, but no gauge invariance. Hence we still count \( 1 + 4 = 5 \) degrees of freedom. Also the number of physical degrees of freedom has not changed. Originally we had two scalar particles and a massless spin-1 particle; since the latter has two physical degrees of freedom, the total number of physical degrees of freedom is four. In the spontaneously broken realization we have one scalar and one massive spin-1 particle. Massive spin-1 particles have three polarizations, so that we count again four physical degrees of freedom.

It is possible to understand the above phenomenon in more physical terms. The gauge field \( A_\mu \) mediates a force between charged particles which is of long range. However, when this field is generated in a medium, it is not obvious that it will still manifest itself as a long-range force. The medium may polarize under the influence of an electromagnetic field, so that the electromagnetic forces will be screened. The characteristic screening length is then inversely proportional to the mass of the gauge field. In fact this phenomenon is well known in superconductivity.

13. Massive \( SU(2) \) gauge fields

We will now apply the Brout-Englert-Higgs mechanism to an \( SU(2) \) gauge theory. Consider \( SU(2) \) gauge fields \( W^a_\mu \) coupled to a doublet of spinless fields denoted by \( \phi \). The relevant Lagrangians were already given in the previous sections (cf. (10.6) and (9.22)) and we find

\[
\mathcal{L} = - \frac{i}{4} (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2 - g \epsilon_{abc} W^a_\mu W^b_\nu \partial_\mu W^c_\nu \\
- \frac{1}{4} g^2 \epsilon_{abcde} W^a_\mu W^b_\nu W^c_\rho W^d_\sigma \\
- \left[ \partial_\mu \phi \right]^2 + \mu^2 |\phi|^2 - \lambda |\phi|^4 \\
- \frac{i}{2} g W^a_\mu \left( \phi^* \tau_a \partial_\mu \phi \right) - \frac{1}{4} g^2 (W^a_\mu)^2 |\phi|^2.
\]

(13.1)

With \( \mu^2, \lambda > 0 \) the potential acquires a minimum for a nonzero value of the field \( \phi \). Following the example of the previous section we decompose \( \phi \) according to

\[
\phi(x) = \left( \begin{array}{c} \phi_1(x) \\ \phi_2(x) \end{array} \right) = \frac{1}{\sqrt{2}} \Phi(x) \left( \begin{array}{c} 0 \\ \rho(x) \end{array} \right),
\]

(13.2)

where \( \Phi(x) \) is an \( x \)-dependent \( SU(2) \) matrix, which is a generalization of the phase factor \( \exp i\theta \) (which is a \( U(1) \) "matrix") used in the previous sections. Here we make use of the fact that the
doublet $\phi$ can be brought into the form $(0, \rho/\sqrt{2})$ by a suitable gauge transformation. The field $\rho$ is gauge invariant and represents the invariant length of the doublet field $\phi$.

In principle, we could now substitute the parametrization (13.2) into the Lagrangian, redefine the gauge fields in a way analogous to (12.5), and observe that the generalized phase factor $\Phi$ disappears from the Lagrangian. However, it is more convenient to adopt the unitary gauge: owing to the local $SU(2)$ invariance we can simply ignore the matrix $\Phi$ and replace $\phi$ by $(0, \rho/\sqrt{2})$. The Lagrangian then takes the form

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)^2 - g\epsilon_{abc}W_\mu^a W_\nu^b \partial_\mu W_\nu^c$$
$$- \frac{3}{4}g^2\epsilon_{abc}\epsilon_{ade} W_\mu^a W_\nu^d W_\nu^e W_\nu^e$$
$$- \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\mu^2 \rho^2 - \frac{1}{4}\lambda \rho^4 - \frac{1}{8}g^2 \rho^2 (W_\mu^a)^2. \quad (13.3)$$

Let us now determine the values $\rho = \pm v$ for which the potential $V(\rho) = -\frac{1}{2}\mu^2 \rho^2 + \frac{1}{4}\lambda \rho^4$ acquires a minimum. The derivative of $V(\rho)$ vanishes whenever $\rho = 0$ or $-\mu^2 + \lambda \rho^2 = 0$. At $\rho = 0$ we have a local maximum, while the minima are at $\rho = \pm v$ with

$$v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (13.4)$$

The mass of the so-called Higgs particle, which is associated with $\rho$, equals

$$m_\rho^2 = 2\lambda v^2, \quad (13.5)$$

while the $W$-masses follow from substituting $\rho^2 = v^2$ into the last term in the Lagrangian (13.3),

$$M_W^2 = \frac{1}{4}g^2 v^2. \quad (13.6)$$

The field $\rho$ seems to play a minor role. It only interacts with the gauge fields through the $\rho^2 W^2$ interaction. In the limit $\lambda \to \infty$, keeping $v$ fixed, the degrees of freedom associated with $\rho$ are suppressed, and one is left with the standard Lagrangian for $SU(2)$ gauge fields with an extra mass term. At the classical level this procedure is harmless and there is no reason why one cannot drop the Higgs field. However, for the quantum theory, the situation is quite different, at least for the nonabelian case. With an explicit mass term the theory is not renormalizable, so that there is no way to obtain sensible predictions. As it turns out the presence of the extra scalar field has a smoothening effect on the quantum corrections and makes the theory renormalizable. This aspect, and not so much the gauge invariance of the original theory which is no longer manifest in (13.3) anyway, forms the prime motivation for constructing theories according to the Brout-Englert-Higgs recipe.

14. The prototype model for $SU(2) \otimes U(1)$ electroweak interactions

We will now extend the model of the previous section in two respects. First we introduce an extra $U(1)$ gauge group, so that the resulting gauge group is $SU(2) \otimes U(1)$. Secondly we introduce two fermions denoted by $p$ and $n$, which will also transform under the combined gauge group in a way that we will specify shortly. Our goal is to exhibit the essential features of the standard
model for electroweak interactions, which is based on this gauge group. As the gauge group is of dimension 4, there will be four gauge fields; the three gauge fields of SU(2) will be denoted by $W_\mu^a$ and the U(1) gauge field by $B_\mu$. However, the Brout-Englert-Higgs mechanism now introduces a novel feature. Initially the three gauge fields of SU(2) and the gauge field of U(1) are massless and have no direct interactions. After the emergence of a mass term, however, it turns out that there is one nontrivial mixture of the gauge fields which remains massless. This field is associated with a nontrivial subgroup of SU(2) $\otimes$ U(1) and will describe the photon.

Let us first comment on the way in which the fermions transform under the gauge transformations. To that order we decompose the fields in chiral components with the help of the projection operators $\frac{1}{2}(1 \pm \gamma_5)$. Their left-handed components are assigned to a doublet representation of SU(2); their right-handed counterparts are singlets:

$$\psi_L = (p_L, n_L); \quad p_R; \quad n_R. \quad (14.1)$$

In the original leptonic version of this model $p$ and $n$ correspond to the neutrino and the electron, respectively. The right-handed neutrino was chosen to decouple in that case, and only occurs as a free field. For hadrons, $p$ and $n$ may for instance correspond to the "up" and the "down" quark, respectively.

Under the additional U(1) group the doublet fields transform as

$$\phi \rightarrow \phi' = e^{\frac{i}{4}\eta \xi} \phi, \quad (14.2)$$

$$\psi_L \rightarrow \psi'_L = e^{\frac{i}{4}\eta \xi} \psi_L, \quad (14.3)$$

and the singlets as

$$p_R \rightarrow p'_R = e^{\frac{i}{4}\eta \xi} p_R, \quad (14.4)$$

$$n_R \rightarrow n'_R = e^{\frac{i}{4}\eta \xi} n_R, \quad (14.5)$$

where $\xi$ is the parameter of the U(1) transformations and, for the moment, $q$, $q_1$, $q_2$ and $q_3$ are arbitrary numbers.

We now assume that the potential is such that the potential acquires a minimum for $\phi \neq 0$. In that case we can decompose $\phi$ according to (13.2). Ignoring the matrix $\Phi$, which amounts to choosing the unitary gauge, gives

$$\phi(x) = (0, \rho(x)/\sqrt{2}), \quad (14.6)$$

with $\rho(x)$ a real scalar field. The form of (14.6) is left invariant under a nontrivial U(1) subgroup of SU(2) $\otimes$ U(1). To identify this subgroup, consider first a somewhat larger subgroup of SU(2) $\otimes$ U(1) consisting of the diagonal matrices. They are parametrized as

$$U(\xi^2, \xi) = \begin{pmatrix} e^{\frac{i}{g}(-g\xi^3 + q\xi)} & 0 \\ 0 & e^{\frac{i}{g}(g\xi^3 + q\xi)} \end{pmatrix}, \quad (14.7)$$

where we have rescaled the SU(2) parameter $\xi^3$ with the SU(2) gauge coupling constant $g$ in accordance with the procedure outlined for the gauge fields in section 10. In order that (14.6) be left invariant under the transformations (14.7), we must obviously have $-g\xi^3 + q\xi = 0$. This motivates the following decomposition of $\xi^3$ and $\xi$,
\begin{equation}
\xi^3 = \cos \theta_W \xi^Z + \sin \theta_W \xi^{EM},
\end{equation}
\begin{equation}
\xi = \cos \theta_W \xi^{EM} - \sin \theta_W \xi^Z,
\end{equation}
where the weak mixing angle \( \theta_W \) satisfies the condition
\begin{equation}
\tan \theta_W = \frac{g}{g'},
\end{equation}
such that the \( U(1) \) subgroup generated by the parameter \( \xi^{EM} \) leaves (14.6) invariant. To see this, substitute (14.8) into (14.7). Using (14.9) it is then obvious that (14.6) is left invariant under transformations (14.7) with \( \xi^Z = 0 \). The group generated by \( \xi^{EM} \), which we denote by \( U(1)^{EM} \) henceforth, thus remains a manifest local gauge symmetry and is not affected by the nonzero value of the field \( \phi \). Hence \( U(1)^{EM} \) corresponds to the electromagnetic gauge transformations in this model, and the weak mixing angle \( \theta_W \) characterizes the embedding of \( U(1)^{EM} \) into the full gauge group \( SU(2) \otimes U(1) \). Observe that, although we have not yet considered a Lagrangian, the symmetry structure of the model is already to a large extent determined by the representation content of the scalar fields. The fact that the model of this section has precisely one massless gauge field is a consequence of choosing a doublet field. For other scalar field configurations one would obtain a different mass spectrum for the gauge fields.

We now redefine the gauge fields \( W^3_\mu \) and \( B_\mu \) in accordance with the decomposition (14.8),
\begin{equation}
W^3_\mu = \cos \theta_W Z_\mu + \sin \theta_W A_\mu,
B_\mu = \cos \theta_W A_\mu - \sin \theta_W Z_\mu.
\end{equation}
Let us now examine how the various gauge fields transform under the two \( U(1) \) transformations parametrized by \( \xi^{EM} \) and \( \xi^Z \). Using the infinitesimal gauge transformations of \( W^a_\mu \) and \( B_\mu \) in terms of the original parameters of \( SU(2) \otimes U(1) \),
\begin{equation}
\delta W^a_\mu = \partial_\mu \xi^a + g \epsilon^{abc} \xi^b W^c_\mu,
\delta B_\mu = \partial_\mu \xi,
\end{equation}
it follows that the fields \( A_\mu \) and \( Z_\mu \) transform according to
\begin{equation}
\delta A_\mu = \partial_\mu \xi^{EM},
\delta Z_\mu = \partial_\mu \xi^Z,
\end{equation}
which identifies \( A_\mu \) as the physical photon field. The other field \( Z_\mu \) corresponds to a neutral massive vector boson, whose mass will be different from that of the fields \( W^{1,2}_\mu \) because of the electroweak-mixing effects. The fields \( W^{1,2}_\mu \) are electrically charged since they transform under electromagnetic gauge transformations. It is convenient to decompose them according to
\begin{equation}
W^\pm_\mu = \frac{1}{2} \sqrt{2} (W^{1}_\mu \mp iW^{2}_\mu).
\end{equation}
Under infinitesimal electromagnetic gauge transformations \( W^\pm_\mu \) transform as
\begin{equation}
\delta W^\pm_\mu = \pm i(g \sin \theta_W) \xi^{EM} W^\pm_\mu,
\end{equation}
which shows that the W bosons associated with \( W^\pm_\mu \) carry an electric charge equal to \( \pm (g \sin \theta_W) \). This charge will be denoted by \( e \), so that we have
\[ e = g \sin \theta_W = q \cos \theta_W. \] (14.15)

The electric charge \( Q^{EM} \), measured in units of \( e \), is defined by the lowest-order coupling of the photon field \( A_\mu \). From (14.10) we can determine the photon coupling in terms of the lowest-order coupling of the fields \( B_\mu \) and \( W^3_\mu \). The former is determined by the constants \( q_i \), defined in (14.2-5), and the latter is given by the \( SU(2) \) generator associated with \( \xi^3 \). We will denote this generator by \( i \) times the hermitean matrix \( T_3 \). Since we are only dealing with singlets or doublets, \( T_3 \) is equal to zero or to the matrix \( \frac{1}{2} \tau_3 \). The charge \( Q^{EM} \) is thus given by

\[
e Q^{EM} = \frac{1}{2} q_i \cos \theta_W + g \sin \theta_W T_3 \\
= e(T_3 + \frac{1}{2} q_i/q) \\
= e(T_3 + \frac{1}{2} Y). \] (14.16)

The operator \( Y \), which is often called the "weak hypercharge", measures the \( U(1) \) "charges" \( q_i \) in units of the coupling constant \( q \) of the field \( \phi \). Hence \( \phi \) has \( Y = 1 \) by definition. According to (14.16) charge differences within \( SU(2) \) multiplets are necessarily multiples of \( e \). Using (14.10) we can also derive a similar expression for the lowest-order coupling of the neutral vector boson \( Z_\mu \) to the other fields, which we denote by \( Q^Z \),

\[
Q^Z = -\frac{1}{2} q_i \sin \theta_W + g \cos \theta_W T_3, \\
= -\frac{1}{2} q \sin \theta_W Y + g \cos \theta_W T_3, \\
= \frac{g}{\cos \theta_W} \left( T_3 - \sin^2 \theta_W Q^{EM} \right). \] (14.17)

The Lagrangian for the gauge fields can now be presented. We define the Lie algebra valued form

\[
g W_\mu = g W^a_\mu (\frac{1}{2} \tau_a) = \frac{1}{2} \begin{pmatrix}
g \cos \theta_W Z_\mu + e A_\mu & g \sqrt{2} W^+_\mu \\
g \sqrt{2} W^-_\mu & -g \cos \theta_W Z_\mu - e A_\mu
\end{pmatrix}, \] (14.18)

and its corresponding field strength

\[
G_{\mu \nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - g [W_\mu, W_\nu], \] (14.19)

or, in component form,

\[
G^a_{\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon_{abc} W^b_\mu W^c_\nu. \] (14.20)

It is convenient to decompose the \( SU(2) \) field strengths according to

\[
G^0_{\mu \nu} \equiv \frac{1}{2} \sqrt{2} (G^1_{\mu \nu} + G^2_{\mu \nu}) \\
= \partial_\mu^E W^\pm_\nu - \partial_\nu^E W^\pm_\mu \pm ig \cos \theta_W (W^\pm_\mu Z_\nu - W^\pm_\nu Z_\mu), \] (14.21)

\[
G^3_{\mu \nu} = \cos \theta_W (\partial_\mu Z_\nu - \partial_\nu Z_\mu) + \sin \theta_W (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu), \] (14.22)

where \( \partial_\mu^E W^\pm_\nu = (\partial_\mu \mp ie A_\mu) W^\pm_\nu \). The \( U(1) \) field strength becomes

\[
G^0_{\mu \nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu \\
= \cos \theta_W (\partial_\mu A_\nu - \partial_\nu A_\mu) - \sin \theta_W (\partial_\mu Z_\nu - \partial_\nu Z_\mu). \] (14.23)
In terms of the above field strengths the gauge field Lagrangian is equal to
\begin{equation}
\mathcal{L}_G = -\frac{1}{2} G^+_{\mu\nu} G^-_{\mu\nu} - \frac{1}{4} G^3_{\mu\nu} G^3_{\mu\nu} - \frac{1}{4} G^0_{\mu\nu} G^0_{\mu\nu}.
\end{equation}

(14.24)
The quadratic terms in (14.24) read
\begin{equation}
\mathcal{L}_0 = -\frac{1}{2} (\partial_\mu W^+_{\mu} - \partial_\nu W^+_{\mu})(\partial_\nu W^-_{\mu} - \partial_\nu W^-_{\mu}) - \frac{1}{4} (\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 - \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2.
\end{equation}

(14.25)
Note that the normalization factor for the $W$-fields is different, because $W^\pm$ is complex. The normalization convention for real and complex fields was already discussed in sections 1 and 2. In addition there are cubic and quartic gauge field interactions. The electromagnetic interactions follow from replacing the derivatives on the charged fields $W^\pm_\mu$ by the covariant derivatives $\partial^{EM}_\mu$ defined above. In addition there is an interaction with the photon field which is separately gauge invariant and follows from (14.22). It gives rise to a magnetic moment for the $W$, and is equal to
\begin{equation}
\mathcal{L}_{\text{max}} = ie (\partial_\mu A_\nu - \partial_\nu A_\mu) W^{+\mu} W^{-\nu}.
\end{equation}

(14.26)
To find the masses of the $W^\pm_\mu$ and $Z_\mu$ bosons we examine the Lagrangian for the scalar field $\phi$,
\begin{equation}
\mathcal{L}_\phi = -|D_\mu \phi|^2,
\end{equation}
where the explicit form for the covariant derivative on $\phi$ is equal to
\begin{equation}
D_\mu \phi = \begin{pmatrix}
\partial_\mu - ie A_\mu - \frac{1}{2} ig \cos^{-1} \theta_W (1 - 2 \sin^2 \theta_W) Z_\mu & -\frac{1}{2} \sqrt{2} g W^+_{\mu} \\
-\frac{1}{2} \sqrt{2} g W^-_{\mu} & \partial_\mu + \frac{1}{2} ig \cos^{-1} \theta_W Z_\mu
\end{pmatrix}
= \begin{pmatrix}
\partial_\mu Z_\mu + \frac{1}{2} i \frac{g}{\cos \theta_W} Z_\mu & -\frac{1}{2} i g W^+_{\mu} \\
-\frac{1}{2} i g W^-_{\mu} & \partial_\mu + \frac{1}{2} i g \cos^{-1} \theta_W Z_\mu
\end{pmatrix}
= \begin{pmatrix}
\partial_\mu - i e A_\mu - \frac{1}{2} \sqrt{2} g W^+_{\mu} \\
-\frac{1}{2} \sqrt{2} g W^-_{\mu}
\end{pmatrix}.
\end{equation}

(14.28)
Substituting this result into (14.27) we find
\begin{equation}
\mathcal{L}_\phi = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{1}{8} g^2 \cos^{-2} \theta_W \rho^2 Z^2_\mu - \frac{1}{4} g^2 \rho^2 |W^+_{\mu}|^2.
\end{equation}

(14.29)
We can now read off the masses of the $W$ and the $Z$ bosons after combining (14.25) with (14.29) and substituting $\rho = v$,
\begin{equation}
M_W = \frac{1}{2} g v, \quad M_Z = \frac{1}{2} \frac{g v}{\cos \theta_W}.
\end{equation}

(14.30)
This leads to the well-known relation
\begin{equation}
\frac{M_W}{M_Z} = \cos \theta_W.
\end{equation}

(14.31)
The gauge boson couplings to the fermions follow directly from substituting covariant derivatives into the free massless Dirac Lagrangians,
\[ L_f = - \bar{\psi}_L \psi_L - \bar{\psi}_R \psi_R - \bar{\psi}_R \psi_R \]
\[ = - \bar{\psi}_L \hat{\psi}_L - \bar{\psi}_R \hat{\psi}_R - \bar{\psi}_R \hat{\psi}_R + \frac{1}{2} \sqrt{2} ig \left( W^+_\mu \bar{\psi}_L \gamma^\mu p_L + W^-_\mu \bar{\psi}_L \gamma^\mu n_L \right) \]
\[ + \frac{ig}{\cos \theta_W} Z^-_\mu \left( \frac{1}{2} \bar{\psi}_L \gamma^\mu p_L - \frac{1}{2} \bar{\psi}_R \gamma^\mu n_L \right), \]

where we have used (14.17) and the definition
\[ \hat{\psi}_\mu = \partial_\mu - i e A_\mu - g Z^-_\mu \frac{\sin^2 \theta_W}{\cos \theta_W} Q^Q. \]

The Fermi coupling constant \( G_F \) is defined by the strength of the four-fermion coupling caused by the exchange of a charged intermediate \( W \)-boson in the limit of zero momentum transfer. In this model its value is given by
\[ \frac{G_F}{\sqrt{2}} = \frac{1}{4} \left( \frac{1}{4} \sqrt{2} g \right)^2 \]
\[ = \frac{1}{2v^2} \]
\[ = \frac{e^2}{8M_W^2 \sin^2 \theta_W}. \]

We now discuss the generation of fermion masses in this model. Mass terms are constructed from the product of a right-handed and a left-handed fermion field. However, the right- and left-handed fermions belong to different \( SU(2) \) representations, so that a direct construction of an invariant mass term is excluded. Therefore the only way for the fermions to acquire masses is via a Yukawa coupling of the scalar doublet \( \phi \) to products of a right- and a left-handed fermion field. Expanding \( \phi \) about \( v \) will then lead to fermionic mass terms. In order to construct the necessary Yukawa couplings we first form two left-handed \( SU(2) \) singlets, \( \psi_1 \) and \( \psi_2 \), by taking the invariant products of \( \psi_L \) with \( \phi \),
\[ \psi_1 = \phi^a \psi_{L_a} = \phi_1^a p_L + \phi_2^a n_{L}, \]
\[ \psi_2 = -e^{ab} \phi_a \psi_{L_b} = -\phi_1 n_{L} + \phi_2 p_{L}. \]

Using the parametrization (14.6) these singlets take the following form
\[ \psi_1(x) = \frac{1}{\sqrt{2}} \rho(x) n_L(x), \]
\[ \psi_2(x) = \frac{1}{\sqrt{2}} \rho(x) p_L(x). \]

Under \( U(1) \) \( \psi_1 \) and \( \psi_2 \) transform as
\[ \psi_1 \rightarrow \psi'_1 = e^{\frac{i}{2} \delta(x_+ - x) \xi} \psi_1, \]
\[ \psi_2 \rightarrow \psi'_2 = e^{\frac{i}{2} \delta(x + x) \xi} \psi_2. \]

We can now construct two invariant Yukawa couplings, if we assume the following relations among the \( U(1) \) coupling constants,
\[ q_2 = q_1 + q, \quad (14.38a) \]
\[ q_3 = q_1 - q. \quad (14.39a) \]

The corresponding invariants are, respectively,
\[ \mathcal{L}_p = -\sqrt{2} G_p \bar{p}_R \psi_2 + h.c., \quad (14.38b) \]
\[ \mathcal{L}_n = -\sqrt{2} G_n \bar{n}_R \psi_1 + h.c., \quad (14.39b) \]

which, using (14.36), can be written as
\[ \mathcal{L}_p = -G_p \rho \bar{p}_R p_L + h.c., \quad (14.38c) \]
\[ \mathcal{L}_n = -G_n \rho \bar{n}_R n_L + h.c. \quad (14.39c) \]

The coupling constants \( G_p \) and \( G_n \) can be chosen real by absorbing possible phases into the definition of \( p_R \) and \( n_R \). Expanding \( \rho(z) \) about the constant \( v \) then gives rise to the following expression for the masses
\[ m_p = G_p v, \quad m_n = G_n v. \quad (14.40) \]

Combining (14.3) and (14.40) it follows that,
\[ G_{p,n} = \sqrt{2} G_F m_{p,n}, \quad (14.41) \]

which shows that the Higgs field \( \rho \) couples weakly to the fermions and also that its coupling is proportional to the fermion mass. Consequently Higgs bosons are expected to couple more strongly to heavy flavours.

Let us now briefly discuss the two versions of this prototype model. In its leptonic version \( p \) and \( n \) correspond to the neutrino and its corresponding lepton, respectively. The right-handed neutrino is assumed to decouple from the other particles as a free massless field. Therefore one ignores the Yukawa coupling \( \mathcal{L}_p \) and the corresponding restriction (14.38a), and chooses \( q_2 = 0 \). The model then depends on three independent gauge coupling constants \( q_1, \theta_W, \) and \( g \). As follows from (14.16) the requirement that the (left-handed) neutrino is electrically neutral forces one to choose \( q_1 = -q \). The condition (14.39a) then leads to \( q_3 = -2q \). Consequently, the left-handed leptons have \( Y = -1 \) and the right-handed lepton has \( Y = -2 \).

The hadronic version of the model requires the presence of both Yukawa couplings in order to generate masses for both the quarks corresponding to \( p \) and \( n \). Because of the two restrictions (14.38a) and (14.39a) we have again three independent gauge coupling constants, say, \( q_1, \theta_W, \) and \( g \). Since the mass terms (14.40) and (14.41) are invariant under electromagnetic gauge transformations the coupling of the photon must be purely vectorlike. This can be verified explicitly by using (14.16), (14.38a) and (14.39a). The charges of the two quarks are equal to \( \frac{1}{2} e(q_1 / q + 1) \) and \( \frac{1}{2} e(q_1 / q - 1) \). The choice \( q_1 = \frac{1}{2} q \) leads to the desired quark charges \( \frac{2}{3} e \) and \( -\frac{1}{3} e \), and also implies \( q_2 = \frac{3}{2} q, q_3 = -\frac{3}{2} q \). Therefore the left-handed quarks have \( Y = \frac{1}{3} \), and the right-handed quarks have \( Y = \frac{4}{3} \) and \( Y = -\frac{4}{3} \).

Let us now consider this model coupled to a full generation, i.e. to a lepton pair and three (because of colour) quark pairs. An important property of this fermion configuration is
that, provided we make the weak-hypercharge assignments for leptons and quarks that we found above, this model is anomaly-free. By anomaly-free we mean that certain divergent Feynman diagrams, consisting of fermion loops with external gauge fields, that are known to spoil the renormalizability of the theory, are absent (or, at least, their leading divergence cancels). For the model at hand, it is only the $U(1)$ gauge field coupling that may lead to difficulties. In order to eliminate the anomaly, we must satisfy two conditions. According to the first one, the sum of the hypercharges of the left-handed fermion doublets must equal the sum of the hypercharges of the right-handed doublets. Since there are only left-handed doublets, we must have

$$\sum_{\text{fermion doublets}} Y = 0. \quad (14.42)$$

According to the discussion above, a lepton doublet has $Y = -1$, and a quark doublet has $Y = \frac{1}{3}$. Since a generation contains three quark doublets, the hypercharges add up to zero in agreement with (14.42).

The second condition states that the sum of $Y^3$ for all the left-handed fermions must equal the sum of $Y^3$ for all right-handed fermions,

$$\sum_{\text{left-handed fermions}} Y^3 = \sum_{\text{right-handed fermions}} Y^3. \quad (14.43)$$

According to the hypercharge assignments, the leptons contribute $2 \cdot (-1)^3$ to the left-hand side and $(-2)^3$ to the right-hand side of (14.43). For each quark colour, the corresponding contributions are $2 \cdot \left(\frac{1}{3}\right)^3$ and $\left(\frac{4}{3}\right)^3 + \left(-\frac{2}{3}\right)^3$, respectively. With these values it is easy to verify that also (14.43) is satisfied,

$$2 \cdot (-1)^3 + 2 \cdot \left(\frac{1}{3}\right)^3 = (-2)^3 + 3 \left(\left(\frac{4}{3}\right)^3 + \left(-\frac{2}{3}\right)^3\right). \quad (14.44)$$

Suggested reading:
I.J.R. Aitchison and A. Hey, Gauge theories in particle physics (Hilger, 1982)
F. Halzen and A. Martin, Quarks and leptons (Oxford, 1984)
K. Huang, Quarks, leptons and gauge fields (World Scient., 1981)
F. Mandl and G. Shaw, Quantum field theory (Wiley, 1984)
C. Quigg, Gauge theories of the strong, weak and electromagnetic interactions (Benjamin, 1983)
J.C. Taylor, Gauge theories of weak interactions (Cambridge Univ. Press, 1976)
B. de Wit and J. Smith, Field theory in particle physics, Vol 1 (North-Holland, 1986); part of the material in these lectures is based on the forthcoming Volume 2
QCD AND COLLIDER PHYSICS

R.K. Ellis and W.J. Stirling

These lectures were delivered, at Egmond-aan-Zee, by R.K. Ellis. Since they had been given, the preceding year, at the CERN School in Lefkada (Greece), and they have appeared in the corresponding proceedings, they will not be reprinted here. We refer the reader to

Proc. 1988 CERN School of Physics, Lefkada (Greece), 1988

or to the preprint

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5 Summary

Appendix
1 Prologue

Precision tests of the electroweak theory will be a central theme in $e^+e^-$ experiments around the $Z$ resonance [1]. The present theory of the electroweak interaction, known as the “Standard Model”, is a gauge invariant quantum field theory with the symmetry group $SU(2) \times U(1)$ spontaneously broken by the Higgs mechanism. It contains three free parameters to describe the gauge bosons $\gamma, W^\pm, Z$ and their interactions with the fermions. For a comparison between theory and experiment three independent experimental input data are required. The most natural choice for $Z$ physics is given by the electromagnetic fine structure constant $\alpha$, the muon decay constant (Fermi constant) $G_\mu$, and the mass of the $Z$ boson which has meanwhile been measured with high accuracy. Other measurable quantities are predicted in terms of the input data. Each additional precision experiment which allows the detection of small deviations from the lowest order predictions can be considered a test of the electroweak theory at the quantum level. In the Feynman-graphical representation of the scattering amplitude for a given process the higher order terms show up as diagrams containing closed loops. The lowest order amplitudes could also be derived from a corresponding classical field theory whereas the loop contributions can only be obtained from the quantized version. The renormalizability of the Standard Model ensures that it retains its predictive power also in higher orders. Precision experiments will therefore either confirm the Standard Model as a fully fledged quantum field theory, in analogy to QED, or signal the need for some significant modifications. The higher order terms, commonly called radiative corrections, are usually complicated in their concrete form, but they are finally the consequence of the basic Lagrangian with a simple structure. The radiative corrections, better denoted as electroweak quantum effects, thus provide the theoretical basis for electroweak precision tests.

These lectures contain an elementary course on the calculation of electroweak radiative corrections in the minimal model and applications to $Z$ physics in $e^+e^-$ annihilation at LEP/SLC. The discussion of the practically important QED corrections arising from real and virtual photon bremsstrahlung is postponed to part II. Here we deal predominantly with those higher order terms which remain after removing the QED corrections. In general this is not possible in an unambiguous way; for the class of processes covered by these lectures, however, the separation of the QED subset is physically reasonable and formally consistent.

Starting point is the basic electroweak Lagrangian which we will accept as a given theoretical framework, and the set of Feynman rules following from this Lagrangian. To warm up we begin with QED and some general introductory remarks on the description of scattering processes in terms of Feynman diagrams. The most technical part is contained in section 3 which is devoted to the computation of loop integrals entering the propagators as self energies and the problems of divergence and renormalization. The generalization to the electroweak theory is the content of section 4. After a more theoretical part dealing with the 1-loop corrections in the muon lifetime and in the amplitudes for $e^+e^- \rightarrow f\bar{f}$ we discuss the physical predictions for the vector boson masses, the $Z$ width and partial widths, and the on-resonance asymmetries. Comparisons with the experimental data, as far as available, are also shown.
2 Introduction to the electroweak Standard Model

2.1 Elements of Quantum Field Theory; Quantum Electrodynamics (QED)

Quantum Field Theory is the theoretically adequate description of particles and their fundamental interactions in accordance with the principles of causality and Lorentz invariance. QED, which is part of the electroweak Standard Model and by itself the parade example of a successful quantum field theory, will serve us as an introductory lesson to get familiar with the Feynman rules for the propagation and interaction of spin-1/2 and spin-1 particles. In these lectures there will not be enough time to give detailed derivations; we will just quote the rules on a more intuitive basis and refer to the literature for a formally correct treatment. More general information can also be found in the lectures by de Wit given at this School.

The free electron-positron field:

The basic dynamical quantity in the field theoretical framework is the defining Lagrangian of the system. Non-interacting electrons and positrons are described by a common spinor field $\psi(x)$ in terms of which the free Dirac Lagrangian is expressed as follows:

$$\mathcal{L}_\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \quad (1)$$

The corresponding field equation following from (1) by Hamilton's principle is the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (2)$$

The one-particle states are characterized by the set of quantum numbers

- mass $m$,
- spin $s = 1/2$,
- momentum $\vec{p}$,
- helicity $\sigma = \pm 1/2$.

They correspond to the one-particle solutions of (2):

$$u_\sigma(p)e^{-ipx} \text{ for electrons, } \quad v_\sigma(p)e^{ipx} \text{ for positrons.} \quad (3)$$

The 4-component spinors $u, v$ are given by the solutions of the Dirac equations in momentum space:

$$(\vec{p} - m)u_\sigma(p) = 0 \quad \text{for electrons} \quad (4)$$

$$(\vec{p} + m)v_\sigma(p) = 0 \quad \text{for positrons.}$$

In Feynman diagrams, such free states occur as external lines. These fermion lines carry an arrow denoting the flow of the electric charge of the particle, which is by convention the electron. Consequently, the arrow of a positron line points into the direction opposite to the positron momentum.
External lines may be incoming or outgoing. We have to keep the following rules where \( u \) and \( v \) have to be replaced by their adjoints:

- \( \bar{u}_\sigma \) for an outgoing particle (electron)
- \( \bar{v}_\sigma \) for an incoming anti-particle (positron).

The \( e^\pm \) propagator:

The free particle solutions (3) correspond to monochromatic waves which extend from - to + infinity. The physically more interesting situations, however, are encountered in the creation of a particle/antiparticle at a space-time point \( y \) and its free propagation to another point \( x \) where it may annihilate or scatter. Such a situation is described by the propagator \( S(x - y) \) which obeys the inhomogenous wave equation with point-like source:

\[
(i \gamma^\mu \partial_\mu - m) S(x - y) = \delta(x - y),
\]

in momentum space:

\[
(q - m) S(q) = 1.
\]

The solution yields the propagator

\[
S(q) = \frac{i}{q - m + i\varepsilon} = i \frac{q + m}{q^2 - m^2 + i\varepsilon}
\]

where the \(+ i\varepsilon\) prescription tells us how to treat the pole at \( q^2 = m^2 \) in order to get the correct causal space-time behaviour of the Fourier transformed \( S(x - y) \). The additional factor \( i \) appears by convention.

Formally, the propagator is the inverse of the matrix in that part of the action which is quadratic in the fields (=free field term), expressed in terms of the Fourier transformed fields:

\[
S[\psi] = \int d^4x L_\psi(x) = \int d^4q \bar{\psi}(q)(q - m)\psi(q).
\]

In a graphical representation, the propagator is a line with two endpoints carrying an arrow which denotes the flow of the particle charge:

\[
\begin{array}{c}
\bullet \\
\frac{i}{q - m + i\varepsilon}
\end{array}
\]

In Feynman diagrams for physical scattering amplitudes propagators always occur as internal lines. Since the momentum of internal particles has no direct physical evidence, we can always choose the internal momentum as in the direction of the arrow.

The free photon field:

Photons appear as one-particle states of a massless spin-1 field \( A_\mu(x) \) which corresponds to the electromagnetic vector potential in Maxwell's theory. The dynamical system is described by the Lagrangian

\[
L_A = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]

with the field strengths

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]
Recalling Hamilton's principle we obtain the field equations for $A_\mu$ by

$$\partial^\mu \frac{\partial L_A}{\partial (\partial^\mu A^\nu)} - \frac{\partial L_A}{\partial A^\nu} = 0 \tag{11}$$

to be

$$g^{\mu\nu} \Box A_\nu - \partial^\nu \partial^\nu A_\nu = 0. \tag{12}$$

The physical one-particle states are characterized by the quantum numbers

momentum $\vec{k}$, helicity $\lambda = \pm 1$

(there is no helicity $\lambda = 0$ state for massless spin-1 particles). The corresponding one-particle solutions of (12) are monochromatic waves

$$\epsilon_\mu(\lambda) e^{-ikx} \tag{13}$$

with polarization vectors $\epsilon_\mu(\lambda)$ obeying

$$\epsilon(\lambda) \cdot k = 0, \quad \epsilon^2 = -1 \quad \text{(spacelike).}$$

In Feynman diagrams free photon states appear as external lines for incoming and outgoing photons, graphically denoted by

``````
\epsilon_\mu(\lambda)
``````

The corresponding expression is just given by the polarization vector $\epsilon_\mu(\lambda)$ of the photon with helicity $\lambda$.

*The photon propagator:*

The photon propagator can be interpreted as the solution of the inhomogenous wave equation with a point-like source at a given space-time point $y$. It describes the free propagation of a photon, created at $y$, before it annihilates at a point $x$:

$$(g^{\mu\nu} \Box A_\nu - \partial^\mu \partial^\nu A_\nu) D_{\nu\rho}(x - y) = g^{\rho\sigma}(x - y) \delta(x - y), \tag{14}$$

in momentum space:

$$(q^2 g_{\mu\nu} - g_{\mu q_{\nu}}) D_{\nu\rho}(q) = -g^{\rho\sigma}. \tag{15}$$

Formally, $D_{\nu\rho}$ is defined as the inverse of the matrix in the quadratic (=free) part of the action (after integration by parts and Fourier transformation)

$$S[A] = \int d^4x L_A(x) = -\int d^4 q A^\mu(q)(q^2 g_{\mu\nu} - g_{\mu q_{\nu}})A^\nu(q) \tag{16}$$

which is equivalent to solving (15). Here a difficulty specific for massless vector particles appears: The matrix in (15) resp. in (16) does not have an inverse. This is immediately clear since $q^\mu$ is an eigenvector with eigenvalue $= 0$. The origin of this peculiarity can be found in the invariance of $L_A$ under gauge transformations

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \tag{17}$$
which reads in momentum space:

\[ A_\mu(k) = A_\mu(k) + k_\mu A(k) \]

Due to this symmetry we have less than 4 physical degrees of freedom; or, in other words, the operator of the l.h.s. of (15) projects on a 3-dimensional subspace orthogonal to \( q^\nu \).

The conventional way out is to introduce a gauge condition, which formally breaks the gauge symmetry, by adding a gauge-fixing term

\[ \mathcal{L}_A \rightarrow \mathcal{L}_A - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \]

to the Lagrangian (9) with an arbitrary parameter \( \xi \). The photon propagator is then determined by

\[ \left[ q^2 g_{\mu\nu} - q_\mu q_\nu (1 - \frac{1}{\xi}) \right] D^{\nu\rho} = g_\mu^\nu \]

yielding \(^1\)

\[ D_{\mu\nu} = \left[ -g_{\mu\nu} + (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right] \frac{i}{q^2 + i\varepsilon} \tag{18} \]

A particularly simple form is obtained in the so-called Feynman gauge with \( \xi = 1 \):

\[ D_{\mu\nu} = \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} \tag{19} \]

One may wonder whether the introduction of an unphysical degree of freedom in the propagator leads to physical consequences. Since the photon couples to a conserved electromagnetic current those parts \( \sim q^\mu \) do not contribute to \( S \)-matrix elements and all physical quantities are gauge invariant and independent of the parameter \( \xi \). More complicated theories with non-abelian gauge symmetries, however, need additional effort to cure the effects coming from the unphysical polarization states (see section 2.3).

**Electromagnetic interaction:**

The standard way to introduce the interaction between the photon and the \( e^+e^- \) fields follows the principle of “minimal substitution” where in the free Lagrangian

\[ \mathcal{L}_{\text{free}} = \mathcal{L}_\psi + \mathcal{L}_A, \]

composed from (1) and (9), the usual derivative is replaced by the covariant derivative

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu - ie A_\mu. \tag{20} \]

This leads to the Lagrangian of QED

\[ \mathcal{L}_{\text{QED}} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{\text{int}} \]

with the interaction term

\[ \mathcal{L}_{\text{int}} = e \bar{\psi} \gamma_\mu \psi A^\mu. \tag{21} \]

\(^1\)the \(+i\varepsilon\) ensures the correct treatment of the pole at \( q^2 = 0 \), and the factor \( i \) is by convention
In terms of Feynman rules, this trilinear coupling is represented by a vertex graph with two fermion lines and one photon line

\[ i \in \gamma_\mu \]

where the interaction point is defined by \( ie_\gamma_\mu \) (the factor \( i \) is again a convention). The lines connected in the vertex can either be internal or external, described by propagators or external wave functions, respectively. In addition, momentum conservation is imposed for the momenta associated with these particle lines.

The QED Lagrangian with the interaction (21) is invariant under the local gauge transformations (17) of the vector field if simultaneously the fermion field is transformed according to

\[ \psi(x) \rightarrow e^{ieA(x)} \psi(x) \tag{22} \]

with the gauge function \( A(x) \) of (17). These gauge transformations form the abelian group \( U(1) \) where the charge acts as the generator of the group. In the language of gauge theories, the photon field is the gauge field associated with the group generator.

**Perturbation theory for scattering processes:**

With the basic elements given above we are now able to write down graphically and analytically the amplitude for a given scattering process in perturbative QED. Perturbative calculations of S-matrix elements consist in the expansion in powers of the coupling constant which in our case is given by \( e \). Since the power of \( e \) corresponds to the number of vertices we equivalently have an expansion in the number of vertices. The “lowest order” or “Born” diagram for a given process is thus given by the diagram which connects the initial and final states by the minimum number of vertices.

The simplest process is that of 2-particle \( \rightarrow \) 2-particle scattering. As an example we consider the annihilation of electrons and positrons into muon pairs \( e^+e^- \rightarrow \mu^+\mu^- \). The lowest order diagram contains two vertices and is therefore of order \( e^2 \):

\[ \begin{array}{c}
\bar{\psi}(p_1) \\
\psi(p_2)
\end{array} \quad \begin{array}{c}
\mu^+(p_3) \\
\mu^-(p_4)
\end{array} \]

**Fig. 1:** The lowest order diagram for \( e^+e^- \rightarrow \mu^+\mu^- \) in QED

Analytically the amplitude is given by the expression

\[ \bar{v}_{\sigma_2}(p_2)ie\gamma^\mu u_{\sigma_1}(p_1) \frac{-ig_{\mu\nu}}{(p_1 + p_2)^2 + ie} \bar{v}_{\sigma_3}(p_3)ie\gamma^\nu v_{\sigma_4}(p_4). \]

The next order \( (e^4) \) diagrams for this process contain 4 vertices. They are displayed in Figure 2. In contrast to the lowest order “tree graph” these contain closed loops.
Fig. 2: 1-loop diagrams for $e^+e^- \rightarrow \mu^+\mu^-$ in QED
In section 3 we will discuss how to handle such loop diagrams. It is important to note that the renormalizability of QED, which is a consequence of gauge invariance, is crucial to maintain the predictive power of the theory in higher orders of the perturbative calculation. Before doing this, we want to give the basic formulation of the electroweak Standard Model.

2.2 The Standard Model Lagrangian; Feynman rules

The phenomenological basis for the formulation of the Standard Model is given by the following empirical facts:

- The $SU(2) \times U(1)$ family structure of the fermions:
  The fermions appear as families with left-handed doublets and right-handed singlets:
  \[
  \begin{align*}
  &\left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L, \quad \left( \begin{array}{c} \nu_\mu \\ \mu \end{array} \right)_L, \quad \left( \begin{array}{c} \nu_\tau \\ \tau \end{array} \right)_L, \quad \nu_R, \quad \mu_R, \quad \tau_R \\
  &\left( \begin{array}{c} u \\ d \end{array} \right)_L, \quad \left( \begin{array}{c} c \\ s \end{array} \right)_L, \quad \left( \begin{array}{c} t \\ b \end{array} \right)_L, \quad u_R, \quad d_R, \quad c_R, \cdots
  \end{align*}
  \]
  They can be characterized by the quantum numbers of the weak isospin $I$, $I_3$, and the weak hypercharge $Y$.

- The Gell-Mann-Nishijima relation:
  Between the quantum numbers classifying the fermions with respect to $SU(2) \times U(1)$ and their electric charge $Q$ the relation
  \[
  Q = I_3 + \frac{Y}{2}
  \]  
  is valid.

- The existence of vector bosons:
  There are 4 vector bosons as carriers of the electroweak force
  \[
  \gamma, \quad W^+, \quad W^-, \quad Z
  \]
  where the photon is massless and the $W^\pm, Z$ have masses $M_W \neq 0, M_Z \neq 0$.

This empirical structure can be embedded in a gauge invariant field theory of the unified electromagnetic and weak interactions by interpreting $SU(2) \times U(1)$ as the group of gauge transformations under which the Lagrangian is invariant. This full symmetry has to be broken by the Higgs mechanism down to the electromagnetic gauge symmetry; otherwise the $W^\pm, Z$ bosons would also be massless. The minimal formulation, the Standard Model, requires a single scalar field (Higgs field) which is a doublet under $SU(2)$.

According to the general principles of constructing a gauge invariant field theory with spontaneous symmetry breaking the gauge, Higgs, and fermion parts of the electroweak Lagrangian

\[
\mathcal{L} = \mathcal{L}_G + \mathcal{L}_H + \mathcal{L}_F
\]  
(24)
are specified in the following way:

Gauge fields
SU(2) × U(1) is a non-abelian group which is generated by the isospin operators \( I_1, I_2, I_3 \) and the hypercharge \( Y \) (the elements of the corresponding Lie algebra). Each of these generalized charges is associated with a vector field: a triplet of vector fields \( W^{a}_{\mu} \) with \( I_{1,2,3} \) and a singlet field \( B_\mu \) with \( Y \). The isotriplet \( W^{a}_{\mu} \), \( a = 1, 2, 3 \), and the isosinglet \( B_\mu \) lead to the field strengths

\[
W^{a}_{\mu \nu} = \partial_{\mu}W^{a}_{\nu} - \partial_{\nu}W^{a}_{\mu} + g_2 \epsilon_{abc} W^{b}_{\mu}W^{c}_{\nu},
\]

\[
B_{\mu \nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}.
\] (25)

g_2 denotes the non-abelian SU(2) gauge coupling constant, \( g_1 \) the abelian U(1) coupling. From the field tensors (25) the pure gauge field Lagrangian

\[
\mathcal{L}_G = -\frac{1}{4} W^{a}_{\mu \nu} W^{\mu \nu \cdot a} - \frac{1}{4} B_{\mu \nu} B^{\mu \nu}
\] (26)
is formed which is the generalization of (9) for the non-abelian case.

Fermion fields and fermion-gauge interaction
The left-handed fermion fields of each lepton and quark family (color index is suppressed)

\[
\psi^L_j = \left( \begin{array}{c} \psi^L_{j+} \\ \psi^L_{j-} \end{array} \right)
\]

with family index \( j \) are grouped into SU(2) doublets with component index \( \sigma = \pm \), and the right-handed fields into singlets

\[
\psi^R_j = \psi^R_{j\sigma}.
\]

Each left and right-handed multiplet is an eigenstate of the weak hypercharge \( Y \) such that the relation (23) is fulfilled. The covariant derivative

\[
D_\mu = \partial_\mu - i g_2 I_\alpha W_\mu^\alpha + i g_1 \frac{Y}{2} B_\mu
\] (27)
induces the fermion-gauge field interaction via the minimal substitution rule:

\[
\mathcal{L}_F = \sum_j \psi^L_j \gamma_\mu D_\mu \psi^L_j + \sum_{j, \sigma} \psi^R_{j\sigma} \gamma_\mu D_\mu \psi^R_{j\sigma}
\] (28)

Higgs field, Higgs - gauge field and Yukawa interaction
For spontaneous breaking of the SU(2) × U(1) symmetry leaving the electromagnetic gauge subgroup U(1)_em unbroken a single complex scalar doublet field with hypercharge \( Y = 1 \)

\[
\Phi(x) = \left( \begin{array}{c} \phi^+(x) \\ \phi^0(x) \end{array} \right)
\] (29)
is coupled to the gauge fields

\[
\mathcal{L}_H = (D_\mu \Phi)^+ (D^\mu \Phi) - V(\Phi).
\] (30)
with the covariant derivative
\[ D_\mu = \partial_\mu - i g_2 l_\sigma W_\mu^\sigma + i g_1 \frac{g_2}{2} B_\mu. \]

The Higgs field self interaction
\[ V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \frac{\lambda}{4} (\Phi^\dagger \Phi)^2 \]  
(31)
is constructed in a way that it has a non-vanishing vacuum expectation value \( v \), related to the coefficients of the potential \( V \) by
\[ v = \frac{2\mu}{\sqrt{\lambda}}. \]  
(32)
The field (28) can be written in the following way:
\[ \Phi(x) = \begin{pmatrix} \phi^+(x) \\ (v + H(x) + i\chi(x))/\sqrt{2} \end{pmatrix} \]  
(33)
where the components \( \phi^+, H, \chi \) now have vacuum expectation values zero.

The real component \( H(x) \) describes physical neutral scalar particles with mass
\[ M_H = \mu\sqrt{2}. \]  
(34)
The Higgs field components have triple and quartic self couplings following from \( V \), and couplings to the gauge fields via the kinetic term of (30).

In addition, Yukawa couplings to fermions are introduced in order to make the charged fermions massive. The Yukawa term is conveniently expressed in the doublet field components (33). We write it down for one family of leptons and quarks only, neglecting quark mixing:
\[ \mathcal{L}_{\text{Yukawa}} = -g_l (\bar{\nu}_L \phi^+ l_R + \bar{\nu}_R \phi^- \nu_L + \bar{l}_L \phi^0 l_R + \bar{l}_R \phi^0 l_L) \]  
(35)\[ = -g_d (\bar{u}_L \phi^+ d_R + \bar{d}_R \phi^- u_L + \bar{d}_L \phi^0 d_R + \bar{d}_R \phi^0 d_L) \]  
\[ - g_u (\bar{u}_R \phi^+ d_L + \bar{d}_L \phi^- u_R + \bar{u}_R \phi^0 u_L + \bar{u}_L \phi^0 u_R). \]
The Yukawa coupling constants \( g_{l,d,u} \) are related to the masses of the charged fermions by eq. (44). \( \phi^- \) denotes the adjoint of \( \phi^+ \).

**Physical fields and parameters**
The gauge invariant Higgs-gauge field interaction in the kinetic part of (30) gives rise to mass terms for the vector bosons in the non diagonal form
\[ \frac{1}{2} \left( \frac{g_2}{2} v \right)^2 (W_1^2 + W_2^2) + \frac{v^2}{4} \left( W_3^3, B_\mu \right) \begin{pmatrix} g_2^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix} \begin{pmatrix} W_3^3 \\ B_\mu \end{pmatrix}. \]  
(36)
The physical content becomes transparent by performing a transformation from the fields \( W_\mu^\pm, B_\mu \) (in terms of which the symmetry is manifest) to the "physical" fields
\[ W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \pm W_\mu^2) \]  
(37)
and

\[
Z_\mu = \cos \theta_W W^\mu + \sin \theta_W B_\mu
\]
\[
A_\mu = -\sin \theta_W W^\mu + \cos \theta_W B_\mu
\]

In these fields the mass term (36) is diagonal and has the form

\[
M_W^2 W^+ W^- + \frac{1}{2} (A_\mu, Z_\mu) \begin{pmatrix} 0 & 0 \\ 0 & M_Z^2 \end{pmatrix} \begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix}
\]

with

\[
M_W = \frac{1}{2} g_2 v
\]
\[
M_Z = \frac{1}{2} \sqrt{g_1^2 + g_2^2} v
\]

The mixing angle of (38) is given by

\[
\cos \theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \frac{M_W}{M_Z}.
\]

Identifying \( A_\mu \) with the photon field which couples via the electric charge \( e = \sqrt{4\pi} \alpha \) to the electron, \( e \) can be expressed in terms of the gauge couplings in the following way

\[
e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}.
\]

or

\[
g_2 = \frac{e}{\sin \theta_W}, \quad g_1 = \frac{e}{\cos \theta_W}.
\]

Finally, from the Yukawa coupling terms in (35) the fermion masses are obtained:

\[
m_f = g_f \frac{v}{\sqrt{2}} = \sqrt{2} \frac{g_f}{g_2} M_W.
\]

The relations above allow to replace the original set of parameters

\[
g_2, \quad g_1, \quad \lambda, \quad \mu^2, \quad g_f
\]

by the equivalent set of more physical parameters

\[
e, \quad M_W, \quad M_Z, \quad M_H, \quad m_f
\]

where each of them can (in principle) directly be measured in a suitable experiment.

Expressed in the physical fields and parameters we can write down the Lagrangian in a way which allows us to read off the propagators and the vertices most directly. We
specify them in the $\xi = 1$ gauge where the vector boson propagators have the simple algebraic form $\sim g_{\mu\nu}$.

$$\mathcal{L}_G + \mathcal{L}_H =$$

$$A_\mu \not\partial A^\mu \sim \frac{-i g_{\mu\nu}}{q^2}$$

$$+ W^-_\mu (\not\partial + M_W^2) W^{+\mu} \sim \frac{-i g_{\mu\nu}}{q^2 - M_W^2}$$

$$+ Z_\mu (\not\partial + M_Z^2) Z^\mu \sim \frac{-i g_{\mu\nu}}{q^2 - M_Z^2}$$

$$+ H (\not\partial + M_H^2) H \sim \frac{i}{q^2 - M_H^2}$$

+ interaction terms $\sim$ 

$VV, VH, HH$

+ (unphysical degrees of freedom)

$$\mathcal{L}_F + \mathcal{L}_{\text{Yukawa}} =$$

$$\sum_f \bar{f} (i \partial - m_f) f \sim \frac{i}{q - m_f}$$

$$+ J^\mu_{\text{em}} A_\mu \sim \gamma \not\partial - i e Q_f \gamma_\mu$$

$$+ J^\mu_{\text{NC}} Z_\mu \sim Z \not\partial \frac{e}{2 \sin \theta_W \cos \theta_W} \gamma_\mu (\gamma^\mu - a_f \gamma_5)$$

$$+ J^\mu_{\text{CC}} W_\mu \sim W \not\partial \frac{e}{2 \sqrt{2} \sin \theta_W} \gamma_\mu (1 - \gamma_5)$$

$$+ \frac{g_f}{\sqrt{2}} \bar{f} f H \sim \frac{i}{\sqrt{2}} \frac{g_f}{2 \sin \theta_W} \frac{m_f}{M_W}$$

+ (unphysical degrees of freedom)

We do not give the lengthy list of all interaction vertices but refer to the literature, see e.g. [2].
In order to describe scattering processes between fermions in lowest order we can neglect the exchange of Higgs bosons because of their small Yukawa couplings to the known fermions. The standard processes accessible by the experimental facilities are basically 4-fermion processes. These are mediated by the gauge bosons and, sufficient in lowest order, defined by the vertices for the fermions interacting with the vector bosons. They are given in the Lagrangian above for the electromagnetic, neutral and charged current interactions. The neutral current coupling constants in (47) read

\[ v_f = I_f^I - 2Q_f \sin^2 \theta_W \]
\[ a_f = I_f^A. \]

\( Q_f \) and \( I_f^I \) denote charge and third isospin component of \( f \).

Unphysical degrees of freedom

The specific gauge chosen for the Lagrangian (47) provides us with vector boson propagators \( \sim g_{\mu \nu} \) which describe the propagation of 4 vector field components whereas only 3 polarization states are physical. On the other hand, as we have seen in the case of the photon, it is not possible to define a propagator without imposing a gauge condition, thereby introducing the 4th component. In a general \( R_\xi \) gauge, the \( W \) and \( Z \) propagators have the form \((M = M_W, M_Z)\):

\[ \frac{i}{q^2 - M^2} \left( -g_{\mu \nu} + \frac{(1 - \xi)q_\mu q_\nu}{q^2 - \xi M^2} \right) \]

For \( \xi = 1 \) we recover (47).

In case of non-abelian gauge fields, the introduction of the unphysical components would give rise to consequences like \( \xi \)-dependent (gauge dependent) physical quantities unless additional unphysical states ("ghosts") are introduced which together with the unphysical Higgs components in (33) render physical matrix elements in a given order gauge independent. Without going into details, it is this sample of unphysical degrees of freedom and their interactions which is indicated in (47). Only the unitary gauge, obtained for \( \xi \to \infty \), is free of ghosts and of unphysical gauge and Higgs field components. We do not want to pursue these more formal aspects in our following discussion.

2.3 \( \mu \) decay and the masses of the vector bosons

The Fermi model with an effective coupling constant \( G_\mu \) yields an expression for the muon lifetime \( \tau_\mu \) from which the value of \( G_\mu \) is determined. Applying the Standard Model in lowest order we find the tree graph amplitude for the decay matrix element from our Feynman rules:

\[ \mu \rightarrow W \nu_e \]

\[ e \rightarrow W \bar{\nu}_e \]

\[ = \frac{i}{2\sqrt{2} \sin \theta_W} \left( \frac{\epsilon}{2} \right)^2 \frac{J^{(\mu)}_{CC} \cdot J^{(e)}_{CC}}{q^2 - M_W^2} \]

(49)

\[ \mu \rightarrow W \nu_e \]

\[ e \rightarrow W \bar{\nu}_e \]
with

\[ J_{CC}^{(\mu)} = \bar{u}_\nu \gamma_\mu (1 - \gamma_5) u_\mu \]
\[ J_{CC}^{(e)} = \bar{u}_\nu \gamma_\mu (1 - \gamma_5) v_\nu , \]

whereas in the Fermi model the decay amplitude is given by

\[ = i \frac{G_\mu}{\sqrt{2}} \cdot J_{CC}^{(\mu)} \cdot J_{CC}^{(e)} \]  \hspace{1cm} (50)

Consistency of the Standard Model at \( q^2 \ll M_W^2 \) with the Fermi model requires the identification

\[ \frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8 \sin^2 \theta_W M_W^2} \]  \hspace{1cm} (51)

which allows us, together with (41), to predict the vector boson masses in terms of the parameters \( \alpha, G_\mu, \) and \( \sin^2 \theta_W \) as follows:

\[ M_W^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu} \cdot \frac{1}{\sin^2 \theta_W} \]  \hspace{1cm} (52)
\[ M_Z^2 = \frac{\pi \alpha}{\sqrt{2} G_\mu} \cdot \frac{1}{\sin^2 \theta_W \cos^2 \theta_W} \]  \hspace{1cm} (53)

With \( \alpha = 1/137.03604(11), G_\mu = 1.16637(2) \cdot 10^{-5} \text{GeV}^2 \) from the measured \( \tau_\mu, \) and \( \sin^2 \theta_W = 0.231 \pm 0.006 \) from neutrino scattering \([3]\) we obtain:

\[ M_W = 77.6 \pm 1.0 \text{GeV} \]
\[ M_Z = 88.5 \pm 0.8 \text{GeV} . \]

This has to be compared with the experimentally measured masses \([4,5]\)

\[ M_W^{\exp} = 80.0 \pm 0.62 \text{GeV}, \]
\[ M_Z^{\exp} = 91.152 \pm 0.033 \text{GeV} \]

from \( pp \) colliders and LEP. The deviation of more than \( 2\sigma \) is a strong indication for the presence of higher order effects in the mass formulae which makes the discussion of electroweak "radiative corrections" unavoidable.

We want to conclude this section with an outlook on non-minimal versions of the Higgs sector where the so-called \( \rho \)-parameter

\[ \rho_0 = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \]  \hspace{1cm} (54)

is different from unity. Although we are not going to discuss these more general models in their higher order terms, we want to indicate how the relations of these chapter have to be modified.
The tree level effect of a more complicated Higgs system is the appearance of an independent additional parameter which can e.g. be chosen as the \( \rho_0 \) of (54). \( \rho_0 \neq 1 \) means that the restrictive relation (41) of the mixing angle to the boson masses is loosened. Our set of Feynman rules in (47), however, is formally not changed. Only \( \sin^2 \theta_W \) as an ingredient in (47) is now

\[
\sin^2 \theta_W = 1 - \frac{M_W^2}{\rho_0 M_Z^2}
\]  

(55)

in replacement of (41). Consequently, also the mass relation (53) has to be modified according to

\[
M_Z^2 = -\frac{\pi \alpha}{\sqrt{2} G_F} \frac{1}{\rho_0 \sin^2 \theta_W \cos^2 \theta_W},
\]

(56)

whereas (52) remains unchanged.

3 Computation of 1-loop diagrams

3.1 The photon propagator in QED

In the 1-loop diagrams shown in Figure 2 one identifies two types of loops modifying the fermion and photon propgator and a loop correction to the electromagnetic electron-photon coupling. The loop insertions in the propagators are conventionally denoted as self energies and the loop insertion in the vertex as vertex correction:

![e self energy](image1)

![\gamma self energy](image2)

![vertex correction](image3)

In order to deal with the corresponding analytic expressions we have to augment our set of Feynman rules:
• Inside each loop, one momentum is not fixed by momentum conservation at the vertices. Rule: integrate over the free 4-momentum as follows:

\[ \int \frac{d^4k}{(2\pi)^4} \cdots \]  

(57)

• For a closed loop of fermions calculate the trace over the string of Dirac matrices and multiply by \((-1)\) as a consequence of Pauli’s principle for fermions:

\[ (-1) \cdot \text{Tr}(\cdots) \]  

(58)

As a concrete example, we explicitly calculate the fermion loop contributions to the photon propagator. This will show us the technique of computation, exhibit the problem of divergence, and finally provide us with the results for integrals which we shall encounter later again in the fermionic contribution to the \(W\) and \(Z\) propagators.

The self energy bubble is denoted by \(\Sigma^\gamma\). It has two Lorentz indices from the two vertices and is normalized according to the following convention:

\[ \frac{-ig_{\nu\sigma}}{q^2} (-i \Sigma^\gamma_{\rho\sigma}) \frac{-ig^{\mu\nu}}{q^2} \]  

(59)

The self energy tensor follows from the Feynman rules as \(^2\)

\[ -i \Sigma^\gamma_{\rho\sigma} = (-1) \text{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{i}{q + k - m} i e \gamma_\rho \frac{i}{k - m} i e \gamma_\sigma \]

\[ = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\text{Tr}(k + q + m) \gamma_\rho (k + m) \gamma_\sigma}{[(k + q)^2 - m^2][k^2 - m^2]} . \]

Calculating the trace (see appendix) we find the following decomposition:

\[ \Sigma^\gamma_{\rho\sigma} = -i e^2 \cdot 4 \left\{ g_{\rho\sigma} \cdot m^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2][(k + q)^2 - m^2]} \right. \]

\[ - g_{\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + kq}{[k^2 - m^2][(k + q)^2 - m^2]} \]

\[ + 2 \int \frac{d^4k}{(2\pi)^4} \frac{k_\rho k_\sigma}{[k^2 - m^2][(k + q)^2 - m^2]} \right\} + \text{terms} \sim q_\rho q_\sigma . \]  

(60)

\(^2\)we drop the \(ie\) terms and assume that they are always associated with \(m^2\) as \(m^2 - ie\)
The first integral without the variable $k$ in the numerator is called the scalar 2-point integral. The terms $\sim q_\mu q_\nu$ do not contribute in physical matrix elements since the internal photon always couples to fermions via a conserved current. Hence, for practical purposes, we are allowed to skip them in the following discussion.

Our strategy consists of three steps:
(i) regularization
(ii) reduction to scalar integrals
(iii) calculation of the scalar integrals.

(i) Regularization:
A brief look at the structure of the integrals in (60) shows that they diverge for $k^2 \to -\infty$ (UV divergence). For this reason we need a regularization which is a procedure to redefine the integrals in a way that they become finite and mathematically well defined objects. The widely used regularization procedure for gauge theories is that of dimensional regularization: replace the dimension 4 by a lower dimension $D$ where the integrals are convergent:

$$
\int \frac{d^4k}{(2\pi)^4} - \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D}
$$

An (arbitrary) mass parameter $\mu$ has been introduced in order to keep the coupling constant, here the electric charge, in front of the integrals as dimensionless quantities. At the end of a calculation of a physical quantity the limit $D \to 4$ has to be performed.

(ii) Reduction to scalar integrals:
The method described here has been developed in [6]. It allows to reduce algebraically the more complicated tensor integrals in (60) to the scalar 1- and 2-point integrals which we denote by the symbols $A$ and $B_0$:

$$
\mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m^2} =: \frac{i}{16\pi^2} A(m)
$$

$$
\mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{[k^2 - m_1^2][(k + q)^2 - m_2^2]} =: \frac{i}{16\pi^2} B_0(q^2, m_1, m_2)
$$

We already allow for the more general case of different masses although $m_1 = m_2$ would be sufficient for QED.

The algebraic reduction of the integrals with $k^2$ and $kq$ is very simple (for brevity reasons we skip the $D$-dimensional volume element in the integrals):

$$
\int \frac{k^2}{[k^2 - m_1^2][(k + q)^2 - m_2^2]} = \int \frac{1}{k^2 - m_1^2} + m_2^2 \int \frac{1}{[k^2 - m_1^2][(k + q)^2 - m_2^2]}
$$

$$
= \frac{i}{16\pi^2} \left\{ A(m_2) + m_1^2 B_0(q^2, m_1, m_2) \right\}
$$
\[
\int \frac{kq}{|k^2 - m_1^2||q + q^2 - m_2^2|} = \frac{1}{2} \left\{ \int \frac{1}{k^2 - m_1^2} - \int \frac{1}{k^2 - m_2^2} + (m_2^2 - m_1^2 - q^2) \int \frac{1}{|k^2 - m_1^2||q + q^2 - m_2^2|} \right\}
\]

\[
= \frac{i}{16\pi^2} \left\{ \frac{1}{2} \left\{ A(m_1) - A(m_2) + (m_2^2 - m_1^2 - q^2) B_0(q^2, m_1, m_2) \right\} \right\}
\]

\[
=: \frac{i}{16\pi^2} q^2 B_1(q^2, m_1, m_2)
\]  

(63)

Hereby, in the \(A\)-integrals involving \(m_2\), the translation \(k + q \rightarrow k\) was performed in the integrands.

The \(k_\rho k_\sigma\) integral can be decomposed into Lorentz tensors and scalar coefficients as follows:

\[
\int \frac{k_\rho k_\sigma}{|k^2 - m_1^2||q + q^2 - m_2^2|} = \frac{i}{16\pi^2} \left( g_{\rho\sigma} B_{22} + q_\rho q_\sigma B_{21} \right).
\]  

(64)

Since we are not interested in the \(q_\rho q_\sigma\) part we have to determine only \(B_{22}\). Multiplying (64) by \(g^{\rho\sigma}\), \(q^\rho q^\sigma\) we obtain two equations

\[
\int \frac{k^2}{|k^2 - m_1^2||q + q^2 - m_2^2|} = \frac{i}{16\pi^2} \left( DB_{22} + q^2 B_{21} \right)
\]  

(65)

\[
\int \frac{(kq)^2}{|k^2 - m_1^2||q + q^2 - m_2^2|} = \frac{i}{16\pi^2} \left( q^2 B_{22} + q^4 B_{21} \right).
\]

Note that

\[
g_{\rho\sigma} g^{\rho\sigma} = \text{Tr}(1) = D
\]

in \(D\) dimensions for the coefficient of \(B_{22}\). From (65) we get

\[
(D - 1) \cdot \frac{i}{16\pi^2} B_{22} = \int \frac{k^2}{|k^2 - m_1^2||q + q^2 - m_2^2|} - \frac{1}{q^2} \int \frac{(kq)^2}{|k^2 - m_1^2||q + q^2 - m_2^2|}.
\]  

(66)

Writing

\[
(kq)^2 = \frac{1}{2} (kq)(2kq + k^2 + q^2 - m_2^2) - \frac{1}{2} (kq)(k^2 - m_1^2 + q^2 + m_1^2 - m_2^2)
\]

we obtain the second integral in (66) via

\[
\int \frac{(kq)^2}{|k^2 - m_1^2||q + q^2 - m_2^2|} = \frac{1}{2} \int \frac{kq}{k^2 - m_1^2} - \frac{1}{2} \int \frac{kq}{q^2 + m_1^2 - m_2^2} - \frac{1}{2} \int \frac{kq}{2} \frac{q^2 + m_1^2 - m_2^2}{|k^2 - m_1^2||q + q^2 - m_2^2|}
\]

\[
= \frac{i}{16\pi^2} \left\{ \frac{q^2}{2} A(m_2) - \frac{q^2 + m_1^2 - m_2^2}{2} q^2 B_1(q^2, m_1, m_2) \right\}
\]  

(67)
with $B_1$ from (63). Here we have used
\[ \int \frac{k^\mu}{k^2 - m^2} = 0 \]
implied by the antisymmetry of the integrand. Combining the above results we obtain:
\[ (D - 1) B_{22} = \frac{1}{2} A(m_2) + m_1^2 B_0(q^2, m_1, m_2) + \frac{q^2 + m_1^2 - m_2^2}{2} B_1(q^2, m_1, m_2). \]  
(68)

Let us now return to $\Sigma^{\gamma}_{\rho\sigma}$ in (60) which can be decomposed according to
\[ \Sigma^{\gamma}_{\rho\sigma} = g_{\rho\sigma} \Sigma^{\gamma} + q_\rho q_\sigma \Sigma^{\gamma}_{L}. \]  
(69)
The coefficient $\Sigma^{\gamma}$ of $g_{\rho\sigma}$ is what we finally want to derive. Inserting (63) and (64) into (60) we obtain the following formula for the simpler case $m_1 = m_2 = m$:
\[ \Sigma^{\gamma} = \frac{\alpha}{\pi} \left\{ -A(m) + \frac{q^2}{2} B_0(q^2, m, m) + 2 B_{22}(q^2, m, m) \right\}. \]  
(70)

Our final task is now to calculate the scalar integrals $A$ and $B_0$.

(iii) The scalar integrals:
We start with the integral $B_0$ in (62). With help of the Feynman parametrization
\[ \frac{1}{ab} = \int_0^1 dx \frac{1}{(ax + b(1-x))^2} \]
and after a shift in the $k$-variable, $B_0$ can be written in the form
\[ \frac{i}{16\pi^2} B_0(q^2, m_1, m_2) = \int_0^1 dx \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k}{[k^2 - x^2 q^2 + x(q^2 + m_1^2 - m_2^2) - m_2^2]^2}. \]  
(71)
The advantage of this parametrization is a simpler $k$-integration where the integrand is only a function of $k^2 = (k^0)^2 - \vec{k}^2$. In order to transform it into an euclidean integral we perform the substitution \(^3\)
\[ k^0 = i k_E^0, \quad \vec{k} = k_E, \quad d^D k = i d^D k_E \]
where the new integration momentum $k_E$ has a definite metric:
\[ k^2 = -k_E^2, \quad k_E^2 = (k_E^0)^2 + \cdots + (k_E^{D-1})^2. \]
This leads us to an euclidean integral over $k_E$:
\[ \frac{i}{16\pi^2} B_0 = i \int_0^1 dx \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{(k_E^2 + Q)^2} \]  
(72)
where
\[ Q = x^2 q^2 - x(q^2 + m_1^2 - m_2^2) + m_1^2 - i\epsilon \]  
(73)
\(^3\)the $i\epsilon$-prescription in the masses ensures that this is compatible with the pole structure of the integrand
is a constant with respect to the $k_E$-integration.

Also the 1-point integral $A$ of (62) can be transformed into an euclidean integral:

$$
\frac{i}{16\pi^2} A(m) = -i \frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{k_E^2 + m^2}.
$$

(74)

Both $k_E$-integrals belong to the general type

$$
\int \frac{d^D k_E}{(k_E^2 + Q)^n}
$$

of rotational invariant integrals in a $D$-dimensional euclidean space. They can be evaluated in $D$-dimensional polar coordinates ($k_E^2 = R$)

$$
\int \frac{d^D k_E}{(k_E^2 + Q)^n} = \frac{1}{2} \int d\Omega_D \int_0^\infty dR R^{D-1} \frac{1}{(R + Q)^n},
$$
yielding

$$
\frac{\mu^{4-D}}{(2\pi)^D} \int \frac{d^D k_E}{(k_E^2 + Q)^n} = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} \cdot Q^{-n + \frac{D}{2}}.
$$

(75)

The singularities of our initially 4-dimensional integrals are now recovered as poles of the $\Gamma$-function for $D = 4$ and values $n \leq 2$.

Although the l.h.s of (75) as a $D$-dimensional integral is sensible only for integer values of $D$, the r.h.s. has an analytic continuation in the variable $D$: it is well defined for all complex values $D \neq 2n$, in particular for

$$
D = 4 - \varepsilon \quad \text{with} \quad \varepsilon > 0.
$$

For physical reasons we are interested in the vicinity of $D = 4$. Hence we consider the limiting case $\varepsilon \to 0$ and perform an expansion around $D = 4$ in powers of $\varepsilon$. For this task we need the following properties of the $\Gamma$-function at $x \to 0$:

$$
\Gamma(x) = \frac{1}{x} - \gamma + O(x),
$$

$$
\Gamma(-1 + x) = -\frac{1}{x} + \gamma - 1 + O(x)
$$

with

$$
\gamma = -\Gamma'(1) = 0.577\ldots
$$

known as Euler's constant.

$n = 1$:

Combining (74) and (75) we obtain the scalar 1-point integral for $D = 4 - \varepsilon$:

$$
A(m) = \frac{-\mu^\varepsilon}{(4\pi)^{-\varepsilon/2}} \cdot \frac{\Gamma(-1 + \frac{\varepsilon}{2})}{\Gamma(1)} \cdot (m^2)^{1-\varepsilon/2}
$$

$$
= m^2 \left( \frac{2}{\varepsilon} - \gamma + \log 4\pi - \log \frac{m^2}{\mu^2} + 1 \right) + O(\varepsilon)
$$

$$
\equiv m^2 \left( \Delta - \log \frac{m^2}{\mu^2} + 1 \right) + O(\varepsilon)
$$

(77)
here we have introduced the abbreviation for the singular part
\[ \Delta = \frac{2}{\epsilon} - \gamma + \log 4\pi. \] (78)

\( n = 2 \):

For the scalar 2-point integral \( B_0 \) we evaluate the integrand of the \( x \)-integration in (72) from (75) as follows:
\[
\frac{\mu^4}{(4\pi)^{2-\epsilon/2}} \cdot \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)} \cdot Q^{-\epsilon/2} = \frac{1}{16\pi^2} \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log \frac{Q}{\mu^2} \right) + O(\epsilon)
\]
\[
= \frac{1}{16\pi^2} \left( \Delta - \log \frac{Q}{\mu^2} \right) + O(\epsilon). \] (79)

Since the \( O(\epsilon) \) terms vanish in the limit \( \epsilon \to 0 \) we skip them in the following formulae. Insertion in (72) with \( Q \) from (73) yields:
\[
B_0(q^2, m_1, m_2) = \Delta - \int_0^1 dx \log \frac{x^2q^2 - x(q^2 + m_1^2 - m_2^2) + m_1^2 - i\epsilon}{\mu^2} \] (80)

For the photon self energy we need the special case \( m_1 = m_2 = m \):
\[
B_0(q^2, m, m) = \Delta - \log \frac{m^2}{\mu^2} + \tilde{B}_0(q^2, m, m) \] (81)
\( \tilde{B}_0 \) denotes the finite function
\[
\tilde{B}_0(q^2, m, m) = -\int_0^1 dx \log \left( \frac{x^2q^2 - xq^2 + m^2}{m^2} - i\epsilon \right). \] (82)

Evaluating now \( B_{22} \) in (68) for \( D = 4 - \epsilon \) (see Appendix) and collecting everything in (70) we obtain the transverse photon self energy:
\[
\Sigma^\gamma(q^2) = \frac{\alpha}{3\pi} \left\{ q^2 \left( \Delta - \log \frac{m^2}{\mu^2} \right) + (q^2 + 2m^2) \tilde{B}_0(q^2, m, m) - \frac{q^2}{3} \right\}. \] (83)

Now we can write down the \( g_{\mu\nu} \)-part of the photon propagator up to 1-loop order
\[
\begin{array}{c}
\begin{array}{c}
\text{--------} \\
\text{+} \\
\text{--------}
\end{array}
\end{array}
\]
by summing (19) and (59):
\[
D^\gamma_{\mu\nu} = -i g_{\mu\nu} \left( \frac{1}{q^2} - \frac{1}{q^2} \Sigma^\gamma(q^2) \frac{1}{q^2} \right) \]
\[
= -i \frac{g_{\mu\nu}}{q^2} \left[ 1 - \Pi^\gamma(q^2) \right]. \] (84)
The dimensionless quantity
\[ \Pi'(q^2) = \frac{\Sigma'(q^2)}{q^2} \] (85)
is usually referred to as the photon "vacuum polarization". We list some simple expressions arising from (83) for special situations of practical interest:

- **\( q^2 = 0 \):**
  \[ \Pi'(0) = \frac{\alpha}{3\pi} \left( \Delta - \log \frac{m^2}{\mu^2} \right) \] (86)

- **light fermions (\(| q^2 | \gg m^2 \)):**
  \[ \Pi'(q^2) = \frac{\alpha}{3\pi} \left( \Delta - \log \frac{m^2}{\mu^2} + \frac{5}{3} - \log \frac{|q^2|}{m^2} + i\pi \theta(q^2) \right) \] (87)

- **heavy fermions (\(| q^2 | \ll m^2 \)):**
  \[ \Pi'(q^2) = \frac{\alpha}{3\pi} \left( \Delta - \log \frac{m^2}{\mu^2} + \frac{q^2}{5m^2} \right) \] (88)

Having solved the technical problems of this section we encounter a physical problem: Our result for \( \Pi'(q^2) \) is, by means of regularization, mathematically well defined, but it depends on the unphysical parameters \( D \) and \( \mu \) and diverges for \( D \to 4 \). The next step of getting rid of these quantities without losing the physical information of the 1-loop calculation is the procedure of renormalization.

### 3.2 The concept of renormalization

In a physical matrix element the photon propagator is always connected with a factor \( e^2 \) from the charges at the vertices. The classical charge \( e \) with
\[ \frac{e^2}{4\pi} = \alpha = \frac{1}{137.03604} \]
is experimentally determined (in principle) from the cross section for Thomson scattering
\[ \sigma_{Th} = \frac{e^4}{6m_e^2}. \] (89)

QED in lowest order recovers \( \sigma_{Th} \) from Compton scattering in the classical limit \( k^2 = 0 \), \( k^0 \to 0 \), as a consequence of the \( ee\gamma \) vertex \( ie\gamma_{\mu} \). This vertex is dressed in 1-loop order by the following set of diagrams

\[ \text{Diagrams} \]

In the classical limit the vertex correction cancels against the external electron self energies as a consequence of the QED Ward identity which follows from current conservation

\[ ^4 \text{loop contributions to external particles on their mass shell receive an extra factor 1/2} \]
in QED. Hence we are left with the photon vacuum polarization only which contributes a shift in the coupling constant at the vertex

\[ i e \gamma_\mu \rightarrow i [e - \frac{1}{2} e \Pi^\gamma(0)] \gamma_\mu. \]  

(90)

If we interpreted \( e \) as the classical charge we would immediately run into a contradiction since it is the quantity in the brackets which enters the cross section \( \sigma_{T\alpha} \) in (89). Therefore, in order to solve this problem, we have to re-interpret the quantity \( e \) in the Lagrangian: it is not the physical charge (which is measureable) but some "bare charge" \( e_0 \) (which is not measurable) related to the physical charge \( e \) by

\[ e_0 = e + \delta e. \]  

(91)

The counter term \( \delta e \) is formally of 1-loop order since it is due to self interactions not present at the Born level. The recipe is to replace in the Lagrangian and the Feynman rules \( e \rightarrow e_0 \) and perform a consequent perturbative expansion also in \( \delta e \). Applied to our vertex in the classical limit, (90) is modified in such a way that the coefficient of \( \gamma_\mu \) up to 1-loop order is given by

\[ e + \delta e - \frac{1}{2} e \Pi^\gamma(0). \]  

(92)

In order to maintain the meaning and the value of \( e \), (92) has to be equal to \( e \). This "renormalization condition" fixes the counter term to be

\[ \frac{\delta e}{e} = \frac{1}{2} \Pi^\gamma(0). \]  

(93)

Next we consider a scattering or annihilation process via photon exchange. The tree amplitude (we drop the external fermion spinors) now contains the bare charge:

\[ \frac{e_0^2}{q^2} - \frac{e^2}{q^2} \Pi^\gamma(q^2) = \frac{e^2}{q^2} \left[1 + 2 \frac{\delta e}{e} - \Pi^\gamma(q^2)\right] \]

\[ = \frac{e^2}{q^2} \left[1 + \Pi^\gamma(0) - \Pi^\gamma(q^2)\right] \]

\[ = \frac{e^2}{q^2} \left[1 - \Pi^\gamma(q^2)\right]. \]  

(94)

The subtracted quantity

\[ \Pi^\gamma(q^2) = \Pi^\gamma(q^2) - \Pi^\gamma(0) \]  

(95)

is called the "renormalized vacuum polarization". It is finite also for \( D \rightarrow 4 \), and is expressed in terms of only physical parameters. Special features of interest are:
• vanishing for real photons:
\[ \Pi^\gamma(0) = 0 \]

• logarithmic behaviour for light fermions (\(| q^2 | \gg m^2\)):
\[ \Pi^\gamma(q^2) = \frac{\alpha}{3\pi} \left( \frac{5}{3} - \log \frac{| q^2 |}{m^2} + i\pi \theta(q^2) \right) \]  \hspace{1cm} (96)

• decoupling of heavy fermions (\(| q^2 | \ll m^2\)):
\[ \Pi^\gamma(q^2) = \frac{\alpha}{3\pi} \cdot \frac{q^2}{5m^2} \]  \hspace{1cm} (97)

In a real world we have to allow for several species of fermions with charges \( Q_f \) by summing over the individual contributions. Of particular interest is the \( q^2 \) scale set by the mass of the \( Z \) boson:
\[ \Pi^\gamma(M_Z^2) = \sum_f Q_f^2 \frac{\alpha}{3\pi} \left( \frac{5}{3} - \log \frac{M_Z^2}{m_f^2} + i\pi \right). \]  \hspace{1cm} (98)

For \( q^2 = M_Z^2 \) the total fermionic contribution to the real part is (see 4.2)
\[ \text{Re} \Pi^\gamma(M_Z^2) = -0.0602 \pm 0.0009 \]

The vacuum polarization in (94) can be considered as a correction either to the propagator or to the electric charge. The latter point of view invokes the concept of a "running charge" which is usually written as
\[ e(q^2)^2 = \frac{e^2}{1 + \text{Re} \Pi^\gamma(q^2)}. \]  \hspace{1cm} (99)

This form corresponds to summing up the iterated bubbles

\[ \cdots + \cdots + \cdots + \cdots + \cdots \]

\[ 1 - \Pi^\gamma + (\Pi^\gamma)^2 - \cdots \]

representing a geometrical summation. Arguments from the renormalization group show that this geometrical summation contains the leading terms \((\alpha \log)^n\) involving the large logarithms of \( \Pi^\gamma \) to all orders with the correct coefficients.

4 1-loop calculations in the Standard Model

4.1 Parameters and renormalization

In section 2.2 we have specified the Standard Model Lagrangian and given the Feynman rules which enable us to calculate the amplitudes for 4-fermion processes. Recalling (47) we see that the parameters which enter these amplitudes are given by
$e, M_W, M_Z, \sin^2 \theta_W$, as far as we can neglect the Higgs exchange contribution. In the minimal version the mixing angle is not an independent quantity since it is closely related to the boson masses. In the following discussion we restrict ourselves to the minimal model with

$$\sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2} =: s_W^2 \quad (100)$$

The short symbol $s_W^2$ will always be utilized for this particular meaning of $\sin^2 \theta_W$ in terms of the boson mass ratio. Thus we are left with $e, M_W, M_Z$ as the basic set of free parameters. As we have learnt from our QED discussion for the electric charge the parameters get contributions from the loop diagrams, and it is crucial to specify definitely how each physical measurable parameter is related to a defining experiment. This is what one usually denotes as a renormalization scheme. The specification in terms of the parameters above is conventionally called the electroweak on-shell scheme. We have now to discuss the higher order contributions to these parameters and their renormalization. Since the boson masses are part of the propagators we have to investigate the effects of the $W$ and $Z$ self energies.

We restrict our discussion to the transverse parts $\sim g_{\mu\nu}$ as we did for the photon. In the electroweak theory, however, the longitudinal components do not give zero results in physical matrix elements. But for light external fermions the contributions are suppressed by $(m_f/M_Z)^2$ and we are allowed to neglect them. Writing the self energies as

$$\Sigma^{W,Z}_{\mu\nu} = g_{\mu\nu} \Sigma^{W,Z} + \ldots \quad (101)$$

and choosing the same normalization as in (59) we have for the 1-loop propagators ($V = W, Z$)

$$\frac{-i g_{\mu\sigma}}{q^2 - M_V^2} \left( -i \Sigma^{V}_{\sigma\sigma} \right) \frac{-i g_{\mu\nu}}{q^2 - M_V^2} = \frac{-i g_{\mu\nu}}{q^2 - M_V^2} \left( -\Sigma^{V}_V \right) \quad (102)$$

In QED we had only the fermion contributions. In the electroweak theory there are also the non-abelian gauge boson loops and loops involving the Higgs boson. The Higgs boson and the top quark thus enter the 4-fermion amplitudes at the 1-loop level as experimentally unknown objects. In the graphical representation, the blob for the self energies denotes the sum of all the diagrams displayed in Figure 3.

Resumming all self energy insertions yields a geometrical series for the dressed propagators:

$$\begin{align*}
\frac{-i g_{\mu\nu}}{q^2 - M_V^2} \left[ 1 + \left( \frac{-\Sigma^{V}}{q^2 - M_V^2} \right) + \left( \frac{-\Sigma^{V}}{q^2 - M_V^2} \right)^2 + \ldots \right] \\
= \frac{-i g_{\mu\nu}}{q^2 - M_V^2 + \Sigma^{V}(q^2)}.
\end{align*} \quad (103)$$
Fig. 3: 1-loop diagrams for the Z and W self energies. $\phi, \chi$ are the unphysical charged and neutral Higgs field components of (33), and the $u$'s denote the ghost fields referring to the unphysical polarization states of the gauge bosons.
The self energies have the following properties:

- \( \text{Im} \Sigma^V(M_V^2) \neq 0 \) for both \( W \) and \( Z \). But the imaginary parts do not depend on \( D \) and \( \mu \), i.e. they are finite also in 4 dimensions and free of unphysical parameters. Their appearance in (103) reflects the finite decay widths of \( W, Z \) and removes the poles from the real axis.

- \( \text{Re} \Sigma^V(M_V^2) \neq 0 \) for both \( W \) and \( Z \). Moreover, the real parts do depend on the unphysical parameters \( D, \mu \) and diverge for \( D \to 4 \).

Again, we are confronted with the problem of parameter interpretation now concerning the masses. In Quantum Field Theory the mass of a particle is given by the pole position of the corresponding propagator; in case of an unstable particle the mass is conventionally defined by the real part of the complex pole. Hence, \( M_{W,Z} \) cannot be identified with these physical masses \(^5\) since they do not coincide with the poles of (103). The way out is mass renormalization:

The quantities \( M_W, M_Z \) in the Lagrangian and thus in the Feynman rules are not the physical masses but have to be replaced by “bare masses”, not measurable themselves but related to the physical masses \( M_W, M_Z \) by

\[
\begin{align*}
M_W^{02} &= M_W^2 + \delta M_W^2 \\
M_Z^{02} &= M_Z^2 + \delta M_Z^2
\end{align*}
\tag{104}
\]

with counter terms of 1-loop order. The “correct” propagators according to this prescription are given by

\[
\frac{-i g_{\mu \nu}}{q^2 - M_V^{02} + \Sigma^V(q^2)} = \frac{-i g_{\mu \nu}}{q^2 - M_V^2 - \delta M_V^2 + \Sigma^V(q^2)}
\tag{105}
\]

instead of (103). The renormalization conditions which ensure that \( M_{W,Z} \) are the physical masses fix the mass counter terms to be

\[
\begin{align*}
\delta M_W^2 &= \text{Re} \Sigma^W(M_W^2) \\
\delta M_Z^2 &= \text{Re} \Sigma^Z(M_Z^2)
\end{align*}
\tag{106}
\]

In this way, two of our input parameters and their counter terms have been defined.

Our third input parameter is the electromagnetic charge \( e \). As done in QED, we want to maintain its definition as the classical charge in the Thomson cross section \(^{(89)}\). Accordingly, we have to replace \( e \to e_0 = e + \delta e \) in the basic Lagrangian, quite in analogy to our discussion in 3.2. The only difference is given by the additional electroweak loop contributions to the \( e\gamma \) vertex in the Thomson limit:

\[\begin{align*}
\text{\( e \gamma \) vertex in the Thomson limit:}
\end{align*}\]
These diagrams have to be absorbed by the charge counter term $\delta e$. That is the charge renormalization condition in the electroweak theory. It looks more complicated than it actually is: the generalization of the QED Ward identity implies that those corrections which are related to the external particles cancel each other, and for $\delta e$ only two universal contributions are left:

$$
\frac{\delta e}{e} = \frac{1}{2} \Pi^\gamma(0) - \frac{s_w}{c_w} \frac{\Sigma^{\gamma 2}(0)}{M_Z^2}.
$$

(107)

The first one is, as in QED, given by the vacuum polarization of the photon. The second term contains the mixing between photon and $Z$, in general described as a mixing propagator with $\Sigma^{\gamma 2}$ normalized as

$$
\gamma \quad \bullet \quad \gamma \quad \bullet \quad Z
$$

$$
-ig_{\mu
\nu} \left( \frac{\Sigma^{\gamma 2}(q^2)}{q^2 - M_Z^2} \right).
$$

The fermion loop contributions to $\Sigma^{\gamma 2}$ vanish at $q^2 = 0$ (this is obvious from (83) which, up to coupling constants, is identical to the fermion loop integral in $\Sigma^{\gamma 2}$); only the non-abelian bosonic loops yield $\Sigma^{\gamma 2}(0) \neq 0$.

From the diagonal photon self energy

$$
\Sigma^\gamma(q^2) = q^2 \Pi^\gamma(q^2)
$$

no mass term arises for the photon since, besides the fermion loops (83), also the bosonic loops behave like

$$
\Sigma^\gamma_{bos}(q^2) \simeq q^2 \Pi^\gamma_{bos}(0) \to 0
$$

for $q^2 \to 0$ leaving the pole at $q^2 = 0$ in the propagator. The absence of mass terms for the photon in all orders is a consequence of the unbroken electromagnetic gauge invariance.

Concluding this discussion we summarize the principal structure of electroweak calculations:

- The classical Lagrangian $\mathcal{L}(\epsilon, M_W, M_Z, \ldots)$ is sufficient for lowest order calculations and the parameters can be identified with the physical parameters.

- For higher order calculations, $\mathcal{L}$ has to be considered as the "bare" Lagrangian of the theory $\mathcal{L}(\epsilon_0, M_W^0, M_Z^0, \ldots)$ with "bare" parameters which are related to the physical ones by

$$
\epsilon_0 = \epsilon + \delta \epsilon, \quad M_W^{02} = M_W^2 + \delta M_W^2, \quad M_Z^{02} = M_Z^2 + \delta M_Z^2.
$$

The counter terms are fixed in terms of a certain set of 1-loop diagrams by specifying the definition of the physical parameters.

\footnote{Besides the fermion loops discussed in section 3 it contains also bosonic loop diagrams from $W^+W^-$ virtual states}
For any 4-fermion process we can write down the 1-loop matrix element with the bare parameters and the loop diagrams for this process. With the counter terms fixed once for ever the matrix element is finite when expressed in terms of the physical parameters, i.e. all UV singularities associated with the regularization quantities $D, \mu$ are removed.

4.2 The vector boson masses and "$\Delta r$"

Now we are prepared to attack the problem of the higher order contributions to the mass formulae (52),(53) of section 2.3.

Originally, the $\mu$-lifetime $\tau_\mu$ has been calculated within the framework of the effective 4-point Fermi interaction. If we include the QED corrections

\[ \frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \left( 1 - \frac{8m_e^2}{m_\mu^2} \right) \left[ 1 + \frac{\alpha}{2\pi} \left( \frac{25}{4} - \pi^2 \right) \right]. \] (108)

The leading 2nd order correction is obtained by replacing

\[ \alpha \to \alpha \left( 1 + \frac{2\alpha}{3\pi} \log \frac{m_\mu}{m_e} \right). \]

This formula is used as the defining equation for $G_\mu$ in terms of the experimental $\mu$-lifetime. In lowest order, the Fermi constant is given by the Standard Model expression (49) for the decay amplitude. In 1-loop order, $G_\mu/\sqrt{2}$ coincides with the calculable expression

\[ \frac{G_\mu}{\sqrt{2}} = \frac{e_0^2}{8s_W^2 M_W^2} \left[ 1 + \frac{\Sigma_W^W(0)}{M_W^2} + (\text{vertex, box}) \right]. \] (109)

This equation contains the bare parameters with the bare mixing angle

\[ s_W^0 = 1 - \frac{M_W^0}{M_W^0}. \] (110)

The term (vertex, box) schematically summarizes the vertex corrections and box diagrams in the decay amplitude, more explicitly shown in Figure 4. A set of infrared divergent "QED correction" graphs has been removed from this class of diagrams. These left-out diagrams, together with the real bremsstrahlung contributions, exactly reproduce the QED correction factor of the Fermi model result in (108) and therefore have no influence on the relation between $G_\mu$ and the Standard Model parameters.
Fig. 4: Vertex corrections with external self energies and box diagrams contributing to the 1-loop amplitude for $\mu \rightarrow \nu_\mu e^-\bar{e}_e$. For the $W\nu$-vertex the analogous sample of vertex corrections is present as well. Omitted are the "QED" diagrams with a photon in the external charged lepton lines, and the photonic vertex correction to the Fermi amplitude is subtracted from the box diagram with photon exchange.
Next we evaluate (109) to 1-loop order by expanding the bare parameters

\[ e_0^2 = (e + \delta e)^2 = e^2(1 + 2\frac{\delta e}{e}), \]

\[ M_{W}^0 = M_{W}^2(1 + \frac{\delta M_{W}^2}{M_{W}^2}), \]

\[ s_{W}^0 = 1 - \frac{M_{W}^2 + \delta M_{W}^2}{M_{W}^2 + \delta M_{W}^2} = s_{W}^2 + c_{W}^2 \left( \frac{\delta M_{W}^2}{M_{W}^2} - \frac{\delta M_{W}^2}{M_{W}^2} \right) \]

(111)

\[ (e_{W}^2 = 1 - s_{W}^2) \] and keeping only terms of 1-loop order in (109):

\[ \frac{G_{\mu}}{\sqrt{2}} = \frac{e^2}{8s_{W}^2M_{W}^2} \left[ 1 + 2\frac{\delta e}{e} \frac{s_{W}^2}{s_{W}^2} \left( \frac{\delta M_{Z}^2}{M_{Z}^2} - \frac{\delta M_{W}^2}{M_{W}^2} \right) + \frac{\Sigma_{W}(0) - \delta M_{W}^2}{M_{W}^2} \right] \]

\[ \equiv \frac{e^2}{8s_{W}^2M_{W}^2} [1 + \Delta r] . \]

(112)

which is the 1-loop corrected version of (51) [7].

The quantity \( \Delta r(e, M_{W}, M_{Z}, M_{H}, m_{t}) \) is the finite combination of loop diagrams and counter terms in (112). Since we have already determined the counter terms in the previous subsection in terms of the boson self energies, it is now only a technical problem to evaluate the diagrams of Figure 4 for the final explicit expression of \( \Delta r \). Here we quote only the result:

\[ (vertex, box) = \frac{\alpha}{\pi s_{W}^2} \left( \Delta - \log \frac{M_{W}^2}{\mu^2} \right) + \frac{\alpha}{4\pi s_{W}^2} \left( 6 + \frac{7 - 4s_{W}^2}{2s_{W}^2} \log e_{W}^2 \right) . \]

(113)

The singular part, up to a factor, coincides with the non-abelian bosonic contribution to the charge counter term in (107):

\[ \frac{\alpha}{\pi s_{W}^2} \left( \Delta - \log \frac{M_{W}^2}{\mu^2} \right) = \frac{2}{c_{W}s_{W}} \frac{\Sigma_{W}(0)}{M_{Z}^2} . \]

Together with (107) and (113) we obtain from (112):

\[ \Delta r = \Pi(0) + \frac{s_{W}^2}{\Sigma(0)} \left( \frac{\delta M_{Z}^2}{M_{Z}^2} - \frac{\delta M_{W}^2}{M_{W}^2} \right) + \frac{\Sigma_{W}(0) - \delta M_{W}^2}{M_{W}^2} \]

\[ + 2 \frac{c_{W}}{s_{W}} \frac{\Sigma_{W}(0)}{M_{Z}^2} + \frac{\alpha}{4\pi s_{W}^2} \left( 6 + \frac{7 - 4s_{W}^2}{2s_{W}^2} \log e_{W}^2 \right) . \]

(114)

The first line is of particular interest: via \( \Sigma_{W}(0) \) and the mass counter terms \( \delta M_{W, Z}^2 \) also the experimentally unknown parameters \( M_{H}, m_{t} \) enter \( \Delta r \), whereas the residual terms depend only on the vector boson masses. The somewhat lengthy expressions for the self energies can e.g be found in [8]. In this place we proceed with a more explicit discussion of the gauge invariant subset of fermion loop corrections which involves, among others, the top quark. This subset is also of primordial practical interest since it constitutes the dominating part of \( \Delta r \).
Fermion contributions to $\Delta \tau$:

In the fermionic vacuum polarization entering (114) we split off the subtracted part evaluated at $M_Z^2$:

$$
\Pi^\gamma(0) = -\text{Re}\Pi^\gamma(M_Z^2) + \Pi^\gamma(0) + \text{Re}\Pi^\gamma(M_Z^2)
$$

$$
= -\text{Re}\Pi^\gamma(M_Z^2) + \text{Re}\Pi^\gamma(M_Z^2).
$$

(115)

The finite quantity $\Pi^\gamma(M_Z^2)$ has already been computed in (98); we neglect the top contribution because of the decoupling property (97). Whereas the leptonic content can easily be obtained from (98), no light quark masses are available as reasonable input parameters for the hadronic content. Instead, $\Pi^\gamma_{\text{had}}$ can be derived from experimental data with help of a dispersion relation

$$
\Pi^\gamma_{\text{had}}(M_Z^2) = \frac{\alpha}{3\pi} M_Z^2 \int_{4m_t^2}^{\infty} ds' \frac{R^\gamma(s')}{s'(s' - M_Z^2 - i\varepsilon)}
$$

(116)

with

$$
R^\gamma(s) = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-)}
$$

as an experimental quantity up to a scale $s_1$ and applying perturbative QCD for the tail region above $s_1$. Using $e^+e^-$ data for the energy range below 40 GeV the integral (116) yields [9]

$$
\text{Re}\Pi^\gamma_{\text{had}}(M_Z^2) = -0.0288 \pm 0.0009.
$$

(117)

where the error is almost completely due to the experimental data. Combining this result with the leptonic part we obtain the number

$$
\text{Re}\Pi^\gamma(M_Z^2) = -0.0602 \pm 0.0009.
$$

(118)

For simplicity we restrict the further discussion to a single family, leptons or quarks, with $m_\pm$, $Q_\pm$, $v_\pm$, $a_\pm$ denoting mass, charge, vector and axial vector coupling of the up(+) and down(-) member. At the end, we perform the sum over the various families.

Besides $\Pi^\gamma(M_Z^2)$ we need the $W$ and $Z$ self energies. The fermion loops can be evaluated following the lines of section 3.1. With the notations of 3.1 we get after some algebra (with an additional factor $N_C = 3$ for quarks):

$$
\Sigma^Z(q^2) = \frac{\alpha}{\pi} \sum_{f = u, d, s, c, t} \left\{ \frac{v_f^2 + a_f^2}{4s_W^2c_W^2} \left[ 2B_{Z\ell}(q^2, m_f, m_f) + \frac{q^2}{2} B_0(q^2, m_f, m_f) - A(m_f) \right] \\
- \frac{m_f^2}{8s_W^2c_W^2} B_0(q^2, m_f^2, m_f^2) \right\}
$$

$$
\Sigma^W(q^2) = \frac{\alpha}{\pi} \cdot \frac{1}{4s_W^2} \left\{ 2B_{Z\ell}(q^2, m_+, m_-) - A(m_+) + A(m_-) \right. \\
+ \frac{q^2 - m_+^2 - m_-^2}{2} B_0(q^2, m_+, m_-) \right\}
$$

(119)

We discuss the light and heavy fermions separately:
• Light fermions:

Applying the formulæ of the appendix in the light fermion limit, i.e. neglecting all terms \( \sim m_\pm/M_{W,Z} \), the self energies simplify considerably. The various ingredients of \( \Delta r \) read in this limit:

\[
\frac{\Sigma^W(0)}{M_W^2} = O \left( \frac{m_\pm^2}{M_W^2} \right) \simeq 0 ,
\]

\[
\frac{\delta M_Z^2}{M_Z^2} = \frac{\alpha}{3\pi} \frac{v_+^2 + a_+^2 + v_-^2 + a_-^2}{4s_W^2 c_W^2} \left( \Delta - \log \frac{M_Z^2}{\mu^2} + \frac{5}{3} \right) ,
\]

\[
\frac{\delta M_W^2}{M_W^2} = \frac{\alpha}{3\pi} \frac{1}{4s_W^2} \left( \Delta - \log \frac{M_Z^2}{\mu^2} + \frac{5}{3} \right) - \frac{\alpha}{16\pi s_W^2} \log c_W^2 ,
\]

and from (87), generalized to fermion charges \( Q_\pm \):

\[
\text{Re} \Pi^\gamma(M_Z^2) = \frac{\alpha}{3\pi} (Q_+^2 + Q_-^2) \left( \Delta - \log \frac{M_Z^2}{\mu^2} + \frac{5}{3} \right) .
\]

Inserting everything into (114) yields

\[
\Delta r = -\text{Re} \Pi^\gamma(M_Z^2)
\]

\[
+ \frac{\alpha}{3\pi} \frac{1}{4s_W^2} \left[ Q_+^2 + Q_-^2 - \frac{c_W^2}{s_W^2} \left( \frac{v_+^2 + a_+^2 + v_-^2 + a_-^2}{4s_W^2 c_W^2} - \frac{1}{4s_W^2} \right) - \frac{1}{4s_W^2} \right]
\]

\[
= -\text{Re} \Pi^\gamma(M_Z^2) - \frac{\alpha}{3\pi} \frac{c_W^2 - s_W^2}{4s_W^2} \log c_W^2
\]

\[
\quad = -\text{Re} \Pi^\gamma(M_Z^2) - \frac{\alpha}{3\pi} \frac{c_W^2 - s_W^2}{4s_W^2} \log c_W^2
\]

(120)

The term in brackets \([\ldots]\) is zero with the coupling constants (48). Thus, the main effect from the light fermions comes from the subtracted photon vacuum polarization as the remainder from the renormalization of the electric charge at \( q^2 = 0 \). For this reason, after summing over all light fermions, we write

\[
\Delta \alpha \equiv -\text{Re} \Pi^\gamma(M_Z^2) = 0.0602 \pm 0.0009 .
\]

The left over term \( \simeq 0.0054 \) is small compared to \( \Delta \alpha \).

• Heavy fermions:

Of special interest is the case of a heavy top quark which contributes a large correction \( \sim m_t^2 \) to \( \Delta r \). In order to extract this piece we keep for simplicity only those terms which are either singular or quadratic in the top mass \( m_t \equiv m_+ \) (\( N_C = 3 \)):

\[
\text{Re} \Pi^\gamma(M_Z^2) = N_C \frac{\alpha}{3\pi} (Q_+^2 + Q_-^2) \left( \Delta - \log \frac{m_t^2}{\mu^2} \right) + \cdots
\]

\[
\frac{\delta M_Z^2}{M_Z^2} = N_C \frac{\alpha}{3\pi} \left\{ \frac{v_+^2 + a_+^2 + v_-^2 + a_-^2}{4s_W^2 c_W^2} - \frac{3m_t^2}{8s_W^2 c_W^2 M_Z^2} \right\} \left( \Delta - \log \frac{m_t^2}{\mu^2} \right) + \cdots
\]

83
\[
\frac{\delta M_W^2}{M_W^2} = N_C \frac{\alpha}{3\pi} \left\{ \frac{1}{4s_W^2} \left( \Delta - \log \frac{m_t^2}{\mu^2} \right) - \frac{3m_t^2}{8s_W^2 M_W^2} \left( \Delta - \log \frac{m_t^2}{\mu^2} + \frac{1}{2} \right) \right\} + \ldots
\]

\[
\frac{\Sigma^W(0)}{M_W^2} = -N_C \frac{\alpha}{3\pi} \cdot \frac{3m_t^2}{8s_W^2 M_W^2} \left( \Delta - \log \frac{m_t^2}{\mu^2} + \frac{1}{2} \right).
\]

Inserting into (114) we verify that the singular parts cancel and a finite term \( \sim m_t^2 \) remains:

\[
(\Delta r)_{s,t} = -\text{Re} \tilde{N}_t^0 (M_Z^2) - \frac{c_W^2}{s_W^2} \Delta \rho + \ldots \tag{122}
\]

with

\[
\Delta \rho = N_C \frac{\alpha}{16\pi s_W^2 c_W^2} \frac{m_t^2}{M_Z^2} \tag{123}
\]

which can also be written as the combination

\[
\Delta \rho = \frac{\Sigma^Z(0)}{M_Z^2} - \frac{\Sigma^W(0)}{M_W^2}
\]

entering the NC/CC neutrino cross section ratio.

As a result of our discussion, we have got a simple form for \( \Delta r \) in the leading terms which is valid also after including the full non-fermionic contributions:

\[
\Delta r = \Delta \alpha - \frac{c_W^2}{s_W^2} \Delta \rho + (\Delta r)_{\text{remainder}}.
\tag{124}
\]

\( \Delta \alpha \), eq. (121), contains the large logarithmic corrections from the light fermions and \( \Delta \rho \) the leading quadratic correction from a large top mass. All other terms are collected in the \( (\Delta r)_{\text{remainder}} \). It should be noted that the remainder also contains a term logarithmic in the top mass (for which our approximation above was too crude)

\[
(\Delta r)_{\text{remainder}}^{\text{top}} = \frac{\alpha}{4\pi s_W^2} \left( \frac{c_W^2}{s_W^2} - \frac{1}{3} \right) \log \frac{m_t}{M_Z} + \ldots \tag{125}
\]

Also the Higgs boson contribution is part of the remainder. For large \( M_H \), it increases only logarithmically ("screening" of a heavy Higgs [10]):

\[
(\Delta r)_{\text{remainder}}^{\text{Higgs}} \simeq \frac{\alpha}{16\pi s_W^2} \cdot \frac{11}{3} \left( \log \frac{M_H^2}{M_Z^2} - \frac{5}{6} \right).
\tag{126}
\]

The typical size of \( (\Delta r)_{\text{remainder}} \) is of the order \( \sim 0.01 \).

**Higher order terms:**

Before entering a numerical discussion, we conclude this theoretical part by a short discussion of higher order effects. The replacement of the 1-loop result

\[
1 + \Delta r \to \frac{1}{1 - \Delta r}
\]
in (112) correctly takes into account all orders in the leading logarithmic corrections \((\Delta \alpha)''\), as can be shown by renormalization group arguments [11]. It corresponds to a resummation of the iterated 1-loop vacuum polarization to all orders. The non-leading terms of next order are numerically not significant. Thus, in a situation where large corrections are only due to the renormalization of the electric charge from 0 up to \(M_Z^2\) the resummed form

\[
G_\mu = \frac{\pi \alpha}{\sqrt{2} M_W^2 s_W^2} \frac{1}{1 - \Delta r} = \frac{\pi \alpha}{\sqrt{2} M_Z^2 c_W^2 s_W^2} \frac{1}{1 - \Delta r}
\]  
(127)

with \(\Delta r\) from (124) represents a very good approximation to the full result.

In case of a heavy top, where also \(\Delta \rho\) is large, the powers \((\Delta \rho)^n\) are not correctly resummed in (127). A result correct in the leading terms up to \(O(\alpha^2)\) is instead given by [12]

\[
\left( \frac{1}{1 - \Delta r} \right) \rightarrow \frac{1}{1 - \Delta \alpha} \cdot \frac{1}{1 + \frac{\Delta \rho}{\Delta \alpha}} + (\Delta r)_{\text{remainder}}
\]  
(128)

where

\[
\Delta \rho = N_C \frac{G_\mu m_t^2}{8 \pi^2 \sqrt{2}} \left[ 1 + \frac{G_\mu m_t^2}{8 \pi^2 \sqrt{2}} (19 - 2 \pi^2) \right]
\]

incorporates the result [13] from 2-loop irreducible diagrams like

(128) is compatible with the following form of the \(M_W - M_Z\) interdependence (if we write \(\rho = 1/(1 - \Delta \rho)\)):

\[
G_\mu = \frac{\pi}{\sqrt{2} M_W^2 \left( 1 - \frac{M_Z^2}{\rho M_W^2} \right)} \cdot \frac{\alpha}{1 - \Delta \alpha} \cdot \left[ 1 + (\Delta r)_{\text{remainder}} \right].
\]  
(129)

It is interesting to compare this result with the expressions (55),(56) which represent the \(M_W - M_Z\) lowest order correlation in a more general model with a tree level \(\rho\)-parameter \(\rho_0 \neq 1\): The tree level \(\rho_0\) enters in the same way as the \(\rho\) from a heavy top in the minimal model. Hence, up to the small quantity \((\Delta r)_{\text{remainder}}\), they are indistinguishable from an experimental point of view (\(\Delta \alpha\) is universal). In the minimal model, however, \(\rho\) is calculable in terms of \(m_t\) whereas \(\rho_0\) is an additional free parameter. Further information on \(m_t\) will allow to disentangle the different sources (see section 4.3).

In this place it should also be mentioned that various extensions of the minimal model retaining \(\rho_0 = 1\), like a fourth generation, a second Higgs doublet, Supersymmetry, contribute to \(\Delta \rho\) in the same way as a heavy top if large mass splittings in \(SU(2)\) doublets are present. Also such contributions cannot be separated from the top effect if only the boson mass relation is studied.
Numerical and experimental results

We now return to the Standard Model and discuss the numerical predictions of our calculation. We can define the quantity $\Delta r$ also as a physical observable by

$$
\Delta r = 1 - \frac{\pi \alpha}{\sqrt{2} G_{\mu}} \frac{1}{M_{W}^{2}} \left(1 - \frac{M_{W}^{2}}{M_{Z}^{2}}\right). \tag{130}
$$

Experimentally, it is determined by $M_{Z}$ and $M_{W}$ resp. the ratio $M_{W}/M_{Z}$. Theoretically, it can be computed from $M_{Z}, G_{\mu}, \alpha$ via the expression (127) specifying the masses $M_{H}, m_{t}$, and adjusting $M_{W}$ such that (127) yields the experimental value for $G_{\mu}$. In practice, (127) is solved for $M_{W}$ by iteration. When we use the 2-loop modified expression (129) instead of (127) and (124) for getting $M_{W}$ as ingredient in (130) we obtain the results shown in Figure 5.

![Graph](image)

**Fig. 5:** $\Delta r$ in $O(\alpha)$ (dotted) and in $O(\alpha^{2})$ (full). $M_{Z} = 91.15 \text{ GeV}$, $M_{H} = 100 \text{ GeV}$.

The theoretical prediction for $\Delta r$ for various Higgs and top masses is displayed in Figure 6 as the area between the two curves. For comparison with data, the experimental 1$\sigma$ limits from the direct measurements of $M_{Z}$ (LEP) and $M_{W}$ (CDF) as well as $M_{W}/M_{Z}$ (UA2) are indicated. The present lower limit for $m_{t}$ from direct searches at the Tevatron is $m_{t} \geq 80 \text{ GeV}$.
Fig. 6: $\Delta r$ as a function of the top mass for $24 \leq M_H \leq 1000$ GeV. $M_Z = 91.15 \pm 0.033$ GeV. 1$\sigma$ bounds from $M_W/M_Z = 0.8829 \pm 0.0055$ (---), $M_W = 80.0 \pm 0.62$ GeV (---)

4.3 $Z$ physics in $e^+e^- \rightarrow f\bar{f}$

The measurement of the $Z$ mass from the $Z$ line shape at LEP provides us with an additional precise input parameter besides $\alpha$ and $G_\mu$. Other observable quantities from the $Z$ peak like total and partial decay widths, forward-backward asymmetries, $\tau$-polarization, allow us to perform precision tests of the theory by comparison with the theoretical predictions.

In lowest order, the $Z$ observables are completely fixed in terms of $\alpha$, $G_\mu$, $M_Z$ applying the rules and relations of section 2.2 to compute the Born $\gamma$ and $Z$ exchange diagrams. Since 1-loop terms are of the order $\alpha/\pi$ and typically enhanced by factors $\sim \log M_Z^2/m_f^2$ or $\sim m_t^2/M_Z^2$, the size $\alpha/\pi = 0.0023$ in view of experimental precisions of a few $10^{-3}$ immediately signals the need for dressing the Born amplitudes by next order contributions.

A gauge invariant subset of the 1-loop diagrams to $e^+e^- \rightarrow f\bar{f}$ are the QED corrections of Figure 7. The sum of the virtual photon loop graphs is UV finite but IR (= infra-red) divergent because of the massless photon. The IR-divergence is cancelled by adding the cross section with real photon bremsstrahlung (after integrating over the phase space for experimentally invisible photons) which always accompanies a realistic scattering process. Since the phase space for invisible photons is a detector dependent
Fig. 7: Virtual QED corrections for $e^+e^- \rightarrow f\bar{f}$

Fig. 8: Non-QED corrections for $e^+e^- \rightarrow f\bar{f}$
quantity the QED corrections can in general not be separated from the experimental device. They are conveniently treated with help of Monte Carlo simulation. For details on the QED corrections see the second part of these lectures.

Our discussion will concentrate on the residual set of 1-loop diagrams, the non-QED or weak corrections. This class is free of IR-singularities but sensitive to the details beyond the lowest order amplitudes. The UV-singular terms associated with the loop diagrams are cancelled by our counter terms of section 4.1 as a consequence of renormalizability. Schematically the 1-loop amplitude for $e^+e^- \rightarrow f\bar{f}$ is shown in Figure 8 where the blobs denote the sum of the individual contributions to the self energies and vertex corrections (including the external fermion self energies). The essential steps for getting the total amplitude finite are: expressing the tree diagrams in terms of the bare parameters $^7$ ($\sqrt{s} = \text{ cms energy}$)

\[ e_0^2 \frac{Q_e Q_f}{s} \gamma_\mu \otimes \gamma^\mu \]  
\[ \frac{e_0^2}{4s_W^2 c_W^2} \left[ \gamma_\mu (I_3^e - 2Q_e s_W^2) - I_3^f \gamma_\mu \gamma_5 \right] \otimes \left[ \gamma^\mu (I_3^e - 2Q_f s_W^2) - I_3^f \gamma^\mu \gamma_5 \right] \left( s - M_Z^2 \right) \]

expanding the bare quantities according to (111), and inserting the counter terms given by (106) and (107). After some tedious calculations the total amplitude can be written as the sum of a dressed photon and a dressed $Z$ exchange amplitude plus the contribution from the box diagrams which are numerically not significant around the peak (relative contribution $< 10^{-4}$). For theoretical consistency (gauge invariance) they have to be retained; for practical purposes they can be neglected in $Z$ physics. Resummation of the iterated self energy insertions in the photon and $Z$ propagators brings the finite $Z$ decay width into its correct place, and treats the higher order leading terms in the proper way. Since the leading terms arise from fermion loops only, we do not have problems with gauge invariance; the bosonic loop terms have to be understood as expanded to strict 1-loop order. Numerically their resummation does not yield significant differences but allows a simple and compact notation.

**Dressed photon amplitude:**

The dressed photon exchange amplitude can be written in the following way:

\[ A_\gamma = \frac{e^2}{1 + \tilde{\Pi}^\gamma (s)} \cdot \frac{Q_e Q_f}{s} \cdot \left[ (1 + F_V^{r\gamma}) \gamma_\mu - F_A^{r\gamma} \gamma_\mu \gamma_5 \right] \otimes \left[ (1 + F_V^{r\gamma}) \gamma^\mu - F_A^{r\gamma} \gamma^\mu \gamma_5 \right] \]

$\tilde{\Pi}^\gamma$ is the $\gamma$ self energy subtracted at $s = 0$. Writing it in the denominator takes into account the resummation of the leading log's from the light fermions. The form factors $F_{V,A}(s)$ arise from the vertex correction diagrams together with the external fermion self energies (Figure 9). They vanish for real photons: $F_{V,A}^{re,fi}(0) = 0$.

$^7$\(\otimes\) means contraction of the initial and final currents with the external spinors, according to the example in Figure 1. The overall factor $i$ has been dropped.
Fig. 9: Non-QED vertex corrections energies for the $\gamma, Z\cdot ff$ vertices and fermion self energies. The physical Higgs contributions are neglected. The diagrams (d-g) and (c') are negligible for light fermions $\neq b$. The self energies have to understood as inserted in the external lines with a factor $1/2$. $f'$ denotes the isospin doublet partner of $f$. 
The typical size of the various corrections is (real parts):

\[
\tilde{\Pi}^{\gamma}(M_Z^2) = -0.06
\]

\[
F^{\mu
u}(M_Z^2) \simeq F_A^{\mu
u}(M_Z^2) \simeq 10^{-3}.
\]

For the region around the Z peak, the photon form factors are negligibly small.

**Dressed Z amplitude and effective neutral current couplings:**

More important is the weak dressing of the Z exchange amplitude. Without the box diagrams the corrections factorize and we obtain a result quite close to the Born amplitude:

\[
A_Z = \sqrt{2}G_\mu M_Z^2 (\rho_{\sigma} \rho_f)^{1/2} \left[ \gamma_\mu (I_3^f - 2Q_z s_W^2 \kappa_e) - I_3^e \gamma_\mu \gamma_5 \right] \otimes \left[ \gamma^\nu (I_3^f - 2Q_1 s_W^2 \kappa_f) - I_3^f \gamma^\nu \gamma_5 \right] \frac{s - M_Z^2}{s - M_Z^2 + i \frac{s}{M_Z^2} \cdot M_Z \Gamma_Z}.
\]

(133)

The weak corrections appear in terms of fermion dependent form factors \( \rho \) and \( \kappa \) in the coupling constants and in the width in the denominator.

The \( s \)-dependence of the imaginary part is due to the \( s \)-dependence of \( \text{Im} \Sigma^Z \); the linearisation is completely sufficient in the resonance region. This "\( s \)-dependent width" is responsible for a dislocation of the peak maximum by \( \simeq -35 \text{ GeV} \). We postpone the discussion of the Z width for the moment and continue with the form factors.

The formfactors \( \rho \) and \( \kappa \) in (133) have universal parts (i.e. independent of the fermion species) and non-universal parts which explicitly depend on the type of the external fermions. The universal parts arise from the counter terms and the boson self energies, the non-universal parts from the vertex corrections and the fermion self energies in the external lines:

\[
\rho_{e,f} = 1 + (\Delta \rho)_{\text{univ}} + (\Delta \rho)_{\text{non-univ}}
\]

\[
\kappa_{e,f} = 1 + (\Delta \kappa)_{\text{univ}} + (\Delta \kappa)_{\text{non-univ}}.
\]

In their leading terms the universal contributions are given by

\[
(\Delta \rho)_{\text{univ}} = \Delta \rho + \cdots
\]

\[
(\Delta \kappa)_{\text{univ}} = \frac{c_w^2}{s_w^2} \Delta \rho + \cdots
\]

(135)

with \( \Delta \rho \) from (123). Replaced by the 2-loop quantity \( \Delta \bar{\rho} \) in (128) also the next order leading term is correctly incorporated. Differently from \( \Delta r \), there is no logarithmic \( m_t \) term in the universal \( \rho \)’s and \( \kappa \)’s.

The leading structure of the universal parts can easily be understood from the bare amplitude in (131) and its counter term expansion according to (111):

\[
\frac{c_w^2}{4 s_w^2 e_w^2} = \frac{c_w^2}{4 s_w^2 e_w^2} \left[ 1 + \frac{\delta e}{e} - \frac{c_w^2 - s_w^2}{s_w^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \right]
\]

\[
= \sqrt{2} G_\mu M_Z^2 \left[ 1 + \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} + \cdots \right]
\]

91
and
\[
\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} = \Delta \rho + \ldots
\]
in the quadratic \(m_t\)-term. Thereby, \(G_\mu\) was introduced by means of (127) together with the expression (112) for \(\Delta \tau\). In a similar way one finds from (111):
\[
s_W^2 \left[ 1 + \frac{c_W^2}{s_W^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \right] = s_W^2 \left[ 1 + \frac{c_W^2}{s_W^2} \Delta \rho + \ldots \right]
\]
where we recover \((\Delta \kappa)_{\text{univ}}\).

For completeness we list the full expressions for the form factors: \(^8\)
\[
(\Delta \rho)_{\text{univ}} = -\Delta \tau - \Pi^Z(s) \tag{136}
\]
\[
(\Delta \rho)_{\text{non-univ}} = F_A^Z(s)/a_f
\]
\[
(\Delta \kappa)_{\text{univ}} = -\frac{c_W}{s_W} \Pi^\gammaZ(s)
\]
\[
(\Delta \kappa)_{\text{non-univ}} = -\frac{1}{2s_W^2 Q_f} \left( F_V^Z(s) - \frac{v_f}{a_f} F_A^Z(s) \right)
\]

The building blocks are the following finite combinations of 2-point functions, besides \(\Delta \tau\) in (114)
\[
\Pi^Z(s) = \frac{\text{Re} \Sigma^Z(s) - \delta M_Z^2} {s - M_Z^2} - \Pi^\gamma(0) + \frac{c_W^2}{s_W} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} - 2 \frac{s_W}{c_W} \Sigma^Z(0) \right)
\]
\[
\Pi^\gammaZ(s) = \frac{\Sigma^\gammaZ(s) - \Sigma^\gammaZ(0)} {s} - \frac{c_W}{s_W} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) + 2 \frac{\Sigma^\gammaZ(0)} {M_Z^2} \tag{137}
\]
and the finite vector and axial vector form factors from the vertex corrections, together with the external fermion self energies, after splitting off the singular part \(\sim \Sigma^\gammaZ(0)\):

\[
\pi = \frac{e}{2s_W c_W} \left\{ \gamma_\mu F_V^Z(s) - \gamma_\mu \gamma_5 F_A^Z(s) + I_3^f \gamma_\mu (1 - \gamma_5) \gamma_\mu \right\}
\]
\[
\pi = \frac{e}{2s_W c_W} \left\{ \gamma_\mu F_V^Z(s) - \gamma_\mu \gamma_5 F_A^Z(s) + I_3^f \gamma_\mu (1 - \gamma_5) \gamma_\mu \right\}
\]

With \(\rho\) and \(\kappa\) evaluated at \(s = M_Z^2\) we can define NC vertices at the \(Z\) resonance with effective couplings:
\[
J^\NC_{\mu} = \left( \sqrt{2} G_\mu M_Z^2 \rho_f \right)^{1/2} \left[ (I_3^f - 2 Q_f s_W \kappa_f) \gamma_\mu - I_3^f \gamma_\mu \gamma_5 \right]. \tag{138}
\]

Of particular interest are the effective mixing angles in the vector couplings:
\[
\sin^2 \theta^f_{\text{eff}} := s_W^2 \kappa_f = \frac{s_W^2}{s_W^2 + (\Delta \kappa)_{\text{non-univ}} s_W^2} \tag{139}
\]
\(^8\) up to small terms which are negligible near the \(Z\) peak, they correspond to those of Bardin et al. [14]
(this should be understood as for the real parts only). $\bar{s}_W^2$ denotes the universal mixing angle for all fermion species; in the leading term it is given by

$$\bar{s}_W^2 = s_W^2 + \frac{c_W^2}{s_W^2} \Delta \rho + \cdots \quad (140)$$

It is equivalent to $s_t^2$ of [15] and, up to non-leading terms, also to the quantity $\sin \theta_{\overline{M^2}}(M_Z^2)$ of [16]. Its most striking feature is the dependence on the top mass which is much weaker than that of $s_W^2$, suppressed by a factor $s_W^2/c_W^2$. Both quantities are displayed in Figure 10. Hence, the on-resonance couplings are less sensitive to $m_t$ than the $W$ mass.

---

**Fig. 10:** $s_W^2$ and the universal part of the effective mixing angle at the $Z$ peak, $\bar{s}_W^2$, versus $m_t$ for $24\,\text{GeV} \leq M_H \leq 1000\,\text{GeV}$. $M_Z = 91.15\,\text{GeV}$.

The terms in $(\Delta \rho)_{\text{univ}}$, $(\Delta \kappa)_{\text{univ}}$ entering via $\Delta \rho$ appear exactly the same way as a possible $\Delta \rho_0$ in

$$\rho_0 = 1 + \Delta \rho_0$$

would appear from a tree level $\rho$-parameter $\neq 0$. Like in $\Delta r$, such a tree level effect can experimentally not be distinguished from the top effect in the universal couplings.

The separation of a universal part in the effective couplings is sensible for two reasons: for the light fermions ($f \neq b, t$) the non-universal contributions are small, and (practically) independent of the unknown parameters $m_t, M_H$ which enter only the universal part. This is, however, not true for the $b$-quark where also the non-universal parts
have a strong dependence on $m_t$ [17] resulting from the virtual top quark in the vertex corrections as follows:

$$
(\Delta \rho)^b_{\text{non-univ}} = -\frac{4}{3} \Delta \rho - \frac{\alpha}{4\pi s_W^2} \left( \frac{8}{3} + \frac{1}{6c_W^2} \right) \log \frac{m_t^2}{M_W^2} + \cdots,
$$

$$
(\Delta \kappa)^b_{\text{non-univ}} = -\frac{1}{2} (\Delta \rho)^b_{\text{non-univ}} + \cdots
$$

which overcompensate the top dependence of $(\Delta \rho)_{\text{univ}}$. Also a logarithmic $m_t$-term is not contained in the universal parts. It is numerically significant for top masses < 200 GeV. If we restrict our discussion to the leading terms in the form factors only we have a simple recipe to write down an “improved Born approximation” which contains all large corrections from light and heavy fermions.

In table 1 we put together the effective mixing angles for the various fermion types as functions of $m_t$ and $M_H$, together with the universal $\tilde{s}_W^2$. These mixing angles determine the various on-resonance asymmetries to be discussed later.

<table>
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<th>$M_H$(GeV)</th>
<th>$m_t$(GeV)</th>
<th>$\mu$</th>
<th>$u$</th>
<th>$d$</th>
<th>$b$</th>
<th>$\tilde{s}_W^2$</th>
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Table 1: Effective mixing angles on-resonance for $M_Z = 91.15$ GeV

We are now prepared to discuss some applications of particular experimental interest:
**Z width and partial widths:**

With help of the form factors $\rho_f, \kappa_f$ we now can easily calculate the $Z$ width $\Gamma_Z$ in the $Z$ propagator of (133). It can be written as the sum over the fermionic partial decay widths

$$\Gamma_Z = \sum_f \Gamma(Z \to f \bar{f}) + \cdots$$

(142)

(Other decay channels are not significant except for a light Higgs which meanwhile is experimentally excluded). The partial widths read (with the real parts of the form factors only), simplified by neglecting the finite fermion mass terms:

$$\Gamma(Z \to f \bar{f}) = N_C \frac{M_Z}{48\pi} \sqrt{2} G_F M_Z^2 \rho_f \left[ 1 + (2I_3^f - 4Q_f s_W^2 \kappa_f)^2 \right] \cdot (1 + \delta_{QED}) \cdot (1 + \delta_{QCD})$$

(143)

with $N_C^f = 3$ for quarks and $= 1$ for leptons. $\delta_{QED}$ is the additional photonic QED correction which is very small ($< 0.0017$); $\delta_{QCD}$ is the gluonic correction for the hadronic final states, with $\alpha_s = 0.12 \pm 0.02$ given by

$$1 + \delta_{QCD} = 1.040 \pm 0.007$$

(144)

for $q \neq b$; for $b$-quarks, due to the finite $b$-mass, the correction factor has to be replaced by 1.045. By the error of $\alpha_S$ an uncertainty in the total width of

$$(\Delta \Gamma_Z)^{QCD} = \pm 12 \text{MeV}$$

is induced. Figure 11 contains the total width prediction of the Standard Model together with the 1$\sigma$ bounds from the LEP experiments. The data show a preference for a high top mass.

The difference between the $d$ and $b$-quark effective couplings due to the top quark to the $Zbb$ vertex manifests itself in the difference between the $d$ and $b$ partial widths, shown in Figure 12. The slight decrease of $\Gamma_b$ is the consequence of the non-universal top contribution to the form factors which overcompensates the increase from the universal part responsible for the behaviour of $\Gamma_d$. In $\Gamma_b$ resp. $\Gamma_d$ in the total hadronic width $\Gamma_{had}$ we now have a second observable at our disposal which contains independent information on $m_t$ besides the $\Delta \rho$. Due to cancellations between the universal and the non-universal top contributions the ratio

$$R_Z = \frac{\Gamma_{had}}{\Gamma_c} = 20.81 \pm 0.14$$

is insensitive to $m_t$ in the minimal model. The error is almost completely the QCD uncertainty. Non-standard contributions in $\Delta \rho$ would spoil these subtle cancellations, which makes $R_Z$ a sensitive indicator of possible new physics. The total information from the $M_W$-$M_Z$ correlation and the leptonic and hadronic $Z$ decays thus allow to disentangle a tree level $\rho_0 \neq 1$ and the effects from a heavy top quark [18] or to look for other kinds of new physics [19].
**Fig. 11**: Total $Z$ width $\Gamma_Z$ in the Standard Model for $M_Z = 91.15$ GeV. $M_H = 25$ GeV (\cdots), 100 GeV (---), 1000 GeV (\cdash). Not included is the QCD uncertainty of $\simeq 12$ MeV. 1-$\sigma$ limits from LEP.

**Fig. 12**: Partial $Z$ widths (in MeV) into $d$-quarks (upper curves) and into $b$-quarks (lower curves). $M_Z = 91$ GeV. $M_H = 10$ \cdots, 100 (---), 1000 (\cdash) GeV. The overall normalization is different for $d$ and $b$ due to the finite mass of the $b$-quark.
On-resonance asymmetries:

Further precisely measurable quantities are the forward-backward asymmetries

$$A_{FB}^f = \frac{\sigma^F - \sigma^B}{\sigma^F + \sigma^B}$$  \hspace{1cm} (145)

with

$$\sigma^F = \int_{\theta < \pi/2} d\Omega \frac{d\sigma}{d\Omega}, \quad \sigma^B = \int_{\theta > \pi/2} d\Omega \frac{d\sigma}{d\Omega};$$

the left-right asymmetry

$$A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$$  \hspace{1cm} (146)

where $\sigma_{L(R)}$ denotes the integrated cross section for left (right) handed electrons. $A_{LR}$, in case of lepton universality, is equal to the final state polarization in $\tau$-pair production:

$$A_{pol}^\tau = A_{LR}. \hspace{1cm} (147)$$

The on-resonance asymmetries are simple combinations of the effective coupling constants:

$$A_{FB}^f = \frac{3}{4} A_e A_f + \cdots \hspace{1cm} (148)$$

$$A_{pol}^\tau = A_{LR} = A_e + \cdots$$

where the dots stand for additional small terms due to the photon exchange part of the amplitude. The basic combinations

$$A_f = \frac{2 | 2 I_{3}^f - 4 Q_f \sin^2 \theta_{eff}^f |}{1 + (2 I_{3}^f - 4 Q_f \sin^2 \theta_{eff}^f)^2}$$  \hspace{1cm} (149)

are functions of the effective mixing angles (139) only. If they are calculated in terms of our basic parameter set, the asymmetries become dependent on $M_H, m_t$, and hence are not uniquely predictable so far. As an example, the $\tau$-polarization is shown in Figure 13.

Once $\sin^2 \theta_{eff}^f$ has been determined for a single fermion species from an asymmetry measurement, all the other asymmetries are fixed independently of the values of the unknown parameters. This is immediately clear for the light fermions $\neq b$ because the non-universal part of $\sin^2 \theta_{eff}^f$ is unique when we know the $Z$ mass. In case of the $b$ quark asymmetry, where the non-universal part does depend on $m_t$, the combination $A_b$ in (149) is only weakly varying with $\sin^2 \theta_{eff}^f$ so that it also practically $m_t$-independent. This has the consequence that the relation between the various asymmetries is in practice insensitive to the unknown parameters and can be considered a test of the basic SU(2)$\times$U(1) structure of the theory. This relation, displayed in Figure 14, is also insensitive to changes in the tree level $\rho$-parameter and to all kinds of "New Physics" which contribute only to the gauge boson propagators because such further unknown objects would be eliminated through $s_W^2$. On the other hand, observed violations of this relation would be a signal for new structures in the coupling constants like the admixture of an extra $Z'$. 

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Fig. 13: $A_{pol}^r$ or $A_{LR}$ in the minimal model. $M_Z = 91.15$ GeV; $M_H = 25$ ($\cdots$), 100 ($\longrightarrow$), 1000 ($\ldots$) GeV

Fig. 14: $\tau$-polarization versus forward-backward asymmetries for muons ($\longrightarrow$), c-quarks ($\cdots$), b-quarks ($\ldots$)
5 Summary

Experiments at LEP and SLC have already determined the mass of the \( Z \) boson and its width with high accuracy. Besides further improvements also on the partial widths, a series of on-resonance asymmetries with and without beam polarization will be measured with high accuracy as well. Together with the experimental determination of the \( W \) mass at \( p\bar{p} \) and \( e^+e^- \) (LEP II) colliders, these experiments provide precision tests of the electroweak Standard Model by probing the quantum structure of the electroweak theory as a local quantum field theory, and simultaneously act as sensors for new physics. The experimentally unknown particles of the Standard Model spectrum enter the predictions for the observables in higher order terms; the same would apply to further heavy objects associated with extensions of the minimal model. Before any new physics effect can be identified, the minimal model predictions have to be known with high accuracy and reliability. For the results at the 1-loop level and the treatment of the leading higher order terms the theoretical ingredients are established. Before the interesting physics can be extracted from experimental data, the detector dependent QED corrections have to be disentangled from cross sections and asymmetries. Hence, a precise understanding of the QED effects and their implementation into Monte Carlo generators require a lot of attention. They will be treated in the second part of these lectures.

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Appendix

Conventions:

Metric:

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

4-vectors:

\[ p = (p^\mu) = (p^0, \vec{p}), \quad p^2 = (p^0)^2 - \vec{p}^2 \]
\[ \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \Box = \partial_\mu \partial^\mu \]

Dirac matrices:

\[ \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = g^{\mu\nu} \]
\[ \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^2 = 1 \]

with \( \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \).

\[ \sigma := a_\mu \gamma^\mu, \quad \sigma^2 = a^2 \]

Traces:

\[ \text{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \]
\[ \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4[g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}] \]
\[ \text{Tr}(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma} \]
\[ \text{Tr}(\gamma_5 \gamma^\mu \gamma^\nu) = 0 \]

Cross section:

The scattering matrix elements \( \mathcal{M} \) are normalized in such a way that the differential cross section for \( e^+e^- \rightarrow f \bar{f} \) reads:

\[ \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2s} \sqrt{1 - \frac{4m_f^2}{s}} |\mathcal{M}|^2 \]
Some one-loop integrals

\begin{align*}
\mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m_1^2 + i\epsilon} & = \frac{i}{16\pi^2} A(m) \\
\mu^{4-D} \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu}{(k^2 - m_1^2 + i\epsilon)((k + q)^2 - m_2^2 + i\epsilon)} & = \frac{i}{16\pi^2} B_0(q^2, m_1, m_2) \\
\mu^{4-D} \int \frac{d^D k}{(2\pi)^D} k_\mu k_\nu \frac{1}{(k^2 - m_1^2 + i\epsilon)((k + q)^2 - m_2^2 + i\epsilon)} & = \frac{i}{16\pi^2} \left\{ g_{\mu\nu} B_{22}(q^2, m_1, m_2) + q_\mu q_\nu B_{21}(q^2, m_1, m_2) \right\}
\end{align*}

For $D \to 4$ with $\Delta = \frac{2}{4-D} - \gamma + \log 4\pi$ :

\begin{align*}
A(m) & = m^2 \left( \Delta - \log \frac{m^2}{\mu^2} + 1 \right) \\
B_0(q^2, m_1, m_2) & = \Delta - \int_0^1 \! dx \log \frac{z^2 q^2 - x(q^2 + m_1^2 - m_2^2) + m_1^2 - i\epsilon}{\mu^2} \\
B_1(q^2, m_1, m_2) & = \left( A(m_1) - A(m_2) + (m_2^2 - m_1^2 - q^2) B_0(q^2, m_1, m_2) \right) / 2q^2 \\
B_{22}(q^2, m_1, m_2) & = \frac{1}{3} \left\{ \frac{1}{2} A(m_2) + m_1^2 B_0(q^2, m_1, m_2) + \frac{1}{2} (q^2 + m_1^2 - m_2^2) B_1(q^2, m_1, m_2) + \frac{m_1^2 + m_2^2}{2} - \frac{q^2}{6} \right\}
\end{align*}

Equal masses $m_1 = m_2$ :

\begin{align*}
B_1(q^2, m, m) & = -\frac{1}{2} B_0(q^2, m, m) \\
B_0(q^2, m, m) & = \Delta - \log \frac{m^2}{\mu^2} + \bar{B}_0(q^2, m, m)
\end{align*}

with

\begin{align*}
\bar{B}_0(0, m, m) & = 0 \\
\bar{B}_0(q^2, m, m) & = 2 - \log \frac{q^2}{m^2} + i\pi \quad (q^2 \gg m^2)
\end{align*}
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1 Introduction

In the first part of these lectures, W. Hollik has given an exposition of how quantum effects of the electroweak theory are calculated. Two major issues in his lectures were the computation of $\Delta r$ (that is, the relation between the $W$ and $Z$ masses with other fundamental parameters of the Minimal Standard Model such as $G_\mu$ and $\alpha$), and the application of higher-order corrections to $Z$ physics in processes like $e^+e^- \rightarrow ff$. This last process is currently under study at LEP and SLC, where $f$ stands for leptons, neutrinos or quarks. Using what he has taught us about loop corrections and renormalization we can in principle compute all purely weak corrections to the cross section for $e^+e^- \rightarrow ff$, that is, all those radiative corrections in which particles other than photons have been added to the lowest-order diagrams. Of course, in principle the additional loop diagrams with the photons must be included in this calculation. However in doing so we encounter the so-called infra-red singularities which cannot be done away with by renormalization; instead these singularities must be cancelled by corresponding ones from bremsstrahlung processes like $e^+e^- \rightarrow ff\gamma$. That, however, implies that we have to deal with more particles than just the $ff$ in the final state, which makes the treatment again more complicated. The present lectures, therefore, are devoted to these problems, that have their own technicalities and require a slightly different point of view, as we shall see.

This contribution is divided into several parts. In section 2, I shall discuss some general features of bremsstrahlung processes where one or more (or even an infinity of) additional photons are produced. This will lead us naturally to the conclusion that the best way to describe these effects is not by analytical formulae but by Monte Carlo algorithms. Section 3 will be devoted to an elementary overview of the most important Monte Carlo techniques, culminating in a schematic lay-out of typical Monte Carlo event generators in section 4. Note that I shall not discuss the process $e^+e^- \rightarrow e^+e^-$ except for a cautionary statement at the very end.

2 Bremsstrahlung

2.1 Soft photon bremsstrahlung

Let us consider where we stand at the end of Part I of these lectures on the computation of weak radiative corrections. We can compute weak loop diagrams and perform renormalization, all of which culminates in amplitudes for the process

$$e^+(p_+)e^-(p_-) \rightarrow \mu^+(q_+)\mu^-(q_-),$$

(1)

and if we square and sum these we have the cross section. Some typical diagrams are depicted in fig.1. In these lectures I shall use this case of muon pair production as an example: the extension to other leptons or quarks is straightforward. This process has a
two-particle final state, which is therefore completely specified by the $\mu^+$ scattering angle, $\Omega_\mu$. The cross section, including the weak quantum effects, reads

$$\frac{d\sigma}{d\Omega_\mu} = \Phi(s, \Omega_\mu, \alpha, G_\mu, M_Z, M_H, M_t, \alpha_s, \cdots),$$

(2)

where $\Phi$ is the appropriate function of the beam-beam invariant mass squared $s \equiv 4E^2$, the scattering angle, and the parameters of the theory. Let us for the moment forget what went into the calculation of the function $\Phi$ and just concentrate on its actual form. At fixed beam energies, the only variable under experimental control in $\Phi$ is the scattering angle $\cos \theta$, of the $\mu$ (assuming that there is no transverse beam polarization, the azimuthal angle $\phi$ is irrelevant). Therefore, the whole outcome of all the complicated calculations is just a function of $\cos \theta$, and not a very wild one at that! In fact, it has been proved that for many purposes, including the description of the $\cos \theta$ distribution close to the resonance, the so-called improved Born approximation is quite good: in this approximation, $d\sigma/d\Omega_\mu$ is written just as at Born level, with modified values for the $Z$ width $\Gamma_Z$, $\sin^2 \theta_w$ (or rather $\sin^2 \theta_w$), and the overall normalization. The corresponding $Z$-exchange amplitude is given in Eq.(133) in Part I of these lectures. Now a one-dimensional distribution is practically always simple to handle: once its analytic form has been calculated, it is quite trivial to integrate it over some experimental acceptance to obtain the observed cross section. After giving the analytical cross section to the experimentalists, a theorist would have nothing more to do with it.

However, things are not that simple. As is well known, whenever charged particles scatter (and are therefore accelerated), some energy must be lost to electromagnetic radiation, that is, bremsstrahlung photons. Of course in many cases this bremsstrahlung energy loss is quite small, so that the bremsstrahlung processes $e^+ e^- \rightarrow \mu^+ \mu^- \gamma(\gamma \gamma \cdots)$ are experimentally indistinguishable from the elastic process: in many cases, photons with an energy of around 100 MeV or less will not be resolved by a LEP experiment. Nonetheless these contributions have to be added to the predicted cross section. For the moment we shall restrict ourselves to single photon emission only.

Let us see what a bremsstrahlung cross section looks like. In fig.2, I have depicted a generic Feynman diagram for $e^+ e^- \rightarrow \mu^+ \mu^-$ or so. The blob denotes any number of weak loop graphs. Let us write the amplitude corresponding to this diagram as

$$M_\theta = u(p)A(p),$$

(3)

where we have singled out the outgoing muon with momentum $p$ and charge $-e$ for special treatment: the $A$ denotes the rest of the diagram. Let us now consider the radiation of a photon with momentum $k^\mu$ and polarization $e^\nu$ from this outgoing fermion, as in fig.3.
This diagram can be written as

\[ \mathcal{M}_1 = e\bar{u}(p)\gamma_\mu \frac{p + k + m}{2p\cdot k} A(p + k) . \] (4)

In the limit where \( k^\mu \) is small compared to the other momenta in the problem, we may approximate \( \mathcal{M}_1 \) as follows:

\[
\begin{align*}
\mathcal{M}_1 & \sim e\bar{u}(p)\gamma_\mu \frac{p + m}{2p\cdot k} A(p) \\
& \sim e\bar{u}(p) \left( \frac{p \cdot \epsilon}{p \cdot k} \right) A(p) \\
& = e \left( \frac{p \cdot \epsilon}{p \cdot k} \right) \mathcal{M}_0 ,
\end{align*}
\] (5)

where in the first line we have dropped all terms with \( k \) in the numerator, and in the second line have used the Dirac equation for the spinor \( u(p) \). As we see, the amplitude factorizes into the original, non-radiative one and a radiation factor. Of course, the other external lines contribute as well: if we perform the same derivation for the general case, we obtain the famous soft-photon approximation

\[ \mathcal{M}_1 = e \left( \frac{p^+ \cdot \epsilon}{p^+ \cdot k} - \frac{p^- \cdot \epsilon}{p^- \cdot k} - \frac{q^+ \cdot \epsilon}{q^+ \cdot k} + \frac{q^- \cdot \epsilon}{q^- \cdot k} \right) \mathcal{M}_0 , \] (6)

where each of the external legs can be recognized. Some remarks are in order here. In the first place, we have neglected any terms with \( k \) in the numerator. This is usually allowed but if \( A(p) \) varies rapidly with \( p \), as is the case if we are in the neighbourhood of a resonance, then this approximation may break down unless we go to really very small \( k \). In the second place, we have only considered radiation from external lines. Obviously if a charged particle were present inside the blob in fig.2 then it also could radiate some bremsstrahlung. It can be proved, however, that such contributions do not behave as \( k^{-1} \) as do the terms in Eq.(6). This is reasonable: photons with small \( k \) are those with long wavelengths which are sensitive to the global charge distributions at large space-time distances (i.e. the external lines) but should not be able to resolve small quantum effects (internal lines) in the tiny region where the hard scattering is taking place.
From the amplitude we can obtain the multidifferential cross section by squaring and including the phase space factors. In squaring the amplitude and summing over the polarizations we can use
\begin{equation}
\sum_{\text{spins}} e_\mu (e_\nu)^* = -g_{\mu\nu},
\end{equation}
so that the matrix element, suitably summed/averaged over spins (and colours, if necessary) reads
\begin{equation}
\sum_{\text{spins}} |M_1|^2 = e^2 R(p_+, p_-, q_+, q_-, k) \times \sum_{\text{spins}} |M_0|^2,
\end{equation}
with
\begin{equation}
R(p_+, p_-, q_+, q_-, k) = \times \left[ \frac{2(p_+ \cdot p_-)}{(p_+ \cdot k)(p_- \cdot k)} - \frac{m^2_e}{(p_+ \cdot k)^2} - \frac{m^2_\mu}{(p_- \cdot k)^2} \right.
+ \left. \frac{2(q_+ \cdot q_-)}{(q_+ \cdot k)(q_- \cdot k)} - \frac{m^2_\mu}{(q_+ \cdot k)^2} - \frac{m^2_\mu}{(q_- \cdot k)^2} \right.
+ \left. \frac{2(p_+ \cdot q_+)}{(p_+ \cdot k)(q_+ \cdot k)} - \frac{2(p_+ \cdot q_-)}{(p_+ \cdot k)(q_- \cdot k)} + \frac{2(p_- \cdot q_-)}{(p_- \cdot k)(q_- \cdot k)} - \frac{2(p_- \cdot q_+)}{(p_- \cdot k)(q_+ \cdot k)} \right].
\end{equation}
Now we come to the phase space factor. In the non-radiative case we have
\begin{equation}
dLIPS_{\mu\mu} = \frac{1}{(2\pi)^6} \delta^4(p_+ + p_- - q_+ - q_-) d^4q_+ \delta(q_+^2) d^4q_- \delta(q_-^2)
= \frac{1}{8(2\pi)^6} d\Omega_\mu,
\end{equation}
where \(\Omega_\mu\) is the muons' solid angle in the laboratory frame, and we have assumed the muons to be essentially massless at LEP energies. If an additional bremsstrahlung photon is present we have
\begin{equation}
dLIPS_{\mu\gamma} = \frac{1}{(2\pi)^9} \delta^4(p_+ + p_- - q_+ - q_- - k) d^4q_+ \delta(q_+^2) d^4q_- \delta(q_-^2) d^4k \delta(k^2)
= \frac{1}{16(2\pi)^9} d\Omega_\mu^* k^0 dk^0 d\Omega_\gamma
= \frac{1}{16(2\pi)^9} \frac{(q_+^0)^2 k^0}{(E_b - k^0) (\Omega_b - k^0)} d\Omega_\mu^* d\Omega_\gamma.
\end{equation}
Here we have two alternative forms of the phase space element. In the first alternative, the muon scattering angle \(\Omega^*\) is defined in their CM frame, which is boosted away from the laboratory frame because it recoils against the bremsstrahlung momentum. The photon variables are defined in the laboratory frame. This makes it hard to translate an experimental cut (which is defined in the laboratory frame) into a simple restriction on \(\Omega^*\), as we could do in the non-radiative case. On the other hand, in the CM frame the muons are of course still back-to-back. In the second form, all variables are defined in the laboratory frame. The first disadvantage in this case is that the phase space integration element itself becomes a quite non-trivial function of the momenta: the \(\mu^+\) energy is not a free variable, but is instead given by the photon energy and the angles, as follows:
\begin{equation}
q_+^0 = q_+^0(k^0, \Omega_\gamma, \Omega_+) = \frac{2E_b(E_b - k^0)}{2E_b - k^0 \left(1 - \cos(q_+^0, k)\right)}.
\end{equation}
Integrating this out analytically is non-trivial, even with a constant matrix element! The
second, more serious, disadvantage is that we lose control completely over the $\mu^-$ scattering
angle. Indeed, from Eq.(12) and the fact that $q_+^2 + q_-^2 + k^0 = 2E_b$ we obtain for $\cos \theta_-$, the angle of $\vec{q}_-$ to the beams, the horrible form
\begin{equation}
\cos \theta_- = \frac{k^0 \cos \theta_+ + q_+^0 (k^0, \Omega_\gamma, \Omega_\mu) \cos \theta_+}{2E_b - k^0 - \vec{q}_-^0 (k^0, \Omega_\gamma, \Omega_\mu)},
\end{equation}
where $\theta_+$ and $\theta_-$ are the angles of the photon and the $\mu^+$ with respect to the beams.
Clearly, in this second formulation of $d\text{LIPS}_{\mu\nu\gamma}$, imposing cuts on the photon and $\mu^+$
angles may be trivial but it is practically impossible to impose any realistic cut on the
$\mu^-$.
Especially in the case of Bhabha scattering, where the cross section depends sensitively
on such angular cuts, this makes an analytical integration of the cross section well nigh
impossible.

For not too hard photons, however, we can still do some things analytically. In that
case we have approximately
\begin{equation}
d\text{LIPS}_{\mu\nu\gamma} \sim \frac{1}{16(2\pi)^9} k^0 dk^0 d\Omega_\gamma d\Omega_\mu,
\end{equation}
with the muons still approximately back-to-back in the lab frame. The cross section for
bremsstrahlung is then given by
\begin{equation}
\frac{d\sigma_{\mu\nu\gamma}}{d\Omega_\mu} = \frac{d\sigma_{\mu\mu}}{d\Omega_\mu} \times R(p_+, p_-, q_+, q_-, k_0) \frac{\alpha}{4\pi^2} k^0 dk^0 d\Omega_\gamma.
\end{equation}

This is the soft-photon approximation for the radiative cross section. As stated before, this
is typically a quite good approximation if the photon energy is up to a few per cent of $E_b$
- unless the cross section fluctuates rapidly with energy, as is of course the case at LEPI.
In that case, the approximation is still good whenever $k^0 \ll \Gamma_Z/M_Z$ at the peak, but at
around $\sqrt{s} \sim M_Z \pm \Gamma_Z/2$, that is, on the steep slopes of the $Z$ peak, $k^0$ has to be really
small for the approximation to be valid. All this holds for initial-state radiation, since only
bremsstrahlung emitted from the initial state can influence the resonance, which is created
‘after’ bremsstrahlung emission: for final-state radiation, emitted after the resonance has
decayed (in this case, into a muon pair) there is no such effect. Obviously, the behaviour
of the interference between initial- and final-state radiation is again more complicated.

One can modify the soft-photon approximation to take this behaviour into account [1]
but here we shall just assume that we use the soft-photon approximation for sufficiently
small values of $k^0$.

Let us now study the main features of bremsstrahlung emission. This can be done
by considering a typical term of the cross section: we have
\begin{equation}
\frac{\alpha}{4\pi^2} \frac{2(p_+ \cdot p_-)}{(p_+ \cdot k)(p_- \cdot k)} k^0 dk^0 d\Omega_\gamma = \frac{\alpha}{\pi^2} k^0 dc_\gamma d\phi_\gamma,
\end{equation}
where $c_\gamma = \cos \theta_\gamma$ and $\beta \equiv |\vec{p}_+|/|\vec{p}_+^0| \sim 1 - 2m_e^2/s$ is the electron and positron velocity.
Inspection of Eq.(16) teaches us all the essentials about bremsstrahlung. In the first place,
due to the fact that the electrons are relativistic ($\beta \sim 1$) the cross section peaks hugely
whenever $c_\gamma \sim 1$: these are the so-called collinear peaks. These remain finite as long as
\( m_e \neq 0 \) (but hence get higher and higher as \( s \) increases). Integrating over \( c_\gamma \) and \( \phi_\gamma \) we obtain
\[
\frac{2\alpha}{\pi} \frac{dk^0}{k^0} \frac{1}{\beta} \log \left( \frac{1 + \beta}{1 - \beta} \right) \sim \frac{2\alpha}{\pi} \log \left( \frac{s}{m_e^2} \right) \frac{dk^0}{k^0},
\]
(17)
where we see that the collinear peaks give rise to large logarithms. In the second place, we see that the cross section goes as \((k^0)^{-1}\), and will hence diverge at zero photon energy. For the moment, we may solve this problem by giving the photon a small but finite mass \( \lambda \). In fact, we know of course that the photon is massless (a rather good limit of \( 3 \times 10^{-36} \) GeV exists [2]) so we will have to argue that at the end of the calculation we can sensibly take \( \lambda \to 0 \). At any rate, we can now perform an integration over \( k^0 \) and obtain the soft-bremsstrahlung correction to the Born cross section due to this particular term in \( R \):
\[
\frac{d\sigma^S_{\mu\mu}}{d\Omega_{\mu\mu}} \right|_{\text{one term}} = \frac{d\sigma^0_{\mu\mu}}{d\Omega_{\mu\mu}} \left\{ \frac{2\alpha}{\pi} \log \left( \frac{s}{m_e^2} \right) \log \left( \frac{\Delta E}{\lambda} \right) + \cdots \right\}. \tag{18}
\]
Here we have assumed an upper limit \( \Delta E \) on the allowed integration region for \( k^0 \). For the soft-photon approximation to hold, \( \Delta E \) cannot be too large: we shall return to this point later on. The terms denoted with \( \cdots \) are terms that can contain a \( \log(s/m_e^2) \) but no \( \log(\Delta E/\lambda) \): they are numerically not negligible but unimportant for our discussion here. If we include the other terms in \( R \) as well, we end up with the integrated soft-photon cross section \( d\sigma^S \)
\[
\frac{d\sigma^S_{\mu\mu}}{d\Omega_{\mu\mu}} = \frac{d\sigma^0_{\mu\mu}}{d\Omega_{\mu\mu}} \times \left\{ (\beta_\epsilon + \beta_f + \beta_i) \log \left( \frac{\Delta E}{\lambda} \right) + \cdots \right\},
\]
\[
\beta_\epsilon \equiv \frac{2\alpha}{\pi} \left( \log \left( \frac{s}{m_e^2} \right) - 1 \right),
\]
\[
\beta_f \equiv \frac{2\alpha}{\pi} \left( \log \left( \frac{s}{m_e^2} \right) - 1 \right),
\]
\[
\beta_i \equiv -\frac{8\alpha}{\pi} \log(\tan \frac{\theta_\mu}{2}). \tag{19}
\]
Here we have explicitly indicated the contributions from initial- and final-state radiation and their interference. At LEP1 energies we have \( \beta_\epsilon \sim 11\% \), \( \beta_f \sim 6\% \), and \( \beta_i \) is typically a few per cent, depending of course on the scattering angle \( \theta_\mu \). If we have particles with a charge \( Q \) other than muons in the final state, \( \beta_f \) acquires a factor \((Q/\epsilon)^2\) and \( \beta_i \) a factor \(-Q/\epsilon\). The terms \((-1)\) in the \( \beta \)'s come from the terms with \( m^2 \) in \( R \).

2.2 The infra-red cancellation

Let us now return to the infra-red divergence, which we regularized with a photon mass \( \lambda \). As indicated in Part I of these lectures, when we compute the loop diagrams with

---

1) Some care has to be taken here, since upon the introduction of a photon mass the expressions for both the phase space element and products like \((p_\mu - k)\) change slightly for \( k^0 \) values comparable to \( \lambda \). These technicalities are important for getting the subleading terms correctly, but do not influence the leading terms with which we are concerned here.
virtual rather than real photons, similar divergences come up. The cross section with virtual photon corrections is seen to be

\[
\frac{d\sigma^\nu_{\mu\nu}}{d\Omega_\mu} = \frac{d\sigma^0_{\mu\nu}}{d\Omega_\mu} \times \left\{1 + (\beta_e + \beta_f + \beta_i) \log \left( \frac{\lambda}{E_b} \right) + \cdots \right\},
\]

(20)

where again the (\cdots) stands for terms that are numerically relevant but do not contain \(\log \lambda\). We see that now we can combine \(\sigma^s\) and \(\sigma^v\) and the result will be finite: the first-order corrected cross section reads

\[
\frac{d\sigma^1_{\mu\nu}}{d\Omega_\mu} = \frac{d\sigma^0_{\mu\nu}}{d\Omega_\mu} \times \left\{1 + (\beta_e + \beta_f + \beta_i) \log \left( \frac{\Delta E}{E_b} \right) + \cdots \right\},
\]

(21)

which justifies our regularization procedure: \(\lambda \to 0\) can now be taken in a sensible way. This phenomenon of infra-red cancellation has of course been known for a long time: it has been proved by Yennie, Frautschi and Suura [3] that this cancellation persists to all orders of perturbation theory. The (\cdots) terms are known to first order exactly, and some of them also to higher orders: detailed accounts can for instance be found in [4, 5].

What is the physical picture behind this? When computing a physical cross section, one takes into account loop effects due to virtual states: virtual photons are emitted, and absorbed again before they enter the final state. This is necessary because such channels give rise to the same final state as the Born scattering. However, since the photon is massless, it can be arbitrarily soft, and therefore a final state with such a photon is also indistinguishable from the non-radiative final state. It stands to reason that only if we combine all channels that lead to experimentally identical final states, can we end up with a sensible answer\(^2\). The following observation may be useful in this context. In the virtual corrections, diagrams occur with virtual \(Z\) loops similar to the virtual-photon diagrams. These corrections are characterized by terms such as \(\alpha \log(M_Z^2/s)\) which are typically small at LEP. However we could imagine a collider with \(\sqrt{s} \sim 10^{6}\)GeV: in that case these corrections would become quite enormous, and in fact go to \(-\infty\) as \(s \to \infty\). This ‘infra-red divergence’ would then have to be cured by considering not only the Born process \(e^+e^- \to \mu^+\mu^-\) but also ‘radiative’ processes like \(e^+e^- \to \mu^+\mu^-Z\): after all, at such high energies the experimental energy resolution would be many thousands of GeV, and a ‘soft \(Z\)’ would be as invisible as a ‘soft photon’ at LEP. It is only the fact that the \(Z\) mass is large at LEP scales which allows us to include virtual \(Z\)’s but forget about the bremsstrahlung \(Z\)’s.

What have we learnt so far? In the first place, it is likely (and we shall prove it later) that some soft bremsstrahlung must always occur; indeed, the soft-photon cross section by itself is even infinite! A sensible, finite answer can be obtained by combining the soft and virtual photon corrections, since the virtual corrections by themselves are also infinitely negative. But to do this we have paid a price, namely (a) we have had to perform the soft-photon approximation, and (b) we have integrated the soft photons over some allowed part of phase space. The result contains \(\log(\Delta E/E_b)\), which implies that

\(^2\) In this sense, the situation is similar to that used in gauge-invariance considerations: the physical propagator for the weak neutral current can be viewed either as solely due to the \(Z^0\) field (the unitary gauge) or to a modified \(Z^0\) field plus a ‘ghost’ field (the \(\text{'t} Hooft-Feynman gauge\). A sensible answer is only guaranteed if we always include all fields in the propagator that lead to the same external states.
the answer for the cross section depends on what is defined to be the allowed soft-photon phase space. If we let $\Delta E$ go to zero, again a negatively infinite result arises. For typical scattering angles, the cross section becomes negative for values of $\Delta E$ of around 100 MeV or less. This is in fact not an extremely small value! As stated before, just to the sides of the $Z$ peak the soft-photon approximation may not even be justified there due to the large changes in the cross section for a small shift in energy. But we are committed to values such as this for two reasons. In the first place, making $\Delta E$ smaller would certainly yield an unphysical (because negative) cross section. Making $\Delta E$ larger means that we assume the soft-photon approximation to be valid over a larger range of $k^0$ values, which is usually not justified. Later on, we shall return to experiment to obtain an idea of $\Delta E$.

The soft-photon approximation is quite useful to obtain a rough estimate of most radiative cross sections, at least when the Born cross section does not vary too rapidly with energy, or when the cuts are not too tight. One should keep in mind, however, that LEP is a high-precision apparatus, and if we want to test the standard model to the quantum level, that is, to effects of order $\mathcal{O}(\alpha)$, we need to calculate the bremsstrahlung effects also to well below this level. This leads us to consider two additional questions:

- If $\Delta E$ is as small as one per cent, the first order correction becomes appreciable, of the order of tens of per cent. How about higher order effects in that case?
- The value of $\Delta E$ only tells us which photons can be emitted in all possible directions. It is clear, however, that under realistic experimental cuts quite hard photons will still be admitted by the cut. Certainly, at that point the soft-photon approximation will no longer be adequate. What about these hard-photon effects? And what about experiments where an explicit hard photon is required \(^3\)?

2.3 Multiple soft photon effects

Let us now have a look at higher orders in perturbation theory. Let us suppose that $\Delta E$ is small, either because of the imposition of a strict cut or because of the dynamics of the resonance process (we shall discuss this later on). Then $\beta \log(\Delta E/E_b)$, where $\beta = \beta_e + \beta_f + \beta_1$, can become quite sizeable, and we have to worry about the magnitude of higher orders. In particular, the cross section $\sigma^{V+S}$ can become negative as indicated in Eq.(21); can this be cured by higher order effects? The answer is affirmative, and relies on the so-called exponentiation procedure. To see how this works, let us go to second order in $\alpha$. Obviously there will then be contributions to the cross section from two virtual photons, two real bremsstrahlung photons, and from the mixed case where the one photon is real, and the other one is virtual. Neglecting non-divergent terms as before, we can then write for the various second-order contributions, again in the soft-photon approximation:

\[
\begin{align*}
\frac{d\sigma^{V_1V_2}}{d\Omega_\mu} &= \frac{d\sigma^9}{d\Omega_\mu} \left\{ \frac{1}{2} \beta^2 \log^2 \left( \frac{\lambda}{E_b} \right) + \cdots \right\}, \\
\frac{d\sigma^{V_1S+S_1V_2}}{d\Omega_\mu} &= \frac{d\sigma^9}{d\Omega_\mu} \left\{ \beta^2 \log \left( \frac{\lambda}{E_b} \right) \log \left( \frac{\Delta E}{\lambda} \right) + \cdots \right\}, \\
\frac{d\sigma^{S_1S_2}}{d\Omega_\mu} &= \frac{d\sigma^9}{d\Omega_\mu} \left\{ \frac{1}{2} \beta^2 \log^2 \left( \frac{\Delta E}{\lambda} \right) + \cdots \right\},
\end{align*}
\]

\(^3\) As in searches for excited fermions.
so that the total second-order corrected cross section reads

$$\frac{d\sigma^2}{d\Omega_\mu} = \frac{d\sigma^0}{d\Omega_\mu} \left\{ 1 + \beta \log \left( \frac{\Delta E}{E_b} \right) + \frac{1}{2} \beta^2 \log^2 \left( \frac{\Delta E}{E_b} \right) + \cdots \right\}, \quad (23)$$

with obvious notation. Note the factor 1/2 which arises because of the indistinguishability of the two photons. It is simple to guess how this series must be continued for 3, 4, \cdots photons. If we allow any number of soft photons to be emitted, the cross section will read

$$\frac{d\sigma^\infty}{d\Omega_\mu} = \frac{d\sigma^0}{d\Omega_\mu} \times \left\{ \sum_{k=0}^{\infty} \left( \beta \log \left( \frac{\Delta E}{E_b} \right) \right)^k + \cdots \right\}$$

$$= \frac{d\sigma^0}{d\Omega_\mu} \times \left\{ e^{\beta \log(\Delta E/E_b)} + \cdots \right\}$$

$$= \frac{d\sigma^0}{d\Omega_\mu} \times \left\{ \left( \frac{\Delta E}{E_b} \right)^\beta + \cdots \right\}. \quad (24)$$

The reason for the name ‘exponentiation’ appears obvious. An important observation to make here is that whereas the first-order corrected cross section $\sigma^1$ went to $-\infty$ as $\Delta E \to 0$, and the second-order corrected one even faster to $+\infty$, the resummed cross section behaves quite reasonably, and even goes to zero (in leading order) for vanishing $\Delta E$. This is the famous Bloch-Nordsieck theorem [6]: there is always some bremsstrahlung, in the sense that if we make the energy resolution infinitely good, the elastic cross section goes to zero. This justifies the conjecture of the previous section. Concerning the (\cdots) terms, there is some knowledge of which of them can be absorbed into the exponentiation: a nice recent discussion can be found in [7]. One last remark is the following subtlety. In principle, if an infinity of photons is allowed, each of which can carry away an amount of energy $\Delta E$, the total energy loss can of course still be enormous. In order to avoid this, it is customary to require that the sum of the soft photon energies is no larger than $\Delta E$. For single bremsstrahlung this of course amounts to the same, but if more bremsstrahlung photons are present, each of them restricts the allowed phase space for the rest by some amount. Therefore, a more careful calculation [3] yields that there is actually an overall factor which supresses the exponentiated cross section: it reads

$$\frac{e^{-\gamma_E \beta}}{\Gamma(1 + \beta)} = 1 - \frac{\pi^2}{12} \beta^2 + \mathcal{O}(\beta^3), \quad (25)$$

where $\gamma_E = 0.5772156 \cdots$ is Euler’s constant, and $\Gamma$ is the gamma-function. Actually this deviates from 1 only in second order, as we have argued above. Its numerical value is about 0.9905 at LEP energies, that is about the limit of the experimental accuracy.

### 2.4 Hard photon bremsstrahlung

In actual LEP experiments, photons can of course be observed directly down to some energy, which varies with the experiment. However when for instance the photon is emitted along the beam (and we know that the cross section peaks for such configurations) this is not possible. Also, in luminosity monitors, a produced electron can typically not be distinguished from any accompanying radiation, so also those collinear ones can be quite hard without being seen explicitly. A more sensitive quantity, which can be measured without too much trouble is the acollinearity $\zeta$, defined as the deviation of the muon (or general fermion) pair from the back-to-back configuration expected for the Born channel:

$$\zeta \equiv 180^0 - \zeta(q^+_s, q^-_s). \quad (26)$$
Obviously, when a hard, non-collinear photon is emitted ζ will deviate from zero. Now remember that in order to cancel the negative divergence from the virtual corrections, we have to integrate the soft-photon cross section over all photon angles: otherwise, a coefficient different from β + β + β would occur, and the divergence would be spoilt. The relevant question is therefore: for a given acollinearity cut ζ, which is that value of k0 at which a photon can still be emitted in all possible directions without leading to an acollinearity larger than ζ? This is a simple exercise in 3-body kinematics, which leads to the definition of the maximal isotropic photon energy:

$$k_{\text{max}}^{\text{iso}} = E_b \times \frac{2 \sin(\zeta/2)}{1 + \sin(\zeta/2)}.$$  \hspace{1cm} (27)

This, then, is the maximal value of ΔE which can be admitted in an experiment that uses an acollinearity cut to select back-to-back events. Photons softer than this limit can be emitted in all directions, while photons harder than this can, when emitted in some directions (perpendicular, say, to the muon pair) lead to an acollinearity exceeding the value ζ. For an angular resolution of 0.5 degrees this leads to about ΔE ~ 0.01E_b, which is also about the limit below which the cross section φV+S can start to become negative. As we see, there is not much room left to vary ΔE. It should be borne in mind, however, that the limit given by k_{\text{max}}^{\text{iso}} only pertains to the isotropic photon phase space. For higher photon energies than k_{\text{max}}^{\text{iso}} we simply have that the acollinearity cut may be violated in some directions, however, those are typically not the preferred directions for photon emission. In fact, when the photon is emitted close to one of the outgoing fermions (the muons), then it can have up to about the full beam energy E_b before building up any acollinearity: in such cases, the only limitation on the photon phase space is coming from a usual additional cut on the muon energy, typically E_μ > E_{\text{thr}} with E_{\text{thr}} somewhere around E_b/2. For initial-state radiation, the situation is somewhat more complicated: if the muons are produced at large angles, initial-state radiation tends to build up an acollinearity quickly. In case the muons come out at smallish angles, the limit may increase. We can see this by another result of three-body kinematics: if we assume that the bremsstrahlung photon momentum is parallel to the beams, there is a relation between the photon energy and the angles of the two muons to the beams: if θ_2 is the smallest of these two muon angles, and θ_2 is the largest one, then

$$k^0 = E_b \left[ 1 - \frac{\tan(\theta_1/2)}{\tan(\theta_2/2)} \right].$$  \hspace{1cm} (28)

This relation allows us to make an estimate of the largest allowed photon energy for initial-state radiation. If both muons come out at angles appreciably larger than the acollinearity between them, so that ζ = θ_2 − θ_1 ≪ θ_1, we have approximately k^0 = E_bζ/\sin θ_2. If the muon angles are both small and comparable to ζ we have roughly k^0 ∼ E_bζ/θ_2. As an example, in a luminosity monitor where the angles of the outgoing e^+e^- are restricted between, say, 50 and 150 mrad, we see that a photon energy as high as 0.66E_b can still be accommodated within the cuts.

Let us leave aside these complications for the moment, and see what we can do analytically. In the last few years, considerable theoretical effort has been devoted to the calculation of the line shape, that is, the total cross section for the process e^+e^- → f f, where f is any fermion except an electron or electron neutrino. The fact that one can in that case integrate over all fermion angles simplifies the calculation considerably. In
particular the effects of initial-state radiation are quite well understood in this case. Let us define \( z \equiv (q^+ + q^-)^2/s \) which is a measure of the energy taken away by the bremsstrahlung: in the single-bremsstrahlung case we have \( \kappa/E_0 = 1 - z \). Extending the old result of Bonneau and Martin [8], one nowadays [9, 10] writes the total cross section as a convolution of a 'structure function' with the Born cross section evaluated at the appropriate invariant mass \( z s \); it is presented here to give an impression of what such formulae look like. The convolution integral reads

\[
\sigma_{\text{total}} = \int_{s_0/s}^1 G(z)\sigma_0(z s)
\]  

where \( G \) is the structure function and \( \sigma_0 \) is the Born (or modified Born) cross section. The value \( s_0 \) is the minimal invariant mass of the produced fermions, which is a more or less realistic experimental cut, especially for hadrons. Including all \( \mathcal{O}(\alpha^2) \) terms and exponentiation of soft photons, we have

\[
G(z) = \beta_e(1 - z)^{\beta_e - 1} \left( 1 + \delta_1^{Y_S} + \delta_2^{Y_S} \right) + \delta_1^H + \delta_2^H,
\]

with

\[
\begin{align*}
\beta_e &= \frac{2\alpha}{\pi} (L - 1), \quad L = \log \left( \frac{s}{m_Z^2} \right), \\
\delta_1^{Y_S} &= \frac{\alpha}{\pi} \left( \frac{3}{2} L + 2\zeta(2) - 2 \right), \\
\delta_2^{Y_S} &= \left( \frac{\alpha}{\pi} \right)^2 \left[ L^2 \left( \frac{9}{8} - 2\zeta(2) \right) + L \left( -\frac{45}{16} + \frac{11}{2} \zeta(2) + 3\zeta(3) \right) \right. \\
&\left. - \frac{6}{5} \zeta(2)^2 - \frac{9}{2} \zeta(3) - 6\zeta(2) \log 2 + \frac{3}{8} \zeta(2) + \frac{19}{4} \right], \\
\delta_1^H &= -\frac{\alpha}{\pi} (1 + z)(L - 1), \\
\delta_2^H &= \left( \frac{\alpha}{\pi} \right)^2 \left\{ Y - (1 + z) \left[ 2(L - 1)^2 \log(1 - z) \\
&\quad + (L - 1) \left( \frac{3}{2} L + 2\zeta(2) - z \right) \right] \right\}, \\
Y &= L^2 \left( \frac{1 + z^2}{1 - z} \log z + \frac{1}{2} (1 + z) \log z + z - 1 \right) \\
&\times \left[ \frac{1 + z^2}{1 - z} \left( \text{Li}_2(1 - z) + \log z \log(1 - z) + \frac{7}{2} \log z - \frac{1}{2} \log(z)^2 \right) \\
&\quad + \frac{1}{4} (1 + z) (\log z)^2 - \log z + \frac{7}{2} - 3z \right] \\
&\quad + \frac{1 + z^2}{1 - z} \left\{ -\frac{3}{6} (\log z)^3 + \frac{1}{2} \log z \text{Li}_2(1 - z) + \frac{1}{2} (\log z)^2 \log(1 - z) \\
&\quad - \frac{3}{2} \text{Li}_2(1 - z) - \frac{3}{2} \log z \log(1 - z) + \zeta(2) \log(z) - \frac{17}{6} \log z - (\log z)^2 \right\} \\
&\quad + (1 + z) \left( \frac{3}{2} \text{Li}_3(1 - z) - 2 \text{Si}_1(z)(1 - z) - \log(1 - z) \text{Li}_2(1 - z) - \frac{1}{2} \right) \\
&\quad - \frac{1}{4} (1 - 5z) (\log(1 - z))^2 + \frac{1}{2} (1 - 7z) \log z \log(1 - z) - \frac{25}{6} z \text{Li}_2(1 - z) \\
&\quad + \left( -1 + \frac{13}{3} z \right) \zeta(2) - \left( \frac{3}{2} + z \right) \log(1 - z) + \frac{1}{6} (11 + 10z) \log z + \left( \frac{3}{2} + z \right) \log z + \frac{7}{2} - 3z \right\}.
\end{align*}
\]
\[ + \frac{2}{(1 - z)^2} (\log z)^2 - \frac{25}{11} z (\log z)^2 \]
\[ - \frac{2}{3} \frac{z}{1 - z} \left( 1 + \frac{2}{1 - z} \log z + \frac{1}{(1 - z)^2} (\log z)^2 \right) \]

(31)

Here we have used the poly-logarithmic functions \( Li_2, Li_3, \) and \( S_{1,2} \) that are described in [11], as well as the Riemann function values \( \zeta(2) = \pi^2/6 \) and \( \zeta(3) = 1.202\ldots \).

The properties of a formula like Eq.(31) are of course not very transparent: it is reproduced here merely as an illustration. Nonetheless, the form of Eq.(29) itself already tells us a lot about the qualitative behaviour of the initial-state radiative corrections. Suppose that we are close to the peak, i.e. \( \sqrt{s} \sim M_Z \). In that case, \( \sigma_0(s) \) is large due to the resonance - but \( \sigma_{\gamma}(z s) \) will be a lot smaller as soon as \( z \) differs appreciably from 1. In other words, as soon as a non-negligible amount of energy is lost to initial-state bremsstrahlung the beams can after radiation no longer form the resonance and the corresponding part of the convolution integral will be suppressed. This is the ‘natural’ photon energy cutoff alluded to above: at resonance, almost all photons emitted by the beams will have energies below about \( \Gamma_Z/M_Z \). The corresponding soft-photon correction of Eq.(24) will therefore behave like \( (\Gamma_Z/M_Z)\sigma_{\gamma} \) which implies a correction of about \(-30\% \) [12]. Additionally, since hardly any hard initial-state radiation photons are around, acollinearity cuts are not very tight around the peak, and the total radiative correction does not depend sensitively on the experimental setup. Contrariwise, for energies above \( M_Z \) the beams will ‘like to’ emit bremsstrahlung such as to attain \( z \sim M_Z^2/s \) and recover the resonance. Consequently the radiative correction will be large and positive: the so-called radiative tail of the resonance. Since the dominant event type there has a hard photon along the beam and a sizeable acollinearity, experimental cuts can influence the magnitude of the radiative tail considerably. These qualitative conclusions are borne out by the explicit integration of Eq.(31), which of course can only be done numerically. In fig.4 we give the Born cross section, the first-order corrected, and the fully corrected cross section according to Eq.(31). These results are among the most accurately known ones for LEPI physics: the various calculations (discussed in detail in [9, 10]) agree typically to within a few tenths of a percent.

Concerning the line shape the following observations are in order. Of course also final-state radiation should be considered. It turns out, however, that if the total cross section is considered, all large logs like \( \beta_f \) cancel in the final result. This is known as the KLN theorem [13].

The correction to the total cross section from final-state radiative effects is given by

\[ \sigma_{\text{final}} = \sigma^0 \left( 1 + \frac{3}{4} \frac{\alpha}{\pi} \right) \sim \sigma^0 \times 1.0017 \]

(32)

which is in fact the precise analogue of the famous \( \alpha_s/\pi \) correction to the hadronic \( R \), the different coefficient being due to the different colour factors in QED and QCD. Since final-state radiation is peaked only at small \( k^0 \), loose cuts on the events like the \( \alpha_0 \) cut or an acollinearity cut do not change this result appreciably, and the correction remains

\[ 4) \text{In principle, this holds only for the unrenormalized result: for a spin-1 particle like Z or } \gamma, \text{ however, it also hold in the renormalized case. For a spin-0 particle, as in the decay of a Higgs boson, mass singularities survive into the total decay cross section after renormalization.} \]
small. If we impose a loose cut $(q_+ + q_-)^2 > s_0$ the final-state radiative correction becomes, to first order:

$$
\sigma_{\text{final}} = \sigma^0 \times \left\{ 1 + \frac{3\alpha}{4\pi} - \frac{\alpha}{\pi} \frac{s_0}{s} \left[ (1 + \frac{s_0}{2s}) \left( \log \left( \frac{s_0}{m^2} \right) - 2 \right) + \frac{s_0}{4s} \right] + \mathcal{O}(\frac{s_0^4}{s^4}) \right\}.
$$

(33)

If $\sqrt{s_0/s}$ is about 0.3, the total final-state correction is zero; if tight cuts are imposed, however, the correction may become sizeably negative, so some care is always advisable in this case. Finally, there is the interference between initial- and final-state radiation. It has been shown [14] that at the Z peak, and for loose cuts, this interference is very small, around the $10^{-3}$ level. This can be understood as follows. When the Z resonance is realized, the Z has a finite (nonzero) lifetime, and consequently its creation and decay are separated in space-time. Therefore, the wave functions for a photon emitted by the initial state and that for a photon from the final state have an unusually large space-time separation and will not overlap appreciably. Again, however, if one imposes tight cuts, so that $\Delta E$ is effectively of order $\mathcal{O}(\Gamma_Z/M_Z)$, or moves away from the resonance, these arguments no longer hold and the interference will typically be again of the order of a few per cent.

These considerations bring us to the problem of what happens in realistic experimental situations. How are we to establish that analytical formulae like Eq.(31) are appropriate? How do we estimate the effects of possibly complicated and tight experimental cuts? An additional example of problems in comparing the theoretical results with experiment occurs when for instance one wants to combine the predictions for initial- and
final-state radiation at the same time. Obviously, when an overall cut \((q_+ + q_-)^2 > s_0\) is imposed, the bremsstrahlung phase space for final-state radiation will depend on the amount of energy lost due to initial-state radiation. Therefore, yet another convolution will be necessary to combine Eq.(31) and Eq.(33), and clearly this cannot be done analytically. In order to answer these questions we need information on the hard-photon cross section, and in particular its integral over arbitrary and possibly very complicated regions of phase space. But the phase space cuts imposed by a typical LEP detector cannot even be described analytically, let alone integrated over: this is already clear from the expression for \(\theta_-\) in terms of \(k^0\), \(\Omega_+\) and \(\Omega_\gamma\) of Eq.(13). What is needed is a method of integrating the wildly fluctuating hard-photon cross section (including collinear peaks) over an arbitrarily complicated phase space, which in the \(\mu^+\mu^-\gamma\) case has 4 or 5 dimensions (depending on whether the overall azimuthal orientation of the events is relevant, as it typically is in the case of a not perfectly cylindrical geometry). Also, we want to be able to judge detector responses to the events that are to be expected in order to estimate acceptances and so on. The answer to these questions is that we have to compute our cross sections using Monte Carlo techniques, and in particular with the help of event generators. These are by definition programs with the following characteristics:

- They produce upon every call a Monte Carlo event, that is, a set of particle momenta \((q_+, q_-, k, \cdots)\) that can be fed into a detector simulation or submitted to arbitrary phase space cuts.

- These events are produced at random, that is without correlation between the subsequent events. This is not strictly required but it makes the parallel between simulation and the real physics much closer and gives a better idea of the statistical fluctuations to be expected. The probability distribution in phase space satisfied by these events should be the one given by the theory; if necessary the events can be assigned a weight depending on the event, which corrects for any deviation from the theory's prediction in the event distribution.

- After a number of events has been generated and submitted to cuts, the program should evaluate a Monte Carlo estimate for the cross section corresponding to the generated event sample.

In the next sections, we shall study the properties of such Monte Carlo programs in some more detail.

3 Monte Carlo building

3.1 General strategy of a Monte Carlo calculation

Independently of the actual form that a Monte Carlo takes, its general computational strategy is always the same, and can be formulated in a few sentences. The problem is always the calculation of an integral, in our case of a cross section \(\sigma\) over a certain allowed phase space. Let us denote the total phase space for the particular final state under consideration by \(\mathcal{R}_t\); in general there will be non-trivial cuts that reduce this region to a smaller one \(\mathcal{R}\). The desired integral can then be written as

\[
\sigma = \int_{\mathcal{R}} \frac{d\sigma}{d\phi} (\phi) d\phi ,
\]  

(34)

where the differential cross section has of course to be predicted by some theory (in our case, the standard model) and \(\phi\) denotes a multidimensional set of phase space variables, in general \(3n - 4\) variables for \(n\) outgoing particles, plus possibly additional degrees of
freedom labelling spins, colours, and what not. In the single-bremsstrahlung case, we
would have for instance \( \phi = (k^0, \Omega^\mu, \Omega^-) \). The simplest Monte Carlo recipe for computing
this integral is as follows:

1. Compute the phase space normalization, also called the total approximate cross sec-
tion:

\[
V = \int_{\mathcal{R}_t} d\phi ;
\]

note that the integral runs over the total phase space \( \mathcal{R}_t \), not the restricted one \( \mathcal{R} \). This step has to be done analytically, but since no \( d\sigma/d\phi \) occurs that is usually
possible.

2. Use a source of random numbers to generate a sequence of phase space points \( \phi_i \),
\( i = 1, 2, \cdots, N \), that are independent and uniformly distributed over \( \mathcal{R}_t \). From any
given \( \phi_i \) one can construct the momenta of the outgoing particles. The \( \phi_i \) are therefore
called the Monte Carlo events.

3. For each \( \phi_i \), compute the event weight:

\[
w_i = \begin{cases} 
\frac{d\sigma}{d\phi}(\phi_i) & \text{if } \phi_i \text{ is inside } \mathcal{R}, \\
0 & \text{if } \phi_i \text{ is outside } \mathcal{R}.
\end{cases}
\]

4. Compute the Monte Carlo estimates of the total cross section and its associated error
as follows:

\[
\sigma_{MC} = \frac{V}{N} \sum_{i=1}^{N} w_i ,
\]

\[
\Delta \sigma_{MC} = \frac{V}{\sqrt{N(N-1)}} \left\{ \sum_{i=1}^{N} (w_i)^2 - \frac{1}{N} \left( \sum_{i=1}^{N} w_i \right)^2 \right\}^{1/2}
\]

The result of the Monte Carlo calculation can then be stated as follows:

\[
\sigma = \sigma_{MC} \pm \Delta \sigma_{MC} .
\]

The weak Law of Large Numbers tells us that in the limit \( N \to \infty \) the result \( \sigma_{MC} \)
will go to the real integral with probability one, and the Central Limit Theorem states
that for large enough finite \( N \), the expected deviations from the exact answer satisfy a
Gaussian distribution with width \( \Delta \sigma_{MC} \), that is the error quoted in Eq.(38) establishes
the confidence level of our answer according to the normal distribution. So, the real
answer will be within \( \Delta \sigma_{MC} \) from \( \sigma_{MC} \) in 68\% of the cases, within 2\( \Delta \sigma_{MC} \) in 95\% of the
cases, and so on. A much more extensive discussion of these statements can be found for
instance in [15, 16]. A few observations are useful here. In the first place, the Monte Carlo
recipe is completely independent of the dimensionality of the problem, in contrast to all
other methods of numerical integration: this makes for the usefulness of Monte Carlo in
multiparticle production. In the second place, the imposition of even very complicated
cuts is simple: we just put \( w_i = 0 \) outside the cuts. In the third place, the events \( \phi_i \) can
be interpreted as real physics-like events, that can be fed into detector simulations and so
on, however, with the additional feature that these events carry a weight, unlike physical
events which of course all have unit weight.
3.2 The weight distribution

As we have already remarked, Monte Carlo events are in general weighted, which makes them differ from ordinary events. In particular, so far we have kept open even the possibilities of weight-0 events. For many purposes (in particular when feeding events into a large, time-consuming detector simulation) one would of course like to have Monte Carlo events that can all be assigned the same weight. For other purposes, such as obtaining the total cross section, event weights pose in principle no problem as long as they are consistently taken into account throughout the calculation. There is a simple algorithm to convert a sample of weighted events into one with unweighted events, as follows:

1. Choose a value $W$ which is larger than the largest weight $w_i$ in the weighted sample. Some times $W$ is known exactly, but in many instances it has to be determined empirically by a ‘pilot run’ of the Monte Carlo. Updating $W$ as a simulation proceeds will bias the final sample to some extent towards low-weight events, and is a dangerous practice.

2. For each $\phi_i$, compare $w_i$ with $W \times \rho_i$. Here and in the following, $\rho, \rho_i, \cdots$ denotes a random number uniformly distributed between 0 and 1 (this is the idealized output of most ‘standard’ random number generators). If $w_i < W \rho_i$, reject the event from the sample. Obviously, low-weight events have a larger chance of being rejected than high-weight ones, and weight-0 events are always rejected.

3. The Monte Carlo events in the surviving sample can all be assigned weight 1. The corresponding $\phi_i$ are then of course still random and independent, but now are guaranteed to be distributed according to the probability distribution proportional to $d\sigma/d\phi$.

The quantity of interest here is of course the efficiency $\text{Eff}(w)$, that is, the size of the surviving sample as a function of the original one. Clearly this is given by $< w_i > / W$, where the average is over the original sample. This is a direct measure of the cost in computing time of insisting on an unweighted sample instead of a weighted one.

There is another important aspect to the weight distribution. It can easily be shown that the Monte Carlo error $\Delta\sigma_{MC}$ is for large $N$ given by $(\Delta\sigma_{MC})^2 = \text{Var}(w)/N$, where $\text{Var}(w)$ is the variance of the distribution of the $w_i$ for an infinitely large event sample.

The two measures $\text{Eff}(w)$ and $\text{Var}(w)$ indicate that the crucial quantity in determining the quality of a Monte Carlo calculation is the distribution of the weights. In assessing a Monte Carlo program, it is always of the essence to look at this distribution, and this ought always to be presented in the description of a simulation program. It is instructive to discuss a few histograms of hypothetical weight distributions. In fig.5 appears the perfect weight distribution. Here all Monte Carlo events have the same weight, so that $\text{Eff}(w) = 1$ and $\text{Var}(w) = 0$. It implies that in terms of the variables $\phi$ that we have chosen, the differential cross section $d\sigma/d\phi$ is constant. Of course, in such a case, the cross section was already determined analytically (by $V$) up to an overall constant, so this case is of little practical interest. The next best case is depicted in fig.6. Here the weights are not all equal, but cluster around a well-defined central value. Both $\text{Eff}(w)$ and $\text{Var}(w)$ will be reasonably small, and the result is generally trustworthy. Of course, usually cuts will be imposed (making $R < R_t$) so that a spike at $w = 0$ develops. In addition, in many cases quite low weights can be generated: this will usually make a weight distribution look like the one in fig.7. Such weight distributions can usually be accepted, provided that $\text{Eff}(w)$ and $\text{Var}(w)$ are not too unfavourable. It should be noted, however, that the values of $N$ for which the Central Limit Theorem becomes applicable depend on higher moments.
of the weight distribution than just \( \text{Var}(w) \). If the weight distribution does not show a clear peak away from zero, the Monte Carlo error \( \Delta \sigma_{\text{MC}} \) will usually be on the optimistic side, and it is good to keep this always in mind. Things start to become dangerous in the case of a weight distribution as depicted in fig.8. Here there is a long, low tail extending towards very high weights. For a short run, things might look all right but in the middle of long runs events with enormous weights will crop up irregularly, which destroys the error estimates. This is typically an indication that the cross section \( d\sigma/d\phi \) exhibits high, narrow peaks. This can only be solved by redesigning the Monte Carlo as we shall discuss below. Note that in the case of fig.8 an estimate of \( W \) will be hard to come by, and in any case \( \text{Eff}(w) \) will tend to be extremely low. Finally, a really unacceptable distribution is given in fig.9. Here a number of events occur which have negative weight. In principle the Monte Carlo approach works equally well for negative values of \( d\sigma/d\phi \) as for positive ones, but in virtually all applications cross sections are assumed to be positive. Certainly, in case one wants to convert the weighted sample into an unweighted one using the rejection algorithm described above, negative weights cannot be accounted for in any way, and the result is guaranteed to be biased. In practice, whenever negative weights occur it is almost certain that either the Monte Carlo program is outrageously badly written, or contains some bug or numerical instability (in particular bremsstrahlung amplitudes can contain
large cancellations which, unless treated with proper care, can lead to negative results). It should be stressed again that the weight distribution must be among the first things looked at when running a Monte Carlo.

### 3.3 Variance reduction

As we have seen, the Monte Carlo error is given by $\Delta \sigma_{MC} \sim \sqrt{\text{Var}(w)/\sqrt{N}}$. Improving the error can therefore be done by increasing the statistics, but obviously this quickly becomes inefficient. In most cases, when $\Delta \sigma_{MC}$ is too large, the algorithm must be redesigned so as to make $\text{Var}(w)$ smaller. In particle-physics applications this is almost invariably done by *importance sampling* which we shall now discuss. In essence this is nothing but a variable transformation, or *mapping*:

$$
\sigma = \int_{\mathcal{R}} w(\phi) d\phi \\
= \int_{\mathcal{R}} \frac{w(\phi)}{g(\phi)} g(\phi) d\phi \\
= \int_{\mathcal{R}'} \tilde{w}(\tilde{\phi}) d\tilde{\phi} \quad ,
$$

where we have introduced a new variable $\tilde{\phi}$ such that

$$
\frac{d\tilde{\phi}(\phi)}{d\phi} = g(\phi) \quad .
$$

The idea is now to generate uniformly distributed values of $\tilde{\phi}$ instead of $\phi$, of course also in the appropriately changed phase space volume $\mathcal{R}'$. This method can work well if we can ensure that (a) the function $g(\phi)$, which in principle is more or less arbitrary, 'looks like' $w(\phi)$ in that it peaks where $w(\phi)$ does, by about the same amount, and (b) is analytically integrable over its phase space $\mathcal{R}'$. Note that to construct the Monte Carlo event, we also have to be able to compute $\phi$ from $\tilde{\phi}$, and this is usually the most non-trivial part of the algorithm. Ideally, we could use $g(\phi) = w(\phi)$, and the weights $\tilde{w}$ would all become unity. Again, however, this means that we were able to integrate the cross section analytically in the first place, and would need not Monte Carlo to start with (of course, there are instances where one can in fact integrate the cross section analytically but still wants
Figure 10: Weight distributions in the Monte Carlo integration of $\exp(-x)/x$ from $x = 0.05$ to $x = 1$. (a): the weights for simple Monte Carlo. (b): the weights for importance sampling using $g(x) = 1/x$.

to have a simulation available – however, this simulation will almost always again be submitted to cuts). The function $g(\hat{\phi})$ is called the approximate differential cross section, and its normalization $\hat{V} = \int R_1 d\hat{\phi}$ is called the total approximate cross section. Keep in mind, however, that this is not an approximation in the sense that $\hat{V}$ should equal $\sigma$ to within a few per cent or so: in many Monte Carlo programs one is happy if the peaks in the cross section can be approximated to within an order of magnitude, as long as they are in the same place. Using the $\hat{V}$ given by a particular Monte Carlo as a guesstimate for the real answer is a bad practice.

Let us now see how all this works in a simple one-dimensional example. Suppose we want to integrate the function $w(x) = \exp(-x)/x$ between $x = 0.05$ and $x = 1$. The exact integral, 2.24851..., can of course be very accurately computed with non-Monte Carlo methods, but here we want to study aspects of the Monte Carlo approach. Let us start by simple Monte Carlo. This means generating uniform $x$ values between $0.05$ and $1$, for instance by $x_i = 0.05 + 0.95 \rho_i$, and computing the values $w(x_i)$. The resulting weight distribution for $N = 500$ is given in fig.10a. The weights cluster around zero, but with a long low tail extending out to the maximum of 19.025. The result of this simulation is $2.288 \pm 0.135$, and $\text{Eff}(w) = 0.127$. Now let us apply importance sampling. The main feature of the integrand is of course the $1/x$-like behaviour, and therefore we choose $g(x) = 1/x$. This means that the new variable, $\hat{x}$, will run from 0 to $-\log(0.05) = 2.996$, and $x$ and $\hat{x}$ are related by $x = \exp(-\hat{x})$. Sampling $\hat{x}$ uniformly instead of $x$, and using as a weight $\hat{w} = \exp(-\exp(-\hat{x}))$, we obtain the weight distribution given in fig.10b. The maximum weight is now smaller, 0.951, but more importantly the distribution peaks now around the upper end. Consequently, we have a much improved Monte Carlo estimate $2.259 \pm 0.0578$ for the result, and a better $\text{Eff}(w) = 0.793$.

It is clear that in order to apply importance sampling one needs to have a better understanding of the structure of the integrand than when using simple Monte Carlo. Another remark in order here is that in the above simple case we were able to produce $x$ values by a simple transformation from the $\hat{x}$ values. In other cases, it may not be possible to find such a simple mapping, but in general any algorithm or trick that produces $x$ values distributed according to $g(x)$ is acceptable as long as the integral $\int g(x)dx$ can be
computed. An extraordinarily complete discussion of algorithms to generate non-uniformly distributed random sequences for many different \( g(x) \) can be found in [17].

There exist a number of other methods of variance reduction. One interesting possibility is to use not the usual random numbers \( \rho \) but instead rely on so-called 'superuniform' random number sources, which have the property that the statistical fluctuations they generate are smaller than those expected for truly random ones. In such cases, Monte Carlo estimates can in principle converge as fast as about \( 1/N \) instead of \( 1/\sqrt{N} \). However, these numbers tend to have a much higher correlation between themselves than random ones, and so may not be suited for some purposes. Other possibilities, such as stratified sampling, where the integration region is split up into a number of smaller ones in each of which a predetermined number of Monte Carlo events is generated, or the use of antithetic variates, where a peak in the cross section is compensated by choosing for each event in a peaking region a number of events in regions with low cross section, can in principle be used, but one always has to ensure that the artificially enhanced correlations between the Monte Carlo events do not have unwanted effects. Lastly, in particular in lattice Monte Carlo one invariably uses algorithms based on the Metropolis algorithm where the Monte Carlo events are parts of a random walk in phase space. In the limit of infinite \( N \) such algorithms will indeed generate the correct behaviour, but again they are not well suited for particle physics because the individual points tend to be highly correlated, and moreover such algorithms can only give moments of the generated distribution, not the overall normalization.

### 3.4 A simple example of event generation

Let us now see how a simple Monte Carlo event is generated in practice. For this, let us choose a Monte Carlo that generates single initial-state bremsstrahlung in the soft-photon approximation, with photon energy between \( \Delta E \) and \( E_b \). In this case the differential cross section to be integrated is given by

\[
\frac{d\sigma}{d\phi} = \left( \frac{d\sigma^0}{d\Omega} \right) \frac{\alpha}{4\pi^2 k^0} \left( \frac{1}{1 - \beta^2 c^2_\gamma} - \frac{m_e^2/s}{(1 - \beta c_\gamma)^2} - \frac{m_e^2/s}{(1 + \beta c_\gamma)^2} \right),
\]

while the integration element is

\[
d\phi = dk^0 dc_\gamma d\phi_\gamma d\cos \theta_\mu d\phi_\mu.
\]

We choose for the approximant function

\[
g(\phi) = \left( \frac{\sigma^0}{4\pi} \right) \frac{\alpha}{\pi^2 k^0} \frac{1}{1 - \beta^2 c^2_\gamma},
\]

which is constant in each of the variables \( \phi_\gamma, \cos \theta_\mu \) and \( \phi_\mu \), so that the approximate total cross section reads (in the approximation \( \beta \sim 1 \) where possible):

\[
\hat{V} = \sigma^0 \frac{2\alpha}{\pi} \log \left( \frac{s}{m_e^2} \log \left( \frac{E_b}{\Delta E} \right) \right).
\]

The algorithm by which the various variables are generated is then for instance given by

\[
k^0 = \Delta E \left( \frac{E_b}{\Delta E} \right)^{\mu_1}.
\]
\[ c_\gamma = \frac{1}{\beta} \left\{ 1 - \frac{2(1 - \beta)}{(1 + \beta) \exp(2\rho_2 - 1) + (1 - \beta)} \right\} , \]
\[ \phi_\gamma = 2\pi \rho_3 , \]
\[ \cos \theta_\mu = 2\rho_4 - 1 , \]
\[ \phi_\mu = 2\pi \rho_5 . \] (45)

From the generated variables, the Monte Carlo event can easily be reconstructed: \( q^0_+ \) follows from Eq.(12), and \( q^\mu_+ \) is then given by momentum conservation. For each generated Monte Carlo event the weight is then given by
\[ \tilde{w} = \left( \frac{4\pi}{\sigma^0} d\Omega_\mu \right) \left[ 1 - 2\frac{m_e^2(1 + \beta^2 c_i^2)}{s(1 - \beta^2 c_i^2)} \right] . \] (46)

It is seen here that the weight no longer depends on the photon energy, and that no collinear peaks remain: all these peaks have been accounted for in the approximant \( g(\phi) \). In more realistic cases, the peaks are usually not approximated this well, and the weight is a considerably more complicated function: however, when all is said and done, it is only the weight distribution that determines the quality of the approximants and the event-generating algorithms. For multiple-photon bremsstrahlung, final-state radiation and so on similar algorithms have been developed, while the interference between initial- and final-state radiation is always taken into account by incorporating it into the event weight. For the technical details one must consult the abundant literature on this subject.

3.5 Random number sources

So far we have freely been using numbers \( \rho, \rho_1 \) that are random and uniformly distributed between 0 and 1. These are usually provided for in a given computer environment: but the algorithms to obtain them are far from trivial, and we shall spend some time discussing them. In principle, the concept of a series of such random numbers is intuitively clear, but it is essentially defined by the condition that there is no finite algorithm to produce such a series: by their very definition, such random number sequences cannot be produced by a digital computer! In practice one settles for a pseudo-random series: this is a sequence that is produced by a finite (even very simple) algorithm, but the non-randomness is not supposed to become apparent unless extremely many (of the order of at least \( 10^9 \)) of these numbers have been generated. An extremely informative source on such algorithms is [18]. Here we only mention some well-known pseudo-random number algorithms. They almost invariably determine the next pseudo-random number on the basis of one or more of the preceding ones: if this is done cleverly enough, realistic randomness properties can be obtained.

- **Multiplicative generators.** These are of the form
\[ \rho_k = n_k/m , \quad n_k = (an_{k-1} + c) \mod m . \] (47)

where \( n_k, a, c \) and \( m \) are integers: \( m \) is usually the word length of the machine, \( 2^{32} \) or so. The value of \( c \) can be zero: the choice of \( a \) is crucial. A well-known choice is \( a = 65539 \) which has turned out to be extremely bad: nowadays \( a = 69069 \) is considered much better. I refer to [18] for an explanation and further discussion.

- **Richtmeyer generators.** These are in fact 'superuniform' generators: they have the form
\[ \rho_k = (k\theta) \mod 1 . \] (48)
where \( \theta \) is an irrational number like the root of a prime number, or its best approximation on a finite-wordlength computer. In generators like this, the fact that the numbers (while uniformly distributed) come out strongly correlated makes it necessary to scramble them, that is, one generates numbers into a buffer and then uses a simple generator like a multiplicative one to shuffle the order inside the buffer. In practice the superuniformity appears only for high statistics and one can use them as normal pseudo-random numbers.

- **Additive generators.** These use more than just the current number to get the next one: typically, one has

\[
\rho_k = (\rho_{k-p} \pm \rho_{k-q}) \text{mod} 1 ;
\]  

(49)

provided \( p \) and \( q \) are chosen judiciously, surprisingly good random behaviour can be obtained for these sequences. This algorithm is incorporated in the **RANMAR** sequence advocated by James in [19], and it ought to become the standard for modern high-energy applications.

Clearly, the pseudo-random-number algorithms themselves are not very complicated: in fact it can be shown that a very complicated algorithm will almost certainly yield a very badly-behaved sequence [18]. A lot of research is going on in this field as well, and we can expect many interesting developments. From the practical point of view, it appears that a good Monte Carlo program should not be very sensitive to the quality of the random number algorithm used. It is good practice to switch between various alternative random number sources every now and then, to judge any such dependence; however, when the results depend sensitively on the random number source, it is the Monte Carlo program that should be improved, not the source.

4 Outline of event generator programs

In this section I shall describe how a simple, one-loop-valid Monte Carlo program for the process \( e^+e^- \rightarrow \mu^+\mu^- \) is generally structured, and discuss how it can be extended in principle to account for higher order effects. Any Monte Carlo run consists essentially of three stages: initialization, generation, and evaluation.

4.1 Initialization stage

The first thing that has to be done is to supply the program with the physical parameters of the theory. These have of course been discussed in great detail in part I of these lectures, by W. Hollik. For our purposes it is enough to notice that, given a sufficient set of input parameters \( (M_Z, M_t, M_H, \alpha_s, \text{the number of neutrinos} \ldots) \), the program should compute any remaining ones such as \( \sin^2 \theta_w, M_W, \Gamma_Z \), and so on. Of course also the total invariant mass \( \sqrt{s} \) and the cut-off between soft and hard photons, \( \Delta E \), have to be given at this point. In some programs, the differential cross section \( d\sigma/d\Omega \) is computed and tabulated in some form, which makes its evaluation in the generation step quite fast. This will allow the program to initialize quite complicated differential cross sections. Note, however, that for this to work it is crucial that the cross section is essentially one-dimensional (tabulating a multidimensional cross section would consume much more time than a reasonable generation run itself). A last and important point is that also the approximate cross sections \( \sigma_0^{\text{app}} \) and \( \sigma_1^{\text{app}} \) have to be computed. Here \( \sigma_0^{\text{app}} \) is the total approximate cross section for the Born result together with the virtual corrections and soft bremsstrahlung up to the predetermined limit \( \Delta E \), and \( \sigma_1^{\text{app}} \) is the
approximate total cross section for single bremsstrahlung harder than \( \Delta E \). One can then define \( P_1 = \frac{\sigma_1^{\text{app}}}{\sigma_0^{\text{app}} + \sigma_1^{\text{app}}} \), which will be used in the next stage.

### 4.2 Generation stage

This is where most of the Monte Carlo work is actually performed. This part of the program typically consists of a subroutine, which upon every call to it delivers one Monte Carlo together with a weight. A call will cause the program to do something like this:

A Decide whether to generate a hard bremsstrahlung event or a ‘soft’ event, according to the appropriate probabilities. In other words, a random number \( \rho \) is compared to \( P_1 \), and a ‘hard’ event is decided upon if \( \rho < P_1 \) (go to B1), otherwise a ‘soft’ event is chosen (go to B0).

B0 Generate a set of non-bremsstrahlung variables. In this case that would only be the muon scattering angle \( \Omega_\mu \). Construct the momenta \( q_\mu^+ \) and \( q_\mu^- \), and put \( k^\mu = 0 \). Usually this generation is very fast, since only \( \cos \theta_\mu \) has a non-trivial behaviour. Computing the weight in this case is also quite simple if \( \frac{d\sigma}{d\Omega_\mu} \) has been tabulated. In some cases, this tabulation may have been used to obtain the exact ‘soft’ cross section rather than the approximate one, and the angle \( \theta_\mu \) follows the exact distribution. In such cases the weight is just unity. Now go to C.

B1 Generate a set of bremsstrahlung variables, according to the approximation chosen. Typically, these are \( k^0, \Omega_+ \), and some other angular variable such as \( \Omega_\mu \). From these, construct the complete set of momenta, in this case \( q_+^\mu, q_-^\mu \), and \( k^\mu \). This is a Monte Carlo event. Compute its weight as the ratio of the exact bremsstrahlung cross section over the approximate one.

C Do some bookkeeping on the event weights, so that \( \sum \omega, \sum \omega_l \), and so on are accumulated during the generation. If weighted events are acceptable, output the Monte Carlo event as-is, together with its weight. If unweighted events are desired, compare the weight with a random number using the rejection algorithm described earlier. If the Monte Carlo event is then accepted, output it with unit weight; if not, start again at A.

Repeating this algorithm a number of times, a sample of Monte Carlo events with weights is generated. These can then be submitted to cuts and analysis, just like the real experimental events.

### 4.3 Evaluation stage

Once a sufficiency of Monte Carlo events has been obtained, the total generated cross section can be computed, using the formulae of Eq.(37). Note that in that case the total approximate cross section \( V \) is given by \( V = \sigma_0^{\text{app}} + \sigma_1^{\text{app}} \). As I have stressed before, it is always a good idea to have some more information on the weight distribution than just its mean and variance, so a histogram of the weights should preferably be included as well, together with any additional useful information.

### 4.4 Towards multiple photon simulation

Let us now turn to the inclusion of multiple-photon effects. In the first place, in step B1 we always end up with hard bremsstrahlung photons with \( k^0 > \Delta E \), while in the alternative step B2 the photon energy is strictly zero. A close-up of the photon energy spectrum as generated by the Monte Carlo will therefore show a ‘gap’ between zero and \( \Delta E \), while of course the experimental events will not display such a gap – it is an artifact.
of the Monte Carlo treatment. As we have seen in the above, $\Delta E$ may actually be large enough to ‘be seen’ experimentally, given a good enough detector resolution. As we have seen, making $\Delta E$ smaller will result in negative event weights, and is hence unacceptable as a way of improving the Monte Carlo simulation. Naively one might think that a smaller $\Delta E$ would be possible if second order effects were included. That, however, turns out to be wrong as a quick look at Eq. (21) and the expression for $d\sigma N_{s_{1} s_{2}}/d\Omega_{\mu}$ in Eq. (22) will make clear: in fact, in any finite order of perturbation theory, the contributions with an odd number of soft/virtual photons will lead to negative cross sections, for about the same value of $\Delta E$ as in the one-loop case. So, inclusion of any finite amount of higher orders does not solve the problem of the photon spectrum gap (which in the jargon is called the ‘$k_0$ problem’, from $k_0 = \Delta E/E_0$). Instead, one has to exponentiate and go to fully infinite order – of course, in some approximation. A small number of ‘exponentiated’ Monte Carlo programs have been written so far. The simplest are just one-loop ones, with a photon spectrum that is (by hand) modified so that it looks like the exponentiated one: essentially this means that the spectrum goes like $\beta(k^0)^{\beta-1}$ instead of just $\beta(k^0)^{-1}$. This is obviously a crude way of approaching the full richness of multiple-photon kinematics. More rigorous approaches in fact extend the above program structure by having in addition double, triple, ⋯ bremsstrahlung and multiple-soft photon effects. One then has $\sigma_{2}^{\text{app}}$, $\sigma_{3}^{\text{app}}$, ⋯ in addition to $\sigma_{0}^{\text{app}}$ and $\sigma_{1}^{\text{app}}$, and a number of $P_n$’s given by

$$P_n = \frac{\sum_{i=0}^{n} \sigma_i^{\text{app}}}{\sum_{i=0}^{\infty} \sigma_i^{\text{app}}}.$$  \hspace{1cm} (50)

In the approximation in which all photons behave according to the soft-photon approximation, the $P_n$ are just the probabilities corresponding to a Poisson distribution with a mean equal to $\beta \log(E_0/\Delta E)$. The decision in the generation step is then not just between 0 or 1 photon, but a choice of any given number of bremsstrahlung photons, as follows: one picks a random number $\rho$ and determines $n$ such that $P_n < \rho < P_{n+1}$; then in a step (Bn), an $n$-tuple bremsstrahlung events must be generated. Since now all sums of (infinitely many) soft-photon loop effects always combine so as to give a factor $(\Delta E/E_0)^{\beta}$ in front of each $n$-photon bremsstrahlung cross section, $\Delta E$ can be put as low as one wishes without jeopardizing the positivity of the result. The only effect of changing $\Delta E$ is a change in the average number of photons in each event, which is always of the order of $\beta \log(E_0/\Delta E)$: the smaller $\Delta E$ is, the ‘finer’ is one’s resolution, and hence the more soft photons one expects to see. The number of photons above a given fixed energy, however, must go to a constant as $\Delta E \to 0$. In practice, $\Delta E \sim 10^{-6}$ is good for all practical purposes. A program setup like this is of course quite complicated: in particular one single code must be prepared to handle an arbitrarily large number of momenta (although of course in practice Monte Carlo events with more than 5 or 6 photons are extremely rare). The best work in this direction so far has been that of S. Jadach, B. Ward and collaborators [20]. Much more information on the various ways to include higher-order QED effects can be found in [21].

4.5 Outlook

In these lectures I have tried to give an elementary introduction to the field of QED effects and their simulation using Monte Carlo event generators. In the course of this and next year, such software will have to prove its usefulness. Of course a number of comparisons between various programs have already been performed: I refer to [21]
where many more details can be found than I could usefully quote here. Some qualitative statements can be made, though. If we consider fermion pair production ($\mu^+\mu^-$, $\tau^+\tau^-$, quark pairs, ...) things look reasonably under control as far as the roughest, most inclusive quantities are involved: these are the line shape, and the forward-backward asymmetry. The line shape can be simulated to within about 0.3% of the semianalytical predictions of [10], with the exception of purely one-loop Monte Carlo programs, which fail miserably! Here of all places it is of the essence that exponentiation in one form or another is taken into account. The asymmetry is obtained slightly less well, but probably the results will be acceptable for the near future. If more cuts are imposed, the quantity one is looking at becomes more and more exclusive, and also the agreement between the various programs becomes worse. Another complication there is that in general no good analytical results exist for cross sections with non-trivial cuts, so that in the case of a discrepancy between programs it is hard to say which one is 'more correct'. A glaring lack of software is present in the case of Bhabha scattering! In the first place, there is as yet no good exponentiated Monte Carlo for large-angle Bhabha scattering, which means that for once one is going to measure the cross section more accurately than it can be predicted. Another problem is the lack of any reliable 'line-shape' calculation in which the scattering angles of the $e^+$ and $e^-$ can be restricted: the work of Caffo et al. in [22] is only a small first step in this direction. For small-angle scattering, used for luminosity monitoring, the situation is a little better in the sense that higher-order effects are argued to be smaller [21], but again no satisfactory higher-order Monte Carlo exists there to verify this statement.

In conclusion: notwithstanding the progress that has undoubtedly been made over the last few years, especially due to the existence of good semianalytical calculations which, although not themselves directly applicable to experiment, can nevertheless serve as an indispensable testing ground for event generators, a lot of effort will still be needed before we can claim that electroweak processes, and in particular multiple-photon effects, can be simulated to the accuracy needed to satisfy all today's and tomorrow's requirements in LEP physics.

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BEYOND THE STANDARD MODEL

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IV References
I Introduction

I would like to invite you to go beyond the Standard Model of the three fundamental interactions of matter. How far we go depends on your patience and intellectual curiosity. We do not know what is waiting us there in terra incognita. We hope to find it out with the help of future experiments. However modern theories give us several guiding ideas, some of which are very exiting. In the present lectures we will discuss some of these ideas looking more attractive today.

1 The Standard Model: Achievements and Drawbacks

The Standard Model (SM) describes the strong, weak and electromagnetic interactions and is based on a gauge principle. According to this principle all the forces of nature are mediated by an exchange of the gauge fields of the corresponding gauge group. The group of the SM is

\[ SU_{\text{colour}}(3) \otimes SU_{\text{left}}(2) \otimes U_{\text{hypercharge}}(1) \]  \hspace{1cm} (1.1)

whereas the field content is the following:

Gauge sector

The gauge bosons are spin 1 vector particles belonging to the adjoint representation of the group (1.1). Their quantum numbers under \( SU(3) \otimes SU(2) \otimes U(1) \) are respectively:

- gluons \( G^a_\mu : (8, 1, 0) \) \( SU_{\text{colour}}(3) \) \( g_a \)
- intermediate \( A^i_\mu : (1, 3, 0) \) \( SU_{\text{left}}(2) \) \( g \)
- weak bosons \( B^i_\mu : (1, 1, 0) \) \( U_Y(1) \) \( g' \)

The coupling constants are usually denoted by \( g_a, g \) and \( g' \) respectively.

Fermion sector

The matter fields are fermions belonging to the fundamental representation of the gauge group. These are believed to be quarks and leptons of at least of three generations. The SM is left-right asymmetric. Left-handed and right-handed fermions have different quantum numbers:

- quarks
  \[ Q^i_\alpha_L = \left( \begin{array}{c} U^i_\alpha \\ D^i_\alpha \end{array} \right)_L = \left( \begin{array}{c} u^i \\ d^i \\ e^i \\ \tau^i \end{array} \right)_L, \hspace{1cm} \left( \begin{array}{c} c^i \\ s^i \\ \mu^i \\ \tau^i \end{array} \right)_L, \hspace{1cm} \left( \begin{array}{c} t^i \\ b^i \end{array} \right)_L, \hspace{1cm} \ldots \] \hspace{1cm} \( (3, 2, 1/3) \)

- leptons
  \[ L_\alpha_L = \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right)_L, \hspace{1cm} \left( \begin{array}{c} e \\ \mu \\ \tau \end{array} \right)_L, \hspace{1cm} \ldots \] \hspace{1cm} \( (1, 2, -1) \)

\( i = 1, 2, 3 \) - colour, \( \alpha = 1, 2, 3, \ldots \) - generation.
Higgs sector

In the minimal version of the SM there is only one doublet of Higgs scalar fields

\[
H = \begin{pmatrix} H^+ \\ H_0 \end{pmatrix} \quad (1, 2, -1),
\]

which is introduced in order to give masses to quarks, leptons and intermediate weak bosons via spontaneous symmetry breaking.

The Lagrangian of the SM is

\[
\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}},
\]

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} G_{\mu \nu}^a G^{a \mu \nu} - \frac{1}{4} A_{\mu \nu}^i A^{i \mu \nu} - \frac{1}{4} B_{\mu \nu} B^{\mu \nu} + i \bar{L}_\alpha \gamma^\mu D_\mu L_\alpha + i \bar{Q}_\alpha \gamma^\mu D_\mu Q_\alpha + i \bar{l}_\alpha \gamma^\mu D_\mu l_\alpha + i \bar{U}_\alpha \gamma^\mu D_\mu U_\alpha + i \bar{D}_\alpha \gamma^\mu D_\mu D_\alpha + (D_\mu H)^\dagger (D_\mu H),
\]

where

\[
G_{\mu \nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c,
\]

\[
A_{\mu \nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g e^{ijk} A_\mu^j A_\nu^k,
\]

\[
B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
\]

\[
D_\mu L_\alpha = (\partial_\mu - ig \frac{\tau^i}{2} A_\mu^i + i g' \frac{\tau^3}{2} B_\mu) L_\alpha,
\]

\[
D_\mu l_\alpha = (\partial_\mu + ig' B_\mu) l_\alpha,
\]

\[
D_\mu Q_\alpha = (\partial_\mu - ig \frac{\tau^i}{2} A_\mu^i - i g' \frac{\tau^3}{2} B_\mu - ig_s \frac{\lambda^a}{2} G_\mu^a) Q_\alpha,
\]

\[
D_\mu U_\alpha = (\partial_\mu - ig' B_\mu - ig_s \frac{\lambda^a}{2} G_\mu^a) U_\alpha,
\]

\[
D_\mu D_\alpha = (\partial_\mu + ig' B_\mu - ig_s \frac{\lambda^a}{2} G_\mu^a) D_\alpha.
\]

\[
\mathcal{L}_{\text{Yukawa}} = f_{\alpha \beta}^i \bar{L}_\alpha l_\beta H + f_{\alpha \beta}^a \bar{Q}_\alpha D_\beta H + f_{\alpha \beta}^v \bar{U}_\alpha U_\beta \hat{H} + \text{h.c.},
\]

where \( \hat{H} = i \tau_2 H^\dagger \).

\[
\mathcal{L}_{\text{Higgs}} = -V = m^2 H^\dagger H - \lambda (H^\dagger H)^2.
\]

Here \( \{f\} \) are the Yukawa and \( \lambda \) is the Higgs coupling constants respectively both dimensionless and \( m \) is the only dimensional mass parameter.

The Lagrangian of the SM contains the following set of free parameters:

- 3 gauge couplings,
- Yukawa couplings,
- Higgs coupling,
- mass parameter,
- quark mixing matrix,
• number of matter fields (generations).

All the particles obtain their masses due to spontaneous symmetry breaking of $SU_{l/e_f}(2)$ group via a non-zero vacuum expectation value of the Higgs field

$$< H >= 1/\sqrt{2} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v = m/\sqrt{\lambda}.$$  

As a result the gauge group of the SM is spontaneously broken down to

$$SU_e(3) \otimes SU_L(2) \otimes U_{Y}(1) \Rightarrow SU_e(3) \otimes U_{EM}(1)$$

The weak bosons now are

$$W^\pm_\mu = \frac{A^1_\mu \pm i A^2_\mu}{\sqrt{2}}, \quad Z_\mu = \sin \theta_W B_\mu - \cos \theta_W A^3_\mu$$

with masses

$$m_W = 1/2gv, \quad m_Z = m_W/\cos \theta_W, \quad \tan \theta_W = g'/g,$$

while the photon field

$$\gamma_\mu = \cos \theta_W B_\mu + \sin \theta_W A^3_\mu$$

remains massless.

The matter fields acquire masses proportional to the corresponding Yukawa couplings:

$$M^u_{\alpha\beta} = f^u_{\alpha\beta} v/\sqrt{2}, \quad M^d_{\alpha\beta} = f^d_{\alpha\beta} v/\sqrt{2}, \quad M^l_{\alpha\beta} = f^l_{\alpha\beta} v/\sqrt{2}, \quad m_H = \sqrt{2}m.$$  

Explicit mass terms in the Lagrangian are forbidden because they are not $SU_{l/e_f}(2)$ symmetrical and would destroy the renormalizability of the Standard model.

The SM has been constructed as a result of numerous efforts both theoretical and experimental. Achievements of the SM are obvious. At present the SM is extraordinary successful, there is no any experiment that contradicts the SM. Moreover there is nothing observed beyond the SM.

However the SM has its natural drawbacks. Among them are:

• large number of free parameters,
• formal unification of strong and electroweak interactions,
• the Higgs particles have not yet been observed and it is not clear whether they are fundamental or composite,
• the problem of CP-violation is not well understood including CP-violation in strong interaction,
• One of the main problems of the SM is the origin of the mass spectrum.

There are also some problems of more fundamental nature:

• why the gauge group is $SU(3) \otimes SU(2) \otimes U(1)$?
• what is the number of generations?
• how to incorporate gravity in unified theory? etc.

The answer to these problems definitely lies beyond the SM.
II Just Beyond the Standard Model

2 Compositeness and Technicolour

One of the main problems of the SM still not truly understood is the symmetry breaking mechanism. Unbroken symmetry means that all fundamental particles are massless. This is because both the fermion mass term $f_L \cdot f_R$ and that of the gauge bosons $W^2_\mu$ and $Z^2_\mu$ are not $SU_{L}\cdot f_{L}(2)$ invariant.

If the symmetry is broken on some scale, these particles can acquire masses $\sim$ the breaking scale. Its typical value is $\sim 250$ GeV (v.e.v. of the Higgs field).

2.1 Symmetry Breaking Without Higgs Fields (A Simple Model)

If the Higgs particles will not be found, we will be faced with the problem of replacing the Higgs mechanism of spontaneous symmetry breaking by some alternative. Such an alternative already exists. This is the so called dynamical symmetry breaking like the chiral symmetry breaking in QCD.

To understand how this mechanism works, we consider QCD with $u$ and $d$ quarks. If they are massless, then the QCD Lagrangian is invariant under chiral group $SU_{L}(2) \otimes SU_{R}(2)$. However quarks can create a vacuum condensate

$$< \bar{u} u + \bar{d} d > \neq 0,$$

which breaks the chiral group down to that of the Isospin $SU_{I}(2)$. The symmetry breaking is spontaneous as far as Lagrangian is still invariant but this is no longer true for the vacuum state. Hence due to the Goldstone theorem one should get massless spin 0 goldstone bosons. Indeed they exist. These are pions $\pi$. (Later they obtain small masses becoming the pseudo-goldstone bosons. However the smallness of pion mass is protected by chiral invariance.)

Due to a non-zero matrix element of the axial isospin current

$$< 0 \mid J_{5a}^{\mu} \mid \pi_b (q^2) > = f_{\pi} q^{\mu} \delta_{ab},$$  \hspace{1cm} (2.1)$$

where $q^{\mu}$ is the momentum of the pion and $f_{\pi} \approx 93 Mev$ is the pion decay constant, the pions appear to be the longitudinal components of massive vector bosons $W$ and $Z$. To see this, consider the propagator of $W$. Because of radiative corrections the total expression has the form

$$\frac{g_{\mu\nu} - q_{\mu} q_{\nu}}{q^2} \rightarrow \frac{g_{\mu\nu} - q_{\mu} q_{\nu}}{q^2(1 + \Pi(q^2))},$$

where $\Pi(q^2)$ is the polarization operator.

To obtain a non-zero mass of $W$ we need a pole term in $\Pi(q^2)$ at $q^2 = 0$. The contribution to $\Pi(q^2)$ due to the interaction with the axial current $\frac{1}{2} W_{\mu}^{\pm} J_{5}^{\mu}$ is given by

$$(g_{\mu\nu} - q_{\mu} q_{\nu}) \Pi(q^2) = \frac{g^2}{4} \int dx e^{i q x} < 0 \mid T J_{5 \mu}^{\pm}(x) J_{5 \nu}^{\pm}(0) \mid 0 > .$$  \hspace{1cm} (2.2)$$

The pole term we are interested in comes from the pion intermediate state (see Fig.1) According to eqs.(2.1,2.2) we get

$$\Pi(q^2) = -\frac{g^2 f_{\pi}}{4 q^2},$$

\[133\]
which leads to a shift of the pole thus giving the desired mass

\[ m_W = \frac{1}{2} g f_\pi. \]

The same mechanism works for the neutral bosons. However one should take into account the mixing between \( B_\mu \) and \( A_\mu^3 \). The mass matrix here has the form

\[ m^2 = \begin{pmatrix} g^2 & g' g \\ g' g & g'^2 \end{pmatrix} \frac{f_\pi}{4}. \]

After diagonalization this leads to

\[ m_W^2 = 0 \]
\[ m_Z^2 = \frac{1}{2} (g^2 + g'^2) f_\pi^2 \]

with a standard ratio

\[ \frac{m_W}{m_Z} = \cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}}. \]

Thus we get what we wanted but...the masses of \( W \) and \( Z \) are of the order of 100 Mev rather than 100 Gev, i.e. 10^3 times smaller than they are actually are.

Hence, the conclusion is the following: the mechanism works, however the scales are wrong. In order to obtain appropriate masses, one needs some new kind of interaction with a different scale. This is just the idea of technicolour.

### 2.2 Technicolour

The technicolour idea is the following: one introduces new strong interactions called Quantum Technicolour Dynamics (QTD) and a new set of fields called technifermions. In a simplest case one has a doublet of technifermions \( T = \begin{pmatrix} A \\ B \end{pmatrix} \). QTD is constructed in full analogy with QCD. The only difference is the scale of interaction

\[ \Lambda_{TC} \rightarrow \frac{F_\pi}{F_\pi} \sim 250 \text{ Gev} \]
\[ \Lambda_{QCD} \sim \frac{F_\pi}{F_\pi} \sim 2600 \]

Hence if \( \Lambda_{QCD} \sim 200 \text{ Mev}, \Lambda_{TC} \sim 500 \text{ Gev} \). This gives \( m_{W,Z} \sim 100 \text{ Gev} \) as it should be.

Therefore we come to the following picture:

The Higgs fields are replaced by the strong interacting technifermions. Instead of a scalar field with a contact self-interaction we introduce a set of fermion fields interacting
through a new gauge force. Instead of a condensate of scalar fields we have a condensate of technifermion pairs which spontaneously breaks the symmetry thus giving masses to the gauge bosons. The appeared goldstone bosons (the technipions) are absorbed by $W$ and $Z$ becoming their longitudinal components exactly like it happens with fundamental scalars in the Higgs mechanism.

The described picture is slightly modified because of the presence of both QCD and QTD, i.e. of ordinary pions as well as technipions. As a result we have two orthogonal combinations

$$|\text{pion absorbed} > = \frac{F_\pi|\text{technipion} > + f_\pi|\text{pion} >}{\sqrt{F_\pi^2 + f_\pi^2}}$$

and

$$|\text{pion physical} > = \frac{F_\pi|\text{pion} > - f_\pi|\text{technipion} >}{\sqrt{F_\pi^2 + f_\pi^2}},$$

with the matrix elements of the full axial current being

$$< 0|J_\mu^S|\text{pion} > = f_\pi q_\mu$$
$$< 0|J_\mu^S|\text{technipion} > = F_\pi q_\mu$$

$$< 0|J_\mu^5|\text{pion absorbed} > = \sqrt{F_\pi^2 + f_\pi^2} q_\mu$$
$$< 0|J_\mu^5|\text{pion physical} > = 0$$

Due to a relative smallness of $f_\pi$ the physical pion is mostly the ordinary one. However its decay modes now are described in a more complicated non-contact way.

2.3 Technispectroscopy

In analogy with QCD one can expect that QTD will predict a spectrum of bound states of techniquarks. It should be very much like the QCD spectrum but rescaled by a factor of $\Lambda_{TC}/\Lambda_{QCD} \sim 2600$. There should exist techni-$\rho$, techni-$\omega$, etc mesons with masses of the order of 1 Tev and with spacings and widths also $\sim 10^3$ larger than in QCD.

These techni-resonances will manifest themselves in $e^+e^-$ annihilation process as the new resonance peaks in the cross-section ratio (see Fig.2) The main difference from the SM here is that the longitudinal parts of $W$ and $Z$ bosons become strongly interacting
particles. In $e^+e^-$ annihilation into a pair of $W^+W^-$ this means that the transverse parts of $W$s will behave like in the SM while the longitudinal ones will produce resonances, which is a specific prediction of technicolour.

Another manifestation of technicolour is high $p_T$ jets in $e^+e^-$ or $p\bar{p}$ collisions. The $p_T$ will be also rescaled by a factor of $10^3$ with respect to QCD.

The spectrum of technistates appears to be strongly model-dependent. It depends on the group of technicolour, representation, etc. For the simplest one-doublet model

$$T = \begin{pmatrix} A \\ B \end{pmatrix}_L, \quad A_R, \quad B_R$$

the spectrum is trivial and contains no (pseudo)goldstone bosons. However already in the one-family model, which was constructed in analogy with the family content of the SM in order to suppress FCNC by GIM mechanism, the situation is more complicated. In this case the techniquarks and technileptons are organised like quarks and leptons in the SM

$$\begin{pmatrix} U^\alpha_c \\ D^\alpha_c \end{pmatrix}_L \begin{pmatrix} U^\alpha_R \\ D^\alpha_R \end{pmatrix} \begin{pmatrix} N^\alpha \\ E^\alpha \end{pmatrix}_L \begin{pmatrix} N^\alpha_R \\ E^\alpha_R \end{pmatrix} \alpha = 1, 2, 3, \ldots - \text{technicolour} \quad c = 1, 2, 3 - \text{colour}$$

The symmetry breaking now is achieved by the condensate of all techniparticles (for simplicity we take them to be equal)

$$<\bar{U}U> = <\bar{D}D> = <\bar{N}N> = <\bar{E}E> \neq 0.$$

In this case

$$\Lambda_{TC} \sim 1300 \sqrt{\frac{3}{N}} \text{GeV} \quad \text{for } SU(N) \quad \text{TC group}$$

This model predicts a rich spectrum of technihadrons with masses $\sim 1$ Tev having an integer (technimesons) or a half-integer (technibaryons) spin. The latter is possible for $N = 2k + 1$.

The lightest states of spin 0 may have masses $\leq 1$ Tev or even $\sim 100$ Gev. These are the pseudo-goldstone bosons with unusual production and decay properties. To find them out, let us first remind QCD. There the $SU_L(2) \otimes SU_R(2)$ chiral symmetry is broken down to $SU_{Isospin}(2)$ by the quark condensate $<\bar{u}u + \bar{d}d> \neq 0$ thus producing a triplet of goldstone bosons - pions $\pi$. They become the pseudo-goldstones acquiring small masses $m_\pi^2 \sim m_0^2$ due to quark mass terms in the QCD Lagrangian.

For the TC one-family model the initial chiral group is $SU_L(8) \otimes SU_R(8)$ which is broken down to $SU_{vector}(8)$. Hence the number of goldstone bosons equals: $2 \times 63 - 63 = 63$, i.e. the number of broken generators. These goldstone bosons become pseudo-goldstones due to $SU(3) \otimes SU(2) \otimes U(1)$ corrections. They can be constructed out of techniquark and technilepton doublets

$$Q^c = \begin{pmatrix} U \\ D \end{pmatrix}^c, \quad L = \begin{pmatrix} N \\ E \end{pmatrix}$$

and can be colour as well as technicolour singlets or multiplets:

$$\Theta^i_a \sim \bar{Q}\gamma^5 \lambda^a \tau_i Q,$$
$$\Theta^i_a \sim \bar{Q}\gamma^5 \lambda^a Q,$$
$$T^i_c \sim \bar{Q}\gamma^5 \tau^i L,$$
$$T^i_c \sim \bar{Q}\gamma^5 L,$$
$$\Pi^i \sim \bar{Q}\gamma^5 \tau^i Q + \bar{L}\gamma^5 \tau^i L,$$
$$P^\pm \sim \bar{Q}\gamma^5 \tau^\pm Q - 3 L\gamma^5 \tau^\pm L,$$
$$P^3 \sim \bar{Q}\gamma^5 \tau^3 Q - 3 L\gamma^5 \tau^3 L,$$
$$P^0 \sim \bar{Q}\gamma^5 Q - 3 L\gamma^5 L,$$
where \( a = 1,2,\ldots 8; i = 1,2,3. \)

The \( \Theta \) -mesons have typical masses \( m_\Theta \sim \sqrt{\frac{4}{N}} \) 200 Gev and the decay modes are:

\[
\begin{align*}
\Theta_a & \rightarrow \bar{q}q, \quad gg, \quad g\gamma, \quad \text{etc}, \\
\Theta^a & \rightarrow \bar{q}q, \quad g\gamma, \quad gZ, \quad \text{etc}, \\
\Theta^{\pm} & \rightarrow \bar{q}q, \quad gW^\pm, \quad \text{etc}.
\end{align*}
\]

For the \( T \)-mesons one has \( m_T \sim 2/3m_\Theta \) and the decay modes are:

\[
\begin{align*}
T^+_c & \rightarrow \bar{b}_c + \nu, \\
T^3_c & \rightarrow \bar{t}_c + \nu, \quad \bar{b}_c + \tau, \\
T^-_c & \rightarrow \bar{t}_c + \nu, \quad \bar{b}_c + \tau, \\
T^0_c & \rightarrow T^3_c + \mu^- + \nu.
\end{align*}
\]

The \( \Pi^i \) mesons are "eaten" by \( W \)s and \( Z \) and become their longitudinal components. \( P^\pm \) are the so-called axions. Their masses are of electromagnetic origin

\[
(m_{P^\pm}^m)^2 \sim \frac{3\alpha}{4\pi} m_Z^2 \ln \frac{M_{TC}^2}{m_\gamma^2} \sim (5 - 10\text{Gev})^2
\]

The decay modes are

\[
P^+ \rightarrow c + \bar{b}, \\
(\bar{t}t) \rightarrow P^+ + \bar{t} + b, \quad \text{if} \quad m_{P^+} < m_t - m_b
\]

\( P^3 \) and \( P^0 \) have masses \( 0 \leq m_{P^3,0}^2 \leq (100\text{Gev})^2 \). They can be produced in \( e^+e^- \) annihilation

\[
e^+e^- \rightarrow \bar{t}t + P^0, \quad P^3 + \gamma,
\]

other modes being suppressed.

### 2.4 Fermion Masses

We have shown how TC can provide us with gauge boson masses. The gauge group of the TC-extended SM now is

\[
SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \otimes TC.
\]

However this is not enough to give masses to fermions. They are still massless. The way out of this puzzle is found along the same lines. One introduces a new type of interactions with a new scale \( \Lambda_E > \Lambda_{TC} \). These new interactions will produce on \( \Lambda_E \) some new effective (nonrenormalizable) operators (four-fermion, etc) that will give us finally the mass terms.

To find out what type of operators plays the main role, let us consider the coupling constant corresponding to an \( n \)-fermion operator. On dimensional grounds it behaves like

\[
g^{(n)} \sim \frac{1}{M^{2n-4}},
\]

where \( M \) is some scale of dimension of the mass. Therefore the larger \( M \) is and the larger \( n \) is the less is the coupling. Hence the main role is played by the operator with minimal \( n \). Because of Lorentz invariance \( n \) is even, so we have

\[
n = 2 : \bar{\psi}\psi. \quad \text{This is a mass term. It is forbidden due to } SU_L(2) \text{ symmetry.}
\]
\( n = 4 \): To construct the four-fermion operator in a one-doublet model \( T = \begin{pmatrix} A \\ B \end{pmatrix} \), we choose the \( 2 \times 2 \) matrix

\[
M_T = 1 \cdot \bar{T}T + i \tau^a \bar{T} \gamma^5 \tau^a T.
\]

Then the effective four-fermion \( SU_L(2) \)-invariant interaction is

\[
\frac{a}{\Lambda^2_E} \bar{q}_L M_{T} q_R + \frac{b}{\Lambda^2_E} \bar{q}_L M_{T} \tau^3 q_R + h.c.
\]

(2.3)

If techniquark condensate exists

\[
< \bar{T}T > = < \bar{A}A + \bar{B}B > \neq 0,
\]

then eq.(2.3) will lead to quark masses

\[
m_u \sim \frac{< \bar{A}A + \bar{B}B >}{\Lambda^2_E} (a + b)
\]

\[
m_d \sim \frac{< \bar{A}A + \bar{B}B >}{\Lambda^2_E} (a - b)
\]

(2.4)

The diagrams contributing to eq.(2.4) are shown in Fig.3

Estimating the value of techniquark condensate for the \( SU(N) \) TC group to be

\[
< \bar{T}T > \sim \sqrt{\frac{3}{N}} \frac{< \bar{q}q >}{\left( \frac{F^*_\pi}{f_\pi} \right)^3} \sim \frac{(600 \text{ Gev})^3}{(300 \text{ Gev})^3} \quad \text{1 doublet model}
\]

\[
\Lambda_E \sim \frac{20 \text{ Tev}}{7 \text{ Tev}}.
\]

we get an estimate for the new scale \( \Lambda_E \) respectively

The same is true for the one family model, however here we have more possibilities. With quark and lepton doublets

\[
q = \begin{pmatrix} u \\ d \end{pmatrix}, l = \begin{pmatrix} \nu \\ e \end{pmatrix}, Q = \begin{pmatrix} U \\ D \end{pmatrix}, L = \begin{pmatrix} N \\ E \end{pmatrix}
\]

we have

\[
M_Q = 1 \cdot \bar{Q}Q + i \tau^a \bar{Q} \gamma^5 \tau^a Q
\]

\[
M_L = 1 \cdot \bar{L}L + i \tau^a \bar{L} \gamma^5 \tau^a L
\]
Figure 4: Effective four-fermion interaction due to gauge boson exchange

Figure 5: The diagram contributing to the quark mass due to a gauge boson exchange in TC theories

And $SU_{Left}(2)$ singlets now are

$$\bar{q}_L M_Q q_R, \quad \bar{q}_L M_Q t^3 q_R, \quad \bar{q}_L M_L q_R, \quad \bar{q}_L M_L t^3 q_R$$

$$\bar{l}_L M_Q l_R, \quad \bar{l}_L M_Q t^3 l_R, \quad \bar{l}_L M_L l_R, \quad \bar{l}_L M_L t^3 l_R$$

Introducing a generation mixing matrix we also get the mixing angles.

Thus to obtain fermion masses one needs an effective four-fermion interaction. The latter can be also obtained from the gauge boson exchange, just like effective four-fermion operators in weak interactions appear due to $W$ and $Z$ exchange (see Fig.4). Then the mass terms arise from the diagrams shown in Fig.5. This is an extended Technicolour idea.

2.5 Extended Technicolour

The guiding idea: Techniparticles and ordinary particles belong to the same representation of a large gauge group that is dynamically broken on some scale

$$G_{ETC} \supset G_{TC}.$$ 

For example

$$SU(N+1) \supset SU(N)$$

and the fundamental representation

$$N + 1 = N + 1 \quad ETC \quad \left\{ \begin{array}{c}
T_1 \\
\vdots \\
T_N \\
q
\end{array} \right\} \quad TC \quad \text{singlet}$$
The other possibilities are

\[ SU(N + m) \supset SU(N), \quad SU(N + 3) \supset SU_{TC}(N) \otimes SU_c(3) \]
\[ N + m = N + m \cdot 1, \quad N + 3 = N + 3 \cdot 1 \]

On the scale \( \Lambda_E \) the ETC group should be dynamically broken

\[ G_{ETC} \xrightarrow{\Lambda_E} U_Y(1) \otimes SU_L(2) \otimes SU_c(3) \otimes G_{TC} \]

The broken generators of the ETC group will lead to massive gauge bosons which will mediate the fermion-technifermion transformations. On \( \Lambda_E \) the ordinary fermions will mix with technifermions and with themselves thus producing rare decays like

\[ K \rightarrow \mu e, \quad \pi^+ \rightarrow e^+ e^+, \quad etc. \]

This is an indication of Extended Technicolour.

### 2.6 GUTs and Technicolour

As you probably noticed, ETC idea is closely related to that of Grand Unified Theories. In the latter case the group of the SM is embedded into a larger group of GUT.

\[ GUT : SU(3) \otimes SU(2) \otimes U(1) \subset G_{GUT} = SU(5). \]

The same can be done with TC.

\[ GUT + TC : G_{TC} \otimes SU(3) \otimes SU(2) \otimes U(1) \subset G_{GUT}. \]

For instance, if \( G_{TC} \) is \( SU(N) \), the natural embedding is to \( SU(N + 5) \):

\[
\left( \begin{array}{cccccc}
\vdots & & & & & \\
SU_{TC}(N) & & & & & \\
& \vdots & & & & \\
& \vdots & \ddots & \vdots & & \\
& & \ddots & \ddots & \ddots & \\
& & \vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots \\
\end{array} \right)
\]

\[ SU_c(3) \]

The pattern of symmetry breaking will then have two stages

\[ G_{GUT} = SU(N + 5) \xrightarrow{\Lambda_E} G_{TC} \otimes SU(3) \otimes SU(2) \otimes U(1) \xrightarrow{\Lambda_E} SU(3) \otimes U(1) \]

To conclude, we mention the successes and problems of Technicolour.

**Successes:**

1) Low energy Higgs bosons are replaced by fermions. These fermions are unified with ordinary fermions;
ii) More than one family of light fermions can be naturally incorporated in a large representation of a unifying group.

Problems:

i) It is difficult to implement the required symmetry breaking in many stages;

ii) Many technifermions may change the $\beta$-functions and ruin successful predictions for the proton decay and low-energy value of $\theta_W$;

iii) Spectra of light fermions is unrealistic.

2.7 Tumbling Theory (Groups Breaking Themselves)

An interesting possibility of dynamical breaking of gauge symmetry, which can be useful to obtain several stages of breaking, is the so called tumbling phenomenon.

Let the gauge group $G$ be asymptotically free. Consider the scattering process shown in Fig.6

The amplitude of this process is proportional to the square of the running coupling $g^2(Q^2)$ which increases as we come down with energy due to the asymptotical freedom. The coefficient $C(f_1, f_2, r)$ depends on the group and representation of fermion fields.

The final fermions may condense in a singlet channel. The condensation will take place if the force between final fermions is attractive and the amplitude is large enough (in AF theory it becomes large as we go down with energy). It will take place first in the most attractive channel. If a condensate is not a singlet under some group, the latter will be dynamically broken.

Thus the chain looks as follows: One starts at a high energy with an unbroken group $G$. All of the particles are massless. As the energy goes down, some fermions condense. This leads to a symmetry breaking. Some particles acquire masses of the order of the symmetry breaking scale, while the others remain massless. Then the procedure is repeated several times.

$$
\begin{align*}
G & \xrightarrow{\text{Condensate}} G_1 \\
\text{massless} & \quad E_1 \\
\text{particles} & \quad m \sim E_1 \\
& \quad \xrightarrow{\text{Condensate}} G_2 \\
& \quad \text{some} E_2 \\
& \quad m \sim E_2 \\
& \quad \rightarrow \ldots \rightarrow G_N \\
& \quad \text{some} E_3 \\
& \quad \text{some} E_N
\end{align*}
$$

We get a chain of symmetry breakings due to condensation. The described picture is called a tumbling gauge theory. This phenomenon may naturally produce a sequence of scales thus solving the problem equally important in TC as well in any other theories.

Another possibility to reach the same goal is the radiative corrections. Suppose that as a result of some symmetry breaking (like in TC theory) some quarks become massive while the others remain massless. Then they will get mass corrections due to the weak interactions (see Fig.7). Hence ordinary fermions will have masses of the order of $\alpha$ and/or
\[ \alpha^2 \times \Lambda_{TC}. \] This mechanism will give masses to the light fermions without introducing a new scale.

However in any case the origin of the fermion spectrum remains unclear. Note that there is no solution of this problem in the Higgs model as well. Understanding of the origin of fermion masses is one of the greatest challenges facing particle physicists today.

3 Supersymmetry

3.1 Motivations of SUSY

Supersymmetry or fermion-boson symmetry have not yet been observed in Nature. This is a purely theoretical invention. Its validity in particle physics follows from the common belief in unification. The general idea is a unification of all forces of Nature. It defines the strategy: increasing unification towards smaller distances up to \( l_{Pl} \sim 10^{-33} \) cm including quantum gravity. However the graviton has spin 2, while the other gauge bosons (photon, gluons, \( W \) and \( Z \) weak bosons) have spin 1. Unification of spin 2 and spin 1 gauge forces within unique algebra is forbidden due to the no-go theorems for any symmetry but SUSY.

If \( Q \) is a generator of SUSY algebra, then

\[ Q|\text{boson} > = |\text{fermion} > \quad \text{and} \quad Q|\text{fermion} > = |\text{boson} > . \]

Hence starting with the graviton spin 2 state and acting by SUSY generators we get the following chain of states

\[ \text{spin} 2 \rightarrow \text{spin} 3/2 \rightarrow \text{spin} 1 \rightarrow \text{spin} 1/2 \rightarrow \text{spin} 0 \]

Thus a partial unification of matter (fermions) with forces (bosons) naturally arises out of an attempt to unify gravity with other interactions.

This very promising unification pattern gives us a justification of research in spite of the absence of even a shred of experimental evidence of SUSY.

The uniqueness of SUSY is due to a strict mathematical statement that algebra of SUSY is the only graded (i.e. containing anticommutators as well as commutators) Lie algebra possible within relativistic field theory.

Theoretical attractiveness of SUSY field theories is explained by remarkable properties of SUSY models. This is first of all a cancellation of ultraviolet divergences in rigid SUSY theories which is the origin of

- possible solution of the hierarchy problem in GUTs;
- construction of a finite field theory;
• possible construction of quantum (super)gravity;

• solution of the problem of vanishing cosmological constant, etc.

What is essential, the standard concepts of QFT allow SUSY without any further assumptions.

3.2 Global SUSY. Algebra and Representations

As can be easily seen, supersymmetry transformations differ from ordinary global transformations as far as they convert bosons into fermions and vice versa. Indeed if we symbolically write SUSY transformation as

\[ \delta B = \varepsilon \cdot f, \]

where \( B \) and \( f \) are boson and fermion fields respectively and \( \varepsilon \) is an infinitesimal transformation parameter, then from the usual (anti)commutation relations for (fermions) bosons

\[ [B, B] = 0, \quad \{f, f\} = 0 \]

we immediately find

\[ \{\varepsilon, \varepsilon\} = 0. \]

This means that all the generators of SUSY must be fermionic, i.e. they must change the spin by a half-odd amount and change the statistics.

Combined with the usual Poincaré and internal symmetry algebra the superPoincaré Lie algebra is

\[
\begin{align*}
[P_\mu, P_\nu] &= 0, \\
[P_\mu, M_{\rho\sigma}] &= i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\mu\nu}M_{\rho\sigma} + g_{\mu\sigma}M_{\nu\rho}), \\
[B_\gamma, B_\delta] &= iC_\gamma^\alpha B_\delta, \\
[B_\gamma, P_\mu] &= [B_\gamma, M_{\mu\nu}] = 0, \\
[Q^i_\alpha, P_\mu] &= [Q^i_\alpha, P_\mu] = 0, \\
[Q^i_\alpha, M_{\mu\nu}] &= \frac{i}{2}(\sigma_{\mu\nu})^\delta_\alpha Q^i_\delta, \\
[Q^i_\alpha, M_{\mu\nu}] &= -\frac{i}{2}Q^i_\delta(\sigma_{\mu\nu})^\delta_\alpha, \\
[Q^i_\alpha, B_\gamma] &= (\sigma^\mu)^i_\alpha Q^\mu_\gamma, \\
[Q^i_\alpha, B_\gamma] &= -\frac{i}{2}Q^\gamma_\delta(\sigma_{\mu\nu})^\delta_\alpha, \\
\{Q^i_\alpha, Q^j_\beta\} &= 2\delta^{ij}(\sigma^\mu)^\alpha_\beta P_\mu, \\
\{Q^i_\alpha, Q^j_\beta\} &= 2\varepsilon_{\alpha\beta}Z^{ij}, \\
\{Z_{ij}, any\thing\} &= 0, \\
\alpha, \beta &= 1, 2, \quad \gamma, \delta = 1, 2, \ldots, N.
\end{align*}
\]

Here \( P_\mu \) and \( M_{\mu\nu} \) are four-momentum and angular momentum operators respectively, \( B_\gamma \) are internal symmetry generators, \( Q^i_\alpha \) and \( \bar{Q}^i_\beta \) are spinorial SUSY generators and \( Z_{ij} \) are the so-called central charges.

A natural question arises: how many SUSY generators are possible, i.e. what is the value of \( N \)? To answer this question, let us consider the massless states in SUSY theory. The ground state defined by \( Q_i|E, \lambda >= 0 \) is labelled by two quantum numbers: energy \( E \) and helicity \( \lambda \). Then a one-particle state is produced by acting of a creation operator

\[ 1 \text{ particle state} \quad \bar{Q}_i|E, \lambda >= |E, \lambda + 1 >_i \]

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It has the same energy $E$ since a SUSY generator commutes with four-momentum (see eq. (3.1)) and helicity $\lambda + 1/2$. The number of one-particle states equals that of SUSY generators, i.e. $\binom{N}{1} = N$.

Two-particle state is constructed by acting of two creation operators

$$2 \text{ particle state} \quad \bar{Q}_i Q_j |E, \lambda > = |E, \lambda + 1 >_{ij}$$

The number of such states is $\binom{N}{2} = \frac{N(N+1)}{2}$. Continuing the procedure we end up with an $N$-particle state

$$N \text{ particle state} \quad \bar{Q}_1 \cdots \bar{Q}_N |E, \lambda + \frac{N}{2} >.$$

The number of these states is $\binom{N}{N} = 1$. Thus the total number of states is

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N = 2^{N-1} \text{bosons} + 2^{N-1} \text{fermions}.$$ 

We see that the number of bosonic states equals that of fermionic. This is a general feature of SUSY. If SUSY is realized linearly bosonic and fermionic states are always equal in number.

Remind now the so-called PCT-theorem valid for any local Lorentz-invariant QFT.

**PCT-Theorem:** In any Lorentz-invariant local field theory for every state with helicity $\lambda$ there should be a parity reflected state with helicity $-\lambda$.

Consider $N = 1$ case with $\lambda = 0$. According to the previous discussion we have

$$\begin{array}{ccc}
N = 1 & \text{helicity} & 0 \quad 1/2 \\
\lambda = 0 & \text{No of states} & 1 \quad 1
\end{array} \quad \overset{PCT}{\Rightarrow} \quad \begin{array}{ccc}
N = 1 & \text{helicity} & -1/2 \quad 0 \\
\text{multiplet} & \text{No of states} & 1 \quad 2 \quad 1
\end{array}$$

Thus a complete $N = 1$ multiplet consists of the following set of states:

This is an example of a self-conjugated multiplet. Some other self-conjugated multiplets are also important

$$\begin{array}{ccc}
N = 4 & \text{helicity} & -1 \quad -1/2 \quad 0 \quad 1/2 \quad 1 \\
\text{SUSY YM} & \text{No of states} & 1 \quad 4 \quad 6 \quad 4 \quad 1
\end{array} \quad \text{SUGRA} \quad \begin{array}{ccc}
N = 8 & \text{helicity} & -2 \quad -3/2 \quad -1 \quad -1/2 \quad 0 \quad 1/2 \quad 1 \quad 3/2 \quad 2 \\
\text{No of states} & 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

As can be seen, the maximal helicity (or spin) is related to the number of SUSY generators $N$ by equation

$$4s \leq N.$$ 

This constraint becomes very essential since any theory containing particles with spin $> 1$ is non-renormalizable, and any theory containing a finite number of particles with spin $\geq 5/2$ has no consistent coupling to gravity. Since we are not able to proceed with non-renormalizable interactions, we get the following constraint on the number of SUSY generators

$$N \leq 4 \quad \text{for renormalizable theories (YM)},$$

$$N \leq 8 \quad \text{for (super)gravity}.$$
3.3 Component Fields and Superfields

Return now to an $N = 1$ supermultiplet. We denote it by $(A, \psi, F)$, where $A$ and $F$ are bosonic and $\psi$ is a fermionic field. Supersymmetry transformations are (we use the two-component spinor notation):

$$
\delta_\varepsilon A = \sqrt{2}\varepsilon \psi, \\
\delta_\varepsilon \psi = i\sqrt{2}\sigma^\mu \varepsilon \partial_\mu A + \sqrt{2}\varepsilon F, \\
\delta_\varepsilon F = i\sqrt{2}\sigma^\mu \varepsilon \partial_\mu \psi, 
$$

(3.2)

where $\sigma^\mu = (1, \sigma^\alpha)$ are Pauli matrices. Transformations (3.2) form a closed algebra. However not all of the fields are the usual ones. To clear it up, let us look at their dimensions. If for $A$ and $\psi$ we have, as usual,

$$[A] = m^1, \quad [\psi] = m^{3/2},$$

then from eq.(3.2) it follows that

$$[\varepsilon] = [\bar{\varepsilon}] = m^{-1/2}, \quad [F] = m^2.$$  

One can see that the field $F$ has a "wrong" dimension. This field is called an auxiliary field, it has no physical meaning and is needed to close the algebra (3.2). As we will see, one can get rid of the auxiliary fields with the help of equations of motion.

The Lagrangian which is invariant under transformations (3.2) (up to a total derivative) is

$$
\mathcal{L} = \mathcal{L}_0 + m\mathcal{L}_m,  \\
\mathcal{L}_0 = i\bar{\psi}\sigma^\mu \partial_\mu \psi + A^* \Box A + F^* F, \\
\mathcal{L}_m = AF + A^* F^* - \frac{1}{2}\bar{\psi}\psi - \frac{1}{2}\bar{\psi}\bar{\psi}.
$$

Equations of motion that follows from eq.(3.3) are

$$i\sigma^\mu \partial_\mu \psi + m\bar{\psi} = 0, \\
\Box A + mF^* = 0, \\
F + mA^* = 0.$$  

One can see that the last equation contains no derivatives, i.e. this is a constraint rather than a dynamical equation. Solving this constraint with respect to an auxiliary field $F$ we get the so-called mass-shell formulation of SUSY. Lagrangian (3.3) becomes

$$\mathcal{L} = i\bar{\psi}\sigma^\mu \partial_\mu \psi - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi}) + A^* \Box A - m^2 A^* A,$$

which is a usual free field Lagrangian for spinor and scalar fields.

*Superspace*

More elegant formulation of supersymmetry transformations and invariants can be achieved in the framework of superspace. Superspace differs from the ordinary Euclidean (Minkowski) space by addition of two new coordinates, which are grassmanian, i.e. anticommuting, variables $\{\theta_\alpha, \bar{\theta}_\dot{\alpha}\} = 0, \quad \theta^2 = 0$. 

$$
\text{Space} \quad \text{Superspace} \\
x_\mu \quad x_\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}
$$

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A SUSY group element can be constructed in superspace in the same way as an ordinary
translation in the usual space

\[ G(x, \theta, \bar{\theta}) = e^{i(-x^\mu P_\mu + \theta Q + \bar{\theta} \bar{Q})}. \]

It leads to the supertranslation in superspace

\[
\begin{align*}
  x_\mu & \rightarrow x_\mu + i\theta \sigma_\mu \xi - i\xi \sigma_\mu \bar{\theta}, \\
  \theta & \rightarrow \theta + \xi, \\
  \bar{\theta} & \rightarrow \bar{\theta} + \bar{\xi},
\end{align*}
\]

where \( \xi \) and \( \bar{\xi} \) are grassmanian transformation parameters.

Now we are in a position to introduce a superfield as a function on a superspace which
is a representation of a superPoincaré group (3.1). The simplest one is a scalar superfield
\( F(x, \theta, \bar{\theta}) \) which is SUSY invariant. Its Taylor expansion in \( \theta \) and \( \bar{\theta} \) has only several terms
due to the nilpotent character of grassmanian parameters.

\[
F(x, \theta, \bar{\theta}) = f(x) + \theta \varphi(x) + \bar{\theta} \bar{\varphi}(x) \\
+ \theta \theta n(x) + \bar{\theta} \bar{\theta} n(x) + \theta \sigma^\mu \bar{\theta} v'_\mu(x) \\
+ \theta \bar{\theta} \lambda(x) + \bar{\theta} \theta \psi(x) + \theta \theta \bar{\theta} d(x).
\]

(3.4)

The coefficients are ordinary functions of \( x \), being the usual fields. Superfield (3.4) is a
reducible representation of SUSY. To get an irreducible one, we consider a function which
depends only on \( \theta \) but not on \( \bar{\theta} \), i.e. obeys the equation

\[
\frac{\partial}{\partial \bar{\theta}} F(x, \theta) = 0.
\]

Unfortunately this equation is not SUSY invariant. To improve it, one has to use the
covariant derivative instead of the ordinary one

\[
\bar{D} F = 0, \quad \text{where } \bar{D} = \frac{\partial}{\partial \bar{\theta}} - i\theta \sigma^\mu \partial_\mu.
\]

(3.5)

The superfield obeying eq.(3.5) is called a chiral superfield

\[
\bar{D} \Phi = 0 \Rightarrow \Phi = \Phi(y, \theta), \quad y = x + i\theta \sigma \bar{\theta}.
\]

Its Taylor expansion looks like

\[
\Phi(y, \theta) = A(y) + \sqrt{2} \psi(y) + \theta \theta F(y) \\
= A(x) + i\theta \sigma^\mu \bar{\theta} \partial_\mu A(x) + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \square A(x) \\
+ \sqrt{2} \psi(x) - \frac{i}{\sqrt{2}} \theta \theta \bar{\theta} \partial_\mu \psi(x) \sigma^\mu \bar{\theta} + \theta \theta F(x)
\]

and manifests the \( N = 1 \) supermultiplet considered above.

The product of chiral superfields \( \Phi^2, \Phi^3, \) etc is also a chiral superfield, while the product
of chiral and antichiral ones \( \Phi^+ \Phi \) is a general superfield.

We are now ready to construct SUSY invariant Lagrangians.
3.4 SUSY Lagrangians

In the superfield notation SUSY invariant Lagrangians are simply polynomials of superfields. Having in mind that for component fields we should have the ordinary terms, the general SUSY invariant Lagrangian has the form

$$\mathcal{L} = \Phi_i^\dagger \Phi_i|_{\theta \bar{\theta} \bar{\theta}} + \left[ (\lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k)_{\theta \bar{\theta}} + h.c. \right]. \tag{3.6}$$

Hereafter the vertical line means the corresponding term of a Taylor expansion. Performing this expansion we get in components

$$\mathcal{L} = i \bar{\psi}_i \sigma^\mu \partial_\mu \psi_i + A_i^* \nabla A_i + F_i^* F_i$$
$$+ \left[ \lambda_i F_i + m_{ij} (A_i F_j - \frac{1}{2} \psi_i \psi_j) + g_{ijk} (A_i A_j F_k - \psi_i \psi_j A_k) + h.c. \right]$$

The last two terms which are additional to that of eq.(3.3) are the interaction ones. To obtain a familiar form of the Lagrangian, we have to solve the constraints:

$$\frac{\partial \mathcal{L}}{\partial F_k^*} = F_k + \lambda_k^* + m_k^* A_i + g_{ijk} A_i A_j^* = 0,$$
$$\frac{\partial \mathcal{L}}{\partial F_k} = F_k^* + \lambda_k + m_k A_i + g_{ijk} A_i A_j = 0.$$

After that we finally get

$$\mathcal{L} = i \bar{\psi}_i \sigma^\mu \partial_\mu \psi_i + A_i^* \nabla A_i - \frac{1}{2} m_{ij} \psi_i \psi_j - \frac{1}{2} m_{ij} \bar{\psi}_i \bar{\psi}_j$$
$$- g_{ijk} \psi_i \psi_j A_k - g_{ijk} \bar{\psi}_i \bar{\psi}_j A_k^* - V(A_i, A_j), \tag{3.7}$$

where $V = F_k^* F_k$. Note that because of the renormalizability constraint $V \leq A^4$ the superpotential should be limited by $W \leq \Phi^3$ as in eq.(3.6).

To construct the gauge invariant interactions, we will need a real vector superfield $V = V^*$. It is not chiral but rather a general superfield

$$V(x, \theta, \bar{\theta}) = C(x) + i \theta \chi(x) - i \bar{\theta} \bar{\chi}(x)$$
$$+ \frac{i}{2} \theta \theta [M(x) + i N(x)] - \frac{i}{2} \bar{\theta} \bar{\theta} [M(x) - i N(x)]$$
$$- \theta \sigma^\mu \bar{\partial}_\mu v(x) + i \theta \theta [\chi(x) + \frac{i}{2} \sigma^\mu \partial_\mu \chi(x)]$$
$$- i \bar{\theta} \bar{\theta} [\lambda + \frac{i}{2} \sigma^\mu \partial_\mu \bar{\chi}(x)] + \frac{1}{2} \theta \bar{\theta} \bar{\theta} [D(x) + \frac{1}{2} \nabla C(x)]. \tag{3.8}$$

The physical degrees of freedom corresponding to a real vector superfield are the vector gauge field $v_\mu$ and majorana spinor field $\lambda$. All other components are unphysical and can be eliminated.

Under the abelian (super)gauge transformation the superfield $V$ is transformed as

$$V \rightarrow V + \Phi + \Phi^*,$$
where \( \Phi \) and \( \Phi^+ \) are some chiral superfields. In components it looks like

\[
\begin{align*}
C & \rightarrow C + A + A^*, \\
\chi & \rightarrow \chi - i\sqrt{2}\psi, \\
M + iN & \rightarrow M + iN - 2iF, \\
v_{\mu} & \rightarrow v_{\mu} - i\partial_{\mu}(A - A^*), \\
\lambda & \rightarrow \lambda, \\
D & \rightarrow D
\end{align*}
\] (3.9)

and corresponds to ordinary gauge transformations for physical components. According to eq.(3.9) one can choose a gauge (Wess-Zumino gauge) where \( C = \chi = M = N = 0 \), leaving us with the physical degrees of freedom except for the auxiliary field \( D \). In this gauge

\[
\begin{align*}
V &= -\theta\sigma^\mu\bar{v}_{\mu}(x) + i\theta\bar{\theta}\bar{\lambda}(x) - i\theta\theta\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\theta\theta\theta\theta D(x), \\
V^2 &= -\frac{1}{2}\theta\theta\theta\theta v_{\mu}(x)v^\mu(x), \\
V^3 &= 0, \text{ etc.}
\end{align*}
\]

One can define also a field strength tensor (as analog of \( F_{\mu\nu} \) in gauge theories)

\[
\begin{align*}
W_\alpha &= -\frac{1}{4}\bar{D}^2 e^\nu D_\alpha e^{-\nu}, \\
\bar{W}_\dot{\alpha} &= -\frac{1}{4}D^2 e^\nu \bar{D}_\alpha e^{-\nu},
\end{align*}
\]

which is a polynomial in the Wess-Zumino gauge. (Here \( Ds \) are the covariant derivatives.) The strength tensor is a chiral superfield

\[
D_\beta W_\alpha = 0, \quad D_\beta \bar{W}_{\dot{\alpha}} = 0.
\]

The gauge invariant Lagrangian now is

\[
\mathcal{L} = \frac{1}{4}(W^\alpha W_\alpha|_{0\theta} + \bar{W}^\dot{\alpha}\bar{W}_{\dot{\alpha}}|_{\dot{0}\dot{\theta}})
= \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\sigma^\mu D_\mu \lambda.
\]

To obtain a gauge-invariant interaction with matter chiral superfields, consider their gauge transformation (abelian)

\[
\Phi \rightarrow e^{-i\gamma^A\Phi}, \quad \Phi^+ \rightarrow \Phi^+e^{i\gamma^A}, \quad V \rightarrow V + i(\Lambda - \Lambda^+),
\]

where \( \Lambda \) is a gauge parameter (chiral superfield).

It is clear now how to construct both the SUSY and gauge invariant interaction

\[
\mathcal{L}_{\text{inv}} = \frac{1}{4}(W^\alpha W_\alpha|_{0\theta} + \bar{W}^\dot{\alpha}\bar{W}_{\dot{\alpha}}|_{\dot{0}\dot{\theta}})
+ \Phi^+_i e^{\gamma^\nu} \Phi_i|_{0\theta\theta\theta}
+ (\frac{1}{2}r_{ij} \Phi_i \Phi_j + \frac{1}{3}g_{ijk} \Phi_i \Phi_j \Phi_k)|_{0\theta} + \text{h.c.}
\]

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In particular the SUSY generalization of QED looks as follows

\[
\mathcal{L}_{\text{SUSY QED}} = \frac{1}{4} (W^\alpha W_\alpha |_{\bar{e}e} + \bar{W}^\dot{\alpha} \bar{W}_{\dot{\alpha}} |_{\bar{e}e})
+ (\Phi_+ e^{\alpha V} \Phi_+ + \Phi_+ e^{-\alpha V} \Phi_-) |_{\bar{e}e\bar{e}e}
+ m \Phi_+ \Phi_- |_{\bar{e}e} + m \Phi_+^\dagger \Phi_-^\dagger |_{\bar{e}e}.
\]

The non-abelian generalization is straightforward

\[
\mathcal{L}_{\text{SUSY YM}} = \frac{1}{4} \int Tr(W^\alpha W_\alpha) \, d^2 \theta + \int \bar{\Phi}_i (e^{\alpha V})_{\dot{\alpha}i} \Phi^\dagger_{\dot{\alpha}} d^2 \theta d^2 \bar{\theta}
+ \int V(\Phi_i) \, d^2 \theta \, + \text{h.c.},
\]

where instead of taking the proper components we use an integration over the superspace according to the rules of grassmanian integration

\[
\int d\theta = 0, \quad \int \theta \, d\theta = 1.
\]

This trick allows one to write down the action as an integral over the whole superspace in exact analogy with an ordinary QFT.

Note that the form of the Lagrangian is practically fixed by symmetry requirements. The only freedom is the field content, the value of the gauge coupling $g$, Yukawa couplings $g_{ijk}$ and the masses. All members of a supermultiplet have the same masses, i.e. bosons and fermions are degenerate in masses. This property of SUSY theories contradicts the phenomenology and requires supersymmetry breaking.

### 3.5 Spontaneous Symmetry Breaking

Apart from non-supersymmetric theories in SUSY models the energy is always nonnegative definite. Indeed, according to quantum mechanics

\[
E = \langle 0 | H | 0 \rangle
\]

and due to SUSY algebra eq.(3.1)

\[
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2 (\sigma^\mu)_{\alpha\dot{\beta}} P_{\mu},
\]

taking into account that $tr(\sigma^\mu P_{\mu}) = 2P_0$, we get

\[
E = \frac{1}{4} \sum_{\alpha=1,2} < 0 | \{Q_\alpha, \bar{Q}_\alpha\} | 0 > = \frac{1}{4} \sum_{\alpha} |Q_\alpha | 0 > |^2 \geq 0.
\]

Hence

\[
E = \langle 0 | H | 0 \rangle \neq 0 \quad \text{if and only if} \quad Q_\alpha \neq 0.
\]

Therefore a supersymmetry is spontaneously broken, i.e. vacuum is not invariant $(Q_\alpha | 0 \neq 0 )$, if and only if the minimum of the potential is positive (i.e. $E > 0$).

The situation is illustrated in Fig.8.

**Mechanisms for SSB**

1) Fayet-Iliopoulos (D-term) mechanism
Let the Lagrangian be SUSY and $U(1)$ gauge invariant and contain two matter superfields $\Phi_1$ and $\Phi_2$. We add to the Lagrangian a linear term

$$\Delta \mathcal{L} = \xi \, V_{\tilde{\theta} \tilde{\theta}},$$

(3.10)

which is a $D$-term of a vector superfield (cf. eq.(3.8)). Eq.(3.10) is SUSY and gauge invariant (up to a total derivative).

Solving now equations of motion for the auxiliary fields we obtain the potential

$$V_{pot} = (m^2 + g\xi)A^*_1 A_1 + (m^2 - g\xi)A^*_2 A_2 + \frac{1}{8} g^2 (A^*_1 A_1 - A^*_2 A_2)^2 + 1/2 \xi^2.$$

There are two possibilities:

i) $g\xi < m^2$. In this case SUSY is spontaneously broken while the gauge invariance is preserved. The masses are

$$m_{A_{1,2}}^2 = m^2 \pm 1/2 g\xi, \quad m_{\psi_{1,2}} = m, \quad m_{A_{\mu,\lambda}} = 0.$$

ii) $g\xi > m^2$. In this case both SUSY and gauge invariance are spontaneously broken. The minimum of the potential is at a finite value for the matter fields. One choice is

$$< A_1 > = 0, \quad < A_2 > = \mu/g \quad \text{with} \quad \mu^2 = 2(\xi g - m^2).$$

A Higgs mechanism will eliminate the goldstone boson, give mass to the vector $A_\mu$ and mix spinors into mass eigenstates $\tilde{\psi}$ and $\tilde{\lambda}$. The masses become

$$m_{\psi_{1,2}} = m^2 + \mu^2, \quad m_{\lambda} = 0,$$

$$m_{A_\mu} = m_{A_1} = \mu, \quad m_{A_2} = \sqrt{2} m.$$

The potential is shown in Fig.9. The spectrum of particles is illustrated in Fig.10. Note that in both cases the following sum rule is always valid

$$\sum_{\text{boson states}} m^2 = \sum_{\text{fermion states}} m^2.$$

The drawback of this mechanism is the necessity of $U(1)$ gauge invariance. It can be used in SUSY generalizations of the SM but not in GUTs.
Figure 9: The potential for Fayet-Iliopoulos model

Figure 10: Mass spectrum of Fayet-Iliopoulos model
2) O’Raifeartaigh (F-term) mechanism

This mechanism works when we have several chiral fields. Let the superpotential be

\[ W(\Phi) = \lambda \Phi_3 + m \Phi_1 \Phi_2 + g \Phi_3 \Phi_1^2. \]

Equations of motion for the auxiliary fields in this case are

\[ F_1^* = mA_2 + 2gA_1A_3, \]
\[ F_2^* = mA_1, \]
\[ F_3^* = \lambda + gA_1^2. \]

Looking for a vacuum solution \(< F_i^* >= 0\) we find that it is absent. This means that SUSY is spontaneously broken since \(V = F_i^* F_i \neq 0\). The potential takes the form

\[ V(A) = \frac{1}{2} |m A_2 + 2g A_1 A_3|^2 + \frac{m^2}{2} |A_1|^2 \]
\[ + \frac{1}{2} |\lambda + g A_1^2|^2. \]

The minimum is achieved by the choice

\[ < A_1 > = < A_2 > = 0, \quad < A_3 > = \frac{\mu}{2g}, \quad \mu \text{ is arbitrary} \]

Again there are two possibilities

i) \(|2g\lambda| < m^2\), then

\[ m_{\psi_{1,2}} = \sqrt{\mu^2/4 + m^2} \pm \mu/2, \quad m_{\psi_3} = 0, \]
\[ m_{A_{1,2}} = \mu^2/2 + m^2 + g\lambda \sqrt{(\mu^2 + 2g\lambda)^2/4 + \mu^2 m^2}, \quad m_{A_3} = 0, \]
\[ m_{\tilde{B}_{1,2}} = \mu^2/2 + m^2 - g\lambda \sqrt{(\mu^2 - 2g\lambda)^2/4 + \mu^2 m^2}, \quad A = \bar{A} + i \bar{B}. \]

ii) \(|2g\lambda| > m^2\), then

\[ m_{\psi_{1,2}} = 4g\lambda - m^2, \quad m_{\psi_3} = 0, \quad m_{A_3} = 0, \]
\[ m_{A_{1,2}} = 4g\lambda - m^2, \quad m_{A_1} = 4g\lambda - 2m^2, \quad m_{\tilde{B}_2} = 4g\lambda. \]

In both cases SUSY is spontaneously broken but \(\sum m_{\text{boson}}^2 = \sum m_{\text{fermion}}^2\) as before. The potential and the spectrum are shown in Figs.11,12. The drawbacks of this mechanism is a lot of arbitrariness in the choice of potential. The mass spectrum also causes some troubles for a phenomenology.

3) A supergravity induced mechanism

This mechanism is based on effective non-renormalizable interactions arising as the low energy limit of supergravity theories. In spite of attractiveness of this mechanism in general, it is not truly substantiated due to the lack of a consistent theory of quantum (super)gravity.
Figure 11: The potential for O'Raifeartaigh model

Figure 12: Mass spectrum of O'Raifeartaigh model
3.6 Phenomenology of Global SUSY Theories

As was already mentioned, in SUSY theories the number of bosonic degrees of freedom equals that of fermionic. Let us count these numbers in the SM. In the minimal version the number of bosonic degrees of freedom equals 28, while that of fermionic equals 90. So the SM is in great deal non-supersymmetric. Trying to add some new particles to supersymmetrize the SM, one should take into account the following observations:

i) There is no fermions with quantum numbers of the gauge bosons;

ii) Higgs fields have a non-zero v.e.v., hence they cannot be superpartners of quarks and leptons since this would induce a spontaneous violation of baryon and lepton numbers;

iii) One needs at least two complex chiral multiplets to give masses to Up and Down quarks. This is due to the form of superpotential and chirality of matter superfields. Therefore the Higgs sector is inevitably enlarged.

\[ SM: \quad 4 \rightarrow 1 \quad SUSY: \quad 8 \rightarrow 5 \]

**Conclusion:** In SUSY models supersymmetry associates known bosons with new fermions and known fermions with new bosons.

### Content of SUSY Particle Theory

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Spin 1</th>
<th>Spin 1/2</th>
<th>Spin 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauge</td>
<td>Photon</td>
<td>Photino</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gluons</td>
<td>Gluinos</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(W^\pm, Z)</td>
<td>&quot;Gauginos&quot;</td>
<td></td>
</tr>
<tr>
<td>Higgs</td>
<td>Spin 1/2 partners of Higgs bosons</td>
<td>Higgs bosons</td>
<td></td>
</tr>
<tr>
<td>Lepton and Quark</td>
<td>Leptons and Quarks</td>
<td>Spin 0 partners of leptons and quarks</td>
<td></td>
</tr>
</tbody>
</table>

### Content of Spontaneously Broken SUSY

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Spin 1</th>
<th>Spin 1/2</th>
<th>Spin 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Massless Gauge</td>
<td>Photon</td>
<td>Photino</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Gluons</td>
<td>Gluinos</td>
<td></td>
</tr>
<tr>
<td>Massive Gauge</td>
<td>(W^\pm)</td>
<td>Heavy fermions (w^\pm, z)</td>
<td>Higgs bosons</td>
</tr>
<tr>
<td></td>
<td>(Wino^\pm, Zino)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lepton and Quark</td>
<td>Leptons and Quarks</td>
<td>Spin 0 sleptons and squarks</td>
<td></td>
</tr>
</tbody>
</table>

Characteristic feature of any supersymmetric generalization of the SM is the presence of superpartners of ordinary particles. The absence of them at modern energies is believed to be explained by their masses being very heavy. This means that if the energy will be high enough, the superpartners will be created.

The interactions of superpartners are essentially the same as in the SM but two of three particles involved into an interaction at any vertex are replaced by superpartners. The reason for this is the so-called *R-parity*, defined by

\[ R - \text{parity} : \quad (-1)^R = (-)^{2s}(-)^{3(B-L)}. \]
Figure 13: Typical vertices involving superpartners

Figure 14: Creation of superpartners in $e^+e^-$ annihilation

For all known particles $R = 0$, whence for all superpartners $R = \pm 1$. A conservation of $R$-parity has two consequences: superpartners are created in pairs and the lightest one is stable. Usually it is supposed that it is a photino $\tilde{\gamma}$, the superpartner of a photon. A typical vertex is shown in Fig.13. The tilde above a letter denotes a corresponding superpartner. Note that the coupling is the same at all the vertices. The above-mentioned rule together with the Feynman rules for the SM enables us to draw diagrams describing creation of superpartners. One of the most promising processes is $e^+e^-$ annihilation (see Fig.14). Creation of superpartners can be accompanied by creation of the ordinary particles as well.

The decay properties of superpartners depend on their masses. For the quark and lepton superpartners the main processes are shown in Fig.15.

Since $R$-parity is conserved, new particles will eventually end up giving photinos (lightest superparticle) whose interactions are comparable to those of neutrinos and they leave undetected. Therefore the way to notice them is to look for the missing transverse mo-
momentum. This is a signature of SUSY in the nearest future experiments.

*Examples.* We end up with some explicit examples of superpartners decays.

- squarks: \( \tilde{q} \rightarrow q + \tilde{\gamma} \) (quark + photino)
- squarks: \( \tilde{q} \rightarrow q + \tilde{\gamma} \) (quark + gluino)
- sleptons: \( \tilde{\ell} \rightarrow l + \tilde{\gamma} \) (lepton + photino)
- gluino: \( \tilde{g} \rightarrow q + \tilde{q} + \tilde{\gamma} \) (quark + antiquark + photino)
- gluino: \( \tilde{g} \rightarrow g + \tilde{\gamma} \) (gluon + photino)
- wino: \( \tilde{w} \rightarrow e + \nu_e + \tilde{\gamma} \) (electron + neutrino + photino)

If \( M_W > M_{\tilde{w}} + M_{\tilde{\gamma}} \) then the following processes are also possible

\[
W \rightarrow \tilde{w} + \tilde{\gamma}, \quad W \rightarrow \tilde{e} + \tilde{\nu}_e \quad \Rightarrow \tilde{\gamma} + \nu_e + e \quad \Rightarrow l + \tilde{\gamma} \quad \Rightarrow \nu_e + \tilde{\gamma}.
\]

Thus if supersymmetry exists in Nature and if it is broken somewhere below 1 Tev, then it will be possible to detect it in the nearest future.

III Far Beyond The Standard Model

4 Grand Unified Theories

4.1 The Idea of GUTs

The philosophy of Grand Unification is based on a *hypothesis*: Gauge symmetry increases with energy. Having in mind unification of all forces of Nature on a common basis and neglecting gravity for the time being due to its weakness the idea of GUTs is the following:

All known interactions are different branches of unique interaction associated with a
simple gauge group. The unification (or splitting) occurs at high energy

\[ \begin{align*}
SU_e(3) & \otimes \quad SU_L(2) & \otimes \quad U_Y(1) \\
gluons & \quad W, Z & \quad \text{photon} \\
quarks & \quad \text{leptons} & \quad \Rightarrow \quad \text{fermions} \\
g_3 & \quad g_2 & \quad g_1 \\
\Rightarrow & \quad G_{\text{GUT}} & \quad (or \ G^n + \text{discrete symmetry})
\end{align*} \]

At first sight this is impossible due to a big difference in the values of the couplings of strong, weak and electromagnetic interactions. The crucial point here is the running coupling constants. According to the renormalization group equations all the couplings depend on the energy scale \( \alpha_i \equiv g_i^2/4\pi \)

\[ RG : \quad \alpha_i = \alpha_i\left(\frac{Q^2}{\Lambda^2}\right) = \alpha_i(\text{distance}). \]

In the SM the strong and weak couplings associated with non-abelian gauge groups decrease with energy, while the electromagnetic one associated with the abelian group on the contrary increases. Thus it becomes possible that on some energy scale they become equal (see Fig.16). According to the GUT idea this equality is not occasional but is a manifestation of unique origin of these three interactions. As a result of spontaneous symmetry breaking, the unifying group is broken and unique interaction is splitted into three branches which we call strong, weak and electromagnetic interactions. This happens at a very high energy of the order of \(10^{15}\) Gev. Of course, this energy is out of the range of accelerators, however, some crucial predictions follow from the very fact of unification.

4.2 General Features of GUTs

While most of the GUT predictions are model dependent, some general features reflect the very idea of a simple gauge group.

i) Prediction for \(\sin^2 \theta_W\). As far as \(\sin^2 \theta_W\) in the SM is expressed through the ratio of \(SU(2)\) and \(U(1)\) couplings and at the unification point they are equal, the value of \(\sin^2 \theta_W\)
can be calculated. Indeed, consider a part of the SM Lagrangian
\[ g\bar{\psi}\gamma^\mu T_3\psi A_\mu^3 + g'\bar{\psi}\gamma^\mu \frac{Y}{2}\psi B_\mu \]
In GUT it looks like
\[ g_{GUT}(\bar{\psi}\gamma^\mu T_3\psi A_\mu^3 + \bar{\psi}\gamma^\mu T_5\psi B_\mu) \]
Comparing these equations we find
\[ g = g_{GUT}, \quad \frac{1}{2}g'Y = g_{GUT}T_0. \]
Evaluating now the electric charge operator \( Q = T_3 + Y/2 \) and having in mind the normalization condition \( \text{Tr} T_i T_j = 1/2\delta_{ij} \) we finally get
\[ \frac{\text{Tr} T_3^2}{\text{Tr} Q^2} = \frac{g'^2}{g^2 + g'^2} = \sin^2\theta_W. \]
Hence calculating the traces of operators for any representation one can calculate the value of \( \sin^2\theta_W \). We will do it for some particular models below.

ii) **Quantization of electric charge.** If \( Q \) belongs to the generators of \( G_{GUT} \) which is a compact group, then \( \text{Tr} Q = 0 \) for any representation. This means that all the charges are comparable, i.e., are the multiplets of electron charge.

iii) **Baryon No non-conservation.** If quarks and leptons belong to the same irreducible representation of a GUT group, then it would be possible to achieve quark-lepton transitions due to the gauge interactions. They will cause a proton decay, neutron-antineutron oscillations and the other processes with baryon No violation. Fortunately, the rate of proton decay happens to be very small so that the life-time does not contradict the reality.

iv) **Grand desert.** The renormalization group plot shown in Fig.16 is based on an extrapolation of the SM from 100 Gev up to \( 10^{15} \) Gev. It is assumed that there is nothing new in this huge energy region, which is usually called the Grand desert. However, it could be that the Grand desert is not empty but inhabited by some animals like superpartners of ordinary particles, additional heavy Higgses, etc. This would influence the rate of the running couplings, changing some low energy predictions of GUTs. One should take care of this in model building.

To construct some GUT we have to fulfill the following requirements:

1. Grand unifying group should include the group of the SM, i.e.
   \[ G_{GUT} \supset SU(3) \otimes SU(2) \otimes U(1) \]
   This means that \( \text{rank } G_{GUT} \geq 4 \) (recall that \( \text{rank } SU(N) = N - 1 \)).

2. Multiplets of \( G_{GUT} \) should include known quarks and leptons.

3. The theory should be renormalizable and contain no anomalies. Recall that this requirement is fulfilled in the SM by cancellation of anomalies between quarks and leptons of each generation.

4. The scalar multiplets (Higgs fields) should provide spontaneous symmetry breaking in several stages
   \[ G_{GUT} \Rightarrow \cdots \Rightarrow SU(3) \otimes SU(2) \otimes U(1) \Rightarrow M_{GUT} \Rightarrow SU(3) \otimes U(1) \]
   Alternatively \( G_{GUT} \) should include \( G_{TC} \) or else.
4.3 SU(5) GUT - Minimal GUT

SU(5) is a minimal group (rank 4) into which SU(3) \( \otimes \) SU(2) \( \otimes U(1) \) can be embedded and which has complex representations needed for chiral fermions. This group satisfies all the requirements mentioned above. The natural embedding of SU(3) \( \otimes \) SU(2) to SU(5) is

\[
\begin{pmatrix}
\vdots \\
SU_c(3) \\
\vdots \\
\vdots \\
\vdots \\
SU_L(2)
\end{pmatrix}
\]

The only generator of SU(5) commuting with SU(3) and SU(2) is the hypercharge

\[
Y = \sqrt{\frac{3}{20}} \begin{pmatrix}
-2/3 & -2/3 & 0 \\
0 & -2/3 & 1 \\
0 & 0 & 1
\end{pmatrix},
\]

which coincides with that of \( U_Y(1) \) of the SM.

Particle content of the SU(5) GUT is the following:

**Gauge sector.** \( W_\mu = W^A_\mu T^A, \ A = 1, 2, \ldots 24, \ T^A \) are the generators of SU(5). It is a 24-plet which can be represented as a traceless 5 \( \times \) 5 matrix

\[
W_\mu = \begin{pmatrix}
\vdots & X_1^1 & Y_1^1 \\
G_\mu^a \frac{1}{2} - \frac{1}{\sqrt{10}} B_\mu \begin{pmatrix} 1 & 2 \end{pmatrix} & X_2^2 & Y_2^2 \\
\vdots & X_3^3 & Y_3^3 \\
\vdots & \vdots & \vdots \\
X_1^* \begin{pmatrix} 1 \\
2 \end{pmatrix} & X_2^* \begin{pmatrix} 2 \\
1 \end{pmatrix} & X_3^* \begin{pmatrix} 3 \\
1 \end{pmatrix} & \frac{1}{2} A_\mu^3 + \sqrt{\frac{3}{20}} B_\mu & W_\mu^+ \\
Y_1^* \begin{pmatrix} 1 \\
3 \end{pmatrix} & Y_2^* \begin{pmatrix} 3 \\
3 \end{pmatrix} & Y_3^* \begin{pmatrix} 1 \\
1 \end{pmatrix} & -\frac{1}{2} A_\mu^3 + \sqrt{\frac{3}{20}} B_\mu
\end{pmatrix}
\]

Among 24 gauge bosons there are 8 gluons \( G_\mu^a \), 3 weak bosons \( W_\mu^\pm \) and \( A_\mu^3 \) and 1 \( U(1) \) boson \( B_\mu \). There are also 12 new fields \( X_\mu \) and \( Y_\mu \). They are usually called lepto-quarks because they mediate lepto-quark transition leading to baryon No violation. The gauge multiplet has the following SU(3) \( \otimes \) SU(2) decomposition

\[
24 = (8, 1) + (1, 3) + (3, 2) + (3, 2)
\]

**Fermion sector.** All fermions are taken to be left-handed. Right-handed particles are replaced by the corresponding left-handed conjugated. The minimal fundamental representation of SU(5) is \( \bar{5} \). However it is more convenient to use the conjugated one which has appropriate SU(3) \( \otimes \) SU(2) \( \otimes U(1) \) quantum numbers

\[
\bar{5}^* = (3, 1, -2/3) + (1, 2, 1)
\]
It is naturally identified with d-quark and electron-neutrino doublet

\[ 5^* = (d_1^c, d_2^c, d_3^c, e^-, \nu_e)_{Left} \]

To find place for the other members of the same family, we have to go beyond the fundamental representation. Surprisingly, the next (after 5) representation, 10 = (5 × 5)_{asym} has precisely correct quantum numbers

\[ 10 = (3, 2, 1/3) + (3^*, 1, -4/3) + (1, 1, -2) \]

It is a 5 × 5 antisymmetric matrix and its fermion assignment is

\[
\begin{pmatrix}
0 & u_3^c & -u_2^c & u_1 & d_1 \\
0 & u_1^c & u_2 & d_2 \\
0 & u_3 & 0 & d_3 \\
0 & e^+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}_{Left}, \quad u_3^c \rightarrow u_R, \quad e^+ \rightarrow e_R.
\]

Thus, all known fermions exactly fit to (5^* + 10) representations of SU(5). Now new fermions appear. Note that there is no room for the right-handed neutrino \( \nu_R \). Hence either neutrino is massless in the SU(5) model or it could be a singlet that does not take part in gauge interaction. In spite of the left-right asymmetry of the model there is no anomalies in the gauge currents. They automatically cancel between contributions of 5^* and 10.

We can now follow all the general features of GUTs with an example of the minimal SU(5) model. Looking at the electromagnetic current

\[ \bar{e}_L \gamma^\mu \left(-\frac{g_{GUT}}{2} A^\mu - g_{GUT} \sqrt{\frac{3}{20}} B^\mu \right) e_L \]

and comparing it with the SM, we find

\[ g = g_{GUT}, \quad g' = \sqrt{\frac{3}{5}} g_{GUT}, \quad \tan \theta_W = \frac{g'}{g} = \frac{\sqrt{3}}{5} \quad \text{or} \quad \sin^2 \theta_W = \frac{3}{8}. \]

Note, however, that this is true only at \( E = E_{GUT} \sim 10^{15} \text{ Gev} \). Charge quantization is also straightforward. From the requirement \( \text{Tr} \ Q = 0 \) under the assumption that \( SU_{\text{Colour}}(3) \) is exact we obtain

\[ Q_d = \frac{1}{3} Q_e, \quad Q_u = -\frac{2}{3} Q_e. \]

It is interesting to estimate the value of the meeting point \( E_{GUT} \). For this purpose consider the RG equations for the running couplings in the SM assuming that no new particles exist. To the one-loop order we have

\[
\frac{d}{d\mu} g_3 = -\frac{g_3^2}{16\pi^2} (11 - \frac{2}{3} f),
\]

\[
\frac{d}{d\mu} g_3 = -\frac{g_3^2}{16\pi^2} (\frac{22}{3} - \frac{2}{3} f),
\]

\[
\frac{d}{d\mu} g_1 = \frac{g_1^2}{16\pi^2} \frac{2}{3} f,
\]

(4.1)
Figure 17: The variation of $\sin^2 \theta_W$ with energy scale

where the couplings $g_3, g_2$ and $g_1$ belong to the gauge groups $SU(3), SU(2)$ and $U(1)$ respectively, and $f$ is the number of flavours. Solving these equations and imposing the boundary conditions $g_1 = g_2 = g_3 = g_{GUT}$ at $\mu = M_{GUT}$, we get the values of $M_{GUT}, g_{GUT}$ and $\sin^2 \theta_W$. Taking the values of $\alpha_s$ and $\alpha_{EM}$ as input we find (note that $e_{EM}^2 = \frac{4\pi}{12\alpha_{EM}}$)

$$\ln \frac{M_{GUT}}{M_W} = \frac{\pi}{11} \left( \frac{1}{\alpha_{EM}} - \frac{8}{3} \frac{1}{\alpha_s(M_W)} \right),$$

$$\sin^2 \theta_W = \frac{1}{6} + \frac{5}{9} \frac{\alpha_{EM}}{\alpha_s(M_W)}.$$ 

This gives

$$M_{GUT} \simeq 10^{15} \text{ Gev}, \quad \sin^2 \theta_W(M_W) \simeq 0.214, \quad \alpha_{GUT} \simeq \frac{1}{40}.$$ 

It should be stressed once more that the value of $\sin^2 \theta_W$ strongly depends on energy scale. The variation of $\sin^2 \theta_W$ with the energy is shown in Figs.17,18.

One of the successes of the minimal $SU(5)$ model is the prediction of $\sin^2 \theta_W$, which is in very good agreement with experimental data.

Spontaneous symmetry breaking in the $SU(5)$ model occurs in two stages. Within the Higgs mechanism of SSB one introduces two Higgs multiplets: $24$ and $5$. The v.e.v are chosen to be

$$<\Phi_{24}> = \begin{pmatrix} V \\ V \\ -3/2 V \\ -3/2 V \end{pmatrix}, \quad V \sim M_{GUT} \sim 10^{15} \text{ Gev},$$
which breaks $SU(5)$ down to $SU(3) \otimes SU(2) \otimes U(1)$ and

$$< H_5 > = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v/\sqrt{2} \end{pmatrix}, \quad v \sim 250 \text{ Gev},$$

which breaks $SU(3) \otimes SU(2) \otimes U(1)$ down to $SU(3) \otimes U(1)$.

4.4 Hierarchy Problem

The appearance of two different scales $V \gg v$ leads to a very serious problem for GUTs, which is called the hierarchy problem. There are two aspects of this problem.

The first one is the very existence of hierarchy. To get the desired spontaneous symmetry breaking pattern, we need

$$m_H \sim v \sim 10^2 \text{ Gev} \quad m_H \sim 10^{-13} \ll 1.$$ (4.2)

The question arises how to get a so small number in a natural way. One needs some kind of fine tuning in a theory, and we don’t know is there anything behind it.

The second aspect of the hierarchy problem is connected with the preservation of a given hierarchy. Even if we choose the hierarchy like eq.(4.2) the radiative corrections will destroy it! To see how this happens, we consider the interaction of Higgs fields. Using the notation $\Phi_{24} \equiv \Phi, H_5 \equiv H$ we write down the interaction terms. They are

$$\alpha (H^1 H)^2, \quad \beta Tr \Phi^4, \quad \gamma (Tr \Phi^2)^2.$$ 

There is no direct interaction between heavy ($\Phi$) and light ($H$) Higgs particles. However, because of the ultraviolet divergences in the diagrams like that shown in Fig.19 and the
Figure 19: Divergent graph leading to the interaction between light and heavy Higgs fields

\[ \Rightarrow \delta m_H^2 \sim \delta \langle \Phi \rangle^2 \sim v^2 \gg v^2 \]

Figure 20: The diagram contributing to the light Higgs field mass due to the interaction with heavy Higgs particles

necessity of renormalization, the corresponding counter terms appear:

\[ \Delta L = \delta H^4 \Phi^2 H. \]

As a result, the radiative corrections to the light Higgs field mass become inevitable. Corresponding diagrams are shown in Fig. 20. Another source of mass corrections is the interaction with heavy gauge bosons (see Fig.21). We see that the mass corrections happen to be much larger than the masses themselves thus violating the chosen hierarchy. The way out of this trouble is the cancellation of radiative corrections. This very accurate cancellation with a precision \( \sim 10^{-13} \) also needs a fine tuning of the coupling constants. It can be naturally achieved in SUSY models. That is why it is usually said that supersymmetry solves the hierarchy problem. However, we see that it solves only the second aspect of the problem. The origin of the hierarchy remains unclear.

\[ \Rightarrow \delta m_H^2 \sim g^2 M_V^2 \sim v^2 \gg v^2 \]

Figure 21: The diagram contributing to the light Higgs field mass due to the interaction with heavy gauge bosons
4.5 Non-minimal Models

Before considering the other Grand Unified Theories let us focus our attention on the problems of the minimal $SU(5)$ GUT. They are:

1. Fermions of the same generation belong to different representations ($\bar{5}^* + 10$);
2. The mass matrix for quarks and leptons is unsatisfactory ($m_e/m_\mu = m_d/m_s = \ldots$);
3. $B - L$ conservation depends on a Higgs sector and looks occasional;
4. Neutrino is massless?
5. Proton life-time is too small;
6. There exists a Grand desert for $E = 10^2 \div 10^{15}$ Gev.

Trying to solve some of these problems, we briefly consider some more complicated models.

$SO(10)$ GUT

For the $SO(10)$ group, which is a next possible group of rank 5, all the fermions of the same generation belong to a single irreducible representation $16$

$$16 = (u_1 u_2 u_3 d_1 d_2 d_3 \nu_e e^- u_1^c u_2^c u_3^c d_1^c d_2^c d_3^c \nu_e^c e^+)_{left}$$

Note that contrary to the $SU(5)$ model the right-handed neutrino (left-handed antineutrino) is present now. This means that the neutrino in the $SO(10)$ model is massive.

The symmetry breaking in the $SO(10)$ model can be achieved in two different ways and needs at least three different scales $M_1 \gg M_2 \gg \cdots M_W$

$$SO(10) \quad M_1 \quad \vdash \quad SU(5) \quad M_2 \quad \vdash \quad SU(3) \otimes SU(2) \otimes U(1) \quad M_W \quad SU(3) \otimes U(1)$$

$$\downarrow \quad SO(6) \otimes SO(4) \sim SU(4) \otimes SU_L(2) \otimes SU_R(2)$$

If one chooses an $SU(5)$ chain, then $SO(10)$ multiplets find their natural decomposition in terms of that of $SU(5)$

$$16 = 5^* + 10 + 1 \quad \text{fermions},$$

$$45 = 24 + 10 + 10^* + 1 \quad \text{gauge bosons}.$$

In $SO(10)$ GUT we solve the problems No 1,3,4,5 and 6 of the above mentioned list. However, the price for this is a more complicated model with a more freedom but a less predictive power.

$E(6)$ GUT

This model is based on the exceptional group $E(6)$ of rank 6. It is left-right symmetric

$$E(6) \supset SU_C(3) \otimes SU_L(3) \otimes SU_R(3).$$

Fermions belong to a single fundamental representation $27$ which has the following decomposition under $SO(10)$

$$27 = 16 + 10 + 1,$$

while the gauge bosons form an adjoint representation $78$. 

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The model contains a lot of new particles. Its attractiveness is mainly due to appearance of $E(6)$ GUT in superstring inspired models (see below).

**SUSY SU(5) GUT**

Another extension of the minimal $SU(5)$ model is SUSY $SU(5)$. In the minimal version the matter content of the theory is the following:

\[
\begin{align*}
\text{Matter superfields:} & \quad \begin{array}{c}
5^* \psi^i \\
10 \psi_{ij}
\end{array} \quad i, j = 1, 2, \ldots 5 \\
\text{Higgs superfields:} & \quad \begin{array}{c}
5 H_i \\
5^* \bar{H}^i \\
24 \Phi_{ij}
\end{array}
\end{align*}
\]

The superpotential contains several terms:

\[
\begin{align*}
\psi_{ij} \psi_{kl} H_m e^{2ikm} & \quad \text{needed for } SU(2) \text{ breaking and to give masses to quarks and leptons} \\
\psi^i \psi_{ij} \bar{H}^j & \\
\text{Tr } \Phi^2 \text{ Tr } \Phi^3 & \quad \text{needed for } SU(5) \text{ breaking and to give masses to leptoquarks.}
\end{align*}
\]

As was already mentioned, the number of fields here is larger than in the minimal $SU(5)$ model. However due to supersymmetry an interaction between light ($H, \bar{H}$) and heavy $\Phi$ Higgs fields is absent, and as a result, the hierarchy is preserved.

In conclusion we list some properties of various GUT models.

<table>
<thead>
<tr>
<th>Model</th>
<th>$SU(5)$</th>
<th>$SO(10)$</th>
<th>$E(6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Automatic absence of anomalies</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>One reducible representation</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Left-Right Symmetry</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Ambiguities in $\sin^2 \theta_W$ due to different patterns of SSB</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$m_\nu$</td>
<td>$0 \sim \frac{M_{W}^2}{M_X}$</td>
<td>$\frac{M_{W}^2}{M_X}$</td>
<td></td>
</tr>
<tr>
<td>Possibility of dynamical SSB</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

### 4.6 Phenomenological Consequences of GUTs

Looking for phenomenological consequences of GUTs at low energies, one should take into account the existence of a large variety of models. The GUT group is not a priori known. This puts the severe problem of choice. Moreover, even if the group is chosen in all cases except $SU(5)$, there exist several patterns of spontaneous symmetry breaking. This also contributes to the arbitrariness of GUT predictions. However, some phenomenological properties like proton decay, the presence of monopoles, the problem of fermion mass spectrum are common to all GUT models.
**Proton Decay**

The leptoquarks $X_\mu$ and $Y_\mu$ present in any GUT model mediate quark-lepton transitions thus leading to baryon No non-conservation. In $SU(5)$ we have the following interaction vertices

\[ 5^* : \ g \frac{1}{\sqrt{2}} X_\mu^i \bar{d}_R \gamma^\mu e_R^i \]
\[ 10 : \ g \frac{1}{\sqrt{2}} \epsilon_{ijk} X_\mu^i \bar{u}_L^j \gamma^\mu u_L^k \]

The corresponding diagrams are shown in Fig.22. A combination of these two vertices leads to proton decay according to leptoquark exchange process shown in Fig.23. As suggested by Fig.23, the dominant mode is

\[ p^+ \rightarrow e^+ \pi^0. \]  \hspace{1cm} (4.3)

The amplitude is proportional to $g_{GUT}^2/\xi_X^4$, where $M_X \sim M_{GUT}$. Hence the proton life-time behaves like $M_{GUT}^4/g_{GUT}^4$. Then from dimensional considerations one has

\[ \tau_p = C \left( \frac{M_{GUT}}{M_p} \right)^4 \frac{1}{\alpha_{GUT}^2} \frac{1}{M_p}, \]

where $M_p$ is the proton mass. A rough estimate is

\[ M_{GUT} \sim 10^{14} \text{ Gev}, \quad \alpha_{GUT} \sim 10^{-2}, \quad \tau_p \sim 10^{36} \text{ sec} \sim 10^{29} \text{ years}. \]

A more accurate calculation taking into account the structure of the proton gives, with some uncertainties

\[ \tau_p = (0.2 \div 10) \cdot 10^{28} \left( \frac{M_{GUT}}{10^{14} \text{ Gev}} \right)^4 \text{ years}. \]
With account of some ambiguities in determining of $M_{GUT}$ the predicted life-time is

$$\tau_p < 2 \cdot 10^{31} \text{ years},$$

while the modern experimental limit on the decay mode (4.3) is

$$\tau_p > 3 \cdot 10^{32} \text{ years}.$$

This seems to rule out the minimal $SU(5)$ GUT. Predictions for other models strongly depend on the particle content, mainly, on the number of light Higgs bosons. Some results are listed below.

\[
\begin{align*}
\text{Minimal } SU(5) & : \quad \tau_p \sim 10^{27} \div 10^{31} \text{ years} \\
(N \to e^+\pi) : (N \to \bar{\nu}\pi) : (N \to \mu^+K) &= 1 : 0.2 : 0.1 \\
\text{Modified } SU(5) & : \\
\tau_p & \sim 10^{31} \div 10^{34} \text{ years} \\
\text{SUSY } SU(5) & : \\
\tau_p & \sim 10^{31} \div 10^{32} \text{ years} \\
\text{with Higgs mediated } & : (N \to \bar{\nu}K, \mu^+K) : (N \to \mu^+\pi, e^+K) : (N \to e^+\pi, \bar{\nu}\pi) = 1 : 0.1 : 0.01 \\
\text{proton decay} & \\
\end{align*}
\]

**Monopoles**

One of the model-independent predictions of GUTs is the presence of monopoles. The reason is that for any semisimple gauge group $G$ which is broken to a subgroup $H \subset G$ containing an $U(1)$ factor $H = h \otimes U(1)$, there exist topologically non-trivial stable configurations of gauge and/or Higgs fields - monopoles.

In our case the GUT group is broken in exactly the needed way

$$G_{\text{GUT}} \to SU(3) \otimes SU(2) \otimes U(1).$$

At large distances the magnetic field of such configurations behaves like that of the Dirac monopole with magnetic charge $g^* = \frac{2\pi}{e}$. The mass of the monopole is

$$M_{\text{Mon}} \sim \frac{M_X}{\alpha} \sim 10^{16} \div 10^{17} \text{ Gev.}$$

Being so heavy the monopoles cannot be produced at accelerators, however, they could have been created at the early stage of the Universe. Hence as far as they are stable, one may hope to detect them now. One of the characteristic processes to look for monopoles is the monopole-catalysed proton decay

$$p + \text{Mon} \rightarrow e^+ + \text{Mon} (+\text{pions}),$$

$$p + \text{Mon} \rightarrow e^+ + \mu^+\mu^- + \text{Mon} (+\text{pions}).$$

The cross-section of the process is of the order of $(1\text{Gev})^{-2}$ while the probability of proton decay in the presence of a monopole depends on the monopole flux. However, magnetic monopoles are not yet found in Nature.
One of the crucial problems of the SM which is not solved in GUTs is the origin of the fermion spectrum. Practically, in all GUTs the situation with fermion masses is the same

\[
\begin{align*}
E & \sim M_{\text{GUT}} \\
m_d &= m_e \\
m_s &= m_\mu \\
m_b &= m_\tau \\
E & \sim M_W \\
\frac{m_e}{m_\mu} &= \frac{1}{200}, \\
\frac{m_d}{m_s} &= \frac{1}{20}. \\
\end{align*}
\]

The last relation evidently contradicts the experiment

\[
\frac{m_e}{m_\mu} \sim \frac{1}{200}, \quad \frac{m_d}{m_s} \sim \frac{1}{20}.
\]

Solution of this problem requires a new insight in the symmetry breaking mechanism.

In conclusion, we mention some of the problems which have found their solution within GUTs or on the contrary have been put forward by GUTs:

* Group and representation  
* Number of generations  
* Unique gauge coupling  
* Prediction of \(\sin^2 \theta_W\)  
* Quantization of charge  
* Quark and lepton masses  
* Mixing matrix  
* Hierarchy problem

5 Supergravity, Superstrings, etc

Before going further towards higher energy let us first have a look at the high energy physics panorama from the present point of view (see Fig.24).

What is shown here is a simplified picture of what we understand today about high energy physics and how we imagine the microworld structure at very small distances.

In this section we are going to discuss possible physics beyond the Planck scale \((10^{19} \text{ Gev} \text{ or } 10^{-33} \text{ cm})\). This is the region where gravity becomes essential and comparable in strength with other interactions. So, one cannot ignore it any more. Because of lack of the quantum theory of gravity due to ultraviolet divergences and much softer ultraviolet behaviour of supersymmetric theories, the hopes are associated with a supersymmetrical generalization of gravity - the supergravity.

5.1 Supergravity

Supergravity (SUGRA) is the theory of local supersymmetry. The motivations for SUGRA are purely theoretical:

i) All fundamental symmetries in Nature are local except supersymmetry. It is natural to gauge supersymmetry as well.

ii) Superalgebra contains translations

\[
[\bar{\epsilon} Q, \bar{Q} \epsilon] = 2 \bar{\epsilon} \sigma^\mu \epsilon P_\mu.
\]

If they are local, i.e. \(\epsilon = \epsilon(x)\), we get local translational invariance with a parameter

\[
a_\mu = \bar{\epsilon} \sigma^\mu \epsilon P_\mu.
\]
Figure 24: High energy physics panorama
But the theory invariant under the general coordinate transformation is General Relativity. Hence, local SUSY is the theory of (super)gravity.

iii) In SUSY GUTs the unification scale approaches the Planck scale where gravity cannot be neglected.

iv) Ultraviolet behaviour of SUGRA is much better than in ordinary gravity.

Anyhow much hopes in construction of quantum gravity have been connected with the development of SUGRA.

As is well known, gauging any symmetry leads to the appearance of appropriate gauge fields. They are proportional to the derivatives of local gauge parameters and have the corresponding quantum numbers:

<table>
<thead>
<tr>
<th>gauge parameter</th>
<th>gauge field</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vartheta \to \vartheta(x)$</td>
<td>$\psi_{\mu}(x) \sim \partial_{\mu}\vartheta(x)$</td>
</tr>
<tr>
<td>scalar (spin 0)</td>
<td>vector (spin 1)</td>
</tr>
<tr>
<td>$a_{\nu} \to a_{\nu}(x)$</td>
<td>$g_{\mu\nu}(x) \sim \partial_{\mu}a_{\nu}(x)$</td>
</tr>
<tr>
<td>vector (spin 1)</td>
<td>tensor (spin 2)</td>
</tr>
<tr>
<td>$\varepsilon^{\mu}<em>{\nu} \to \varepsilon^{\nu}</em>{\nu}(x)$</td>
<td>$\psi^{\nu}<em>{\mu\alpha} \sim \partial</em>{\mu}\varepsilon^{\nu}_{\alpha}(x)$</td>
</tr>
<tr>
<td>spinor (spin 1/2)</td>
<td>spin-tensor (spin 3/2)</td>
</tr>
</tbody>
</table>

Thus, gauging supersymmetry leads to appearance of a new particle of spin 3/2 called the gravitino, which is a superpartner of the graviton. This is a crucial prediction of the SUGRA theory. Gravitinos are always present in any SUGRA model playing an essential role in anomaly and UV divergences cancellation.

$N = 1$ Supergravity

This is the simplest version of SUGRA. It has only one supersymmetry generator, and hence, only one gravitino field. The particle content of the model is the following:
- One spin 2 state $g_{\mu\nu}$ - graviton,
- $N = 1$ spin 3/2 state $\psi_{\mu\alpha}$ - gravitino,
- Matter supermultiplets.

The Lagrangian of pure $N = 1$ SUGRA consists of two parts. The first one is the General Relativity Lagrangian

$$\mathcal{L}_G = -\frac{1}{2K^2}\sqrt{-g} R = \frac{1}{2K^2}eR, \quad (5.1)$$

where $g_{\mu\nu}$ is the metric tensor, $g = det g_{\mu\nu}$, $R$ is a scalar curvature, $K^2 = G$ is the Newton gravitational constant and $e^{\mu}_{\nu}$ is a vierbein defined by

$$g_{\mu\nu} = e^{\mu}_{\rho} e^{\nu}_{\sigma} \eta_{\rho\sigma},$$

$\eta_{mn}$ being the Minkowski metric tensor.

The second part is the Rarita-Shwinger Lagrangian for the spin-tensor gravitino field

$$\mathcal{L}_{RS} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu\nu} \gamma_\rho \gamma_\sigma D_\rho \psi_\sigma, \quad (5.2)$$

where the covariant derivative

$$D_\rho = \partial_\rho + \frac{1}{2} \gamma_{\rho}^{\mu\nu} \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} = \frac{1}{2} \{\gamma_\mu, \gamma_\nu\}$$
and $\omega^m_\rho$ is the spin connection expressed in terms of the vierbein $e^m_\mu$.

The total Lagrangian

$$\mathcal{L}_{\text{SUGRA}} = \mathcal{L}_G + \mathcal{L}_{RS}$$  \hspace{1cm} (5.3)

is invariant under local SUSY transformations

$$\delta e^m_\mu = \frac{K}{2} \bar{\epsilon}(x) \gamma^m \psi_\mu,$$

$$\delta \omega^m_\rho = 0,$$

$$\delta \psi_\mu = \frac{1}{K} D_\mu \bar{\epsilon}(x) = \frac{1}{K} (\partial_\mu + \frac{1}{2} \omega^m_\mu \gamma_m) \bar{\epsilon}(x).$$  \hspace{1cm} (5.4)

SUSY algebra (5.4) is closed only "on-shell". For the "off-shell" formulation like in global SUSY theories, a set of auxiliary fields is needed.

To construct a realistic model, one should add to Lagrangian (5.1 - 5.3) the terms describing a supersymmetric matter. We will not do it here because of complexity of formulae.

As was mentioned above, the quantum theory of gravity is non-renormalizable. In pure gravity the divergences are cancelled only in the one-loop order on mass shell. If matter fields are added, these cancellations take place only in supergravity. However, even there cancellation fails in higher loops. Possible improvement of the UV behaviour is associated with extended supergravities. Maximally extended $N = 8$ SUGRA is believed to be finite up to eight loops. Unfortunately, we have no complete formulation of extended supergravities. Presumably they require an infinite number of auxiliary fields.

From the modern point of view supergravity as well as super-gauge models are only effective theories being the low energy limits of superstrings.

### 5.2 Superstrings

The superstring theory is the most ambitious theory in particle physics. Two years ago it was sometimes called the Theory of Everything. It pretends to describe the origin of particle physics being the ultimate fundamental theory.

What is it, superstring? The basis of a superstring is the classical relativistic one-dimensional object - a string. Like an ordinary violin string, a superstring has a sequence of vibrational modes. A single string possesses an infinite series of such normal modes, i.e. an infinite series of massive states in local quantum field theory. These normal modes are identified with the ordinary particles. The mass spectrum is quantized with

$$\Delta m^2 \sim T,$$

where $T$ is the string tension, the only dimensional parameter of the string theory. In superstring models it is supposed that $T \sim (10^{19} \text{ Gev})^2$.

The strings could be open or closed as shown in Fig. 25. Examining the spectrum of string states, we find that for an open string the massless states contain spin 1 bosons which are identified with the gauge bosons of a corresponding local gauge group. For a closed string such bosons have spin 2 and, therefore, are identified with gravitons. As far as due to the string interactions these two types of strings are transformed into one another, the gravitational and gauge interactions become intimately connected having the same origin. This way in a string theory a unification of all forces takes place.

There exist now several string theory models. Only few of them are mathematically self-consistent. One of the problems is the appearance of tachyons. To avoid unphysical
Figure 25: Examples of open and closed strings

Figure 26: Spectrum of the Heterotic string model
tachyon states, supersymmetry is needed. We present here, for illustration, the spectrum of one of the most popular models, the heterotic string model (see Fig.26).

For $p \ll 10^{19}$ Gev or $x \ll 10^{-33}$ cm all the massive states "decouple" leaving us with effective point-like field theory of (super)gravity and (super)Yang-Mills with fixed parameters and particle content. Observed particles (quarks, leptons, gauge bosons, etc) should be among massless excitations ($m \ll 10^{19}$ Gev).

(Super)String Dynamics

The dynamics of a string theory is described by the least action principle. In analogy with a point-like particle, where the action $S$ is the length of a world line, for the string the action is the square of the world sheet (see Fig. 27). The action for a string is essentially that of the two-dimensional $\sigma$-model

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \eta_{\mu\nu} \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu + \text{0 - terms},$$

where $\eta_{\mu\nu}$ is the metric of $D$-dimensional "target space", $g_{\alpha\beta}$ is the metric of the world sheet, $\alpha, \beta = \sigma, \tau$. This is the so-called first quantized approach to the string theory.

To define a viable quantum field theory, some conditions should be imposed:

i) Modular invariance (reparametrization invariance of the world sheet);
ii) Conformal invariance (the underlying theory is a 2-D conformally invariant field theory);
iii) Absence of anomalies.

The absence of conformal anomalies puts a strong constraint on a target space:

$$D = 26 \quad \text{for bosonic string},$$

$$D = 10 \quad \text{for fermionic or superstring}.$$  

Now it is realized that the additional dimensions may correspond to internal degrees of freedom which eventually give rise to the gauge and other symmetries observed in the $D = 4$ world (remind the Kaluza - Klein idea).

iv) Supersymmetry (provides the absence of tachyons and finiteness(?) of a superstring model).

The Kaluza - Klein Idea

The necessity of considering higher space-time dimensions in the superstring approach renewed the interest in the Kaluza-Klein idea of 50 year ago. To begin with, we consider the simplest case.
Imagine the General Relativity but in 5 dimensions. The interval now is

\[ ds^2 = \hat{g}_{MN} \, dx^M dx^N, \]

where \( x^M = (x^\mu, y) \), \( \mu = 1, 2, 3, 4 \). The \( D = 4 \) interpretation of 5-D metric is the following:

\[
\begin{align*}
\hat{g}_{\mu\nu} &= g_{\mu\nu}(x) \quad \text{tensor} \quad \text{spin} \ 2 \quad \text{graviton} \\
\hat{g}_{\mu5} &= A_\mu(x) \quad \text{vector} \quad \text{spin} \ 1 \quad \text{photon} \\
\hat{g}_{55} &= \phi(x) \quad \text{scalar} \quad \text{spin} \ 0 \quad \text{dilaton}
\end{align*}
\]

(5.5)

If so, then Einstein equations in \( D = 5 \) give Einstein + Maxwell + Klein-Gordon equations in \( D = 4 \). Thus, 5-D gravity theory describes gravity plus the \( U(1) \) gauge theory (electrodynamics) plus the scalar field theory in \( D = 4 \). In the framework of this approach the gauge invariance (charge conservation) is a consequence of general coordinate invariance (energy conservation), i.e. one has a remarkable unification of gravity with electromagnetism.

A natural question arises: what is the meaning of the fifth dimension? Why it is not observed yet? To answer these questions, let us suppose that the topology of space is

\[ R^4 \times S^1, \]

where \( R^4 \) is the Euclidean (Minkowski) 4-D space and \( S^1 \) is one-sphere or a circle. This means that the fifth dimension is compact

\[ -\infty < x^\mu < \infty, \quad 0 \leq my \leq 2\pi, \]

where \( m \sim 1/R \) and \( R \) is the radius of compactification (see Fig.28 ). Performing the Fourier expansion in the fifth coordinate we have

\[ \hat{g}_{MN}(x, y) = \sum_{n=-\infty}^{\infty} \hat{g}^{(n)}_{MN}(x) e^{imny}. \]

According to eq.(5.5), \( n = 0 \) states correspond to massless chargeless particles (graviton, photon, dilaton), while \( n \neq 0 \) states are massive with \( m_n = n \cdot m \) and charges \( q_n = n \cdot K \cdot m \), where \( K = \sqrt{G} \), \( G \) being the Newton gravitational constant. If the fundamental charge is that of electron

\[ e = K \cdot m, \]

then \( m \sim 10^{19} \text{ GeV}. \)

To incorporate the other interactions in the Kaluza-Klein scenario, i.e. to get strong, weak and electromagnetic interactions of the SM from higher-dimensional gravity, we have to substitute

\[ U(1) \Rightarrow G \supset SU(3) \otimes SU(2) \otimes U(1) \]
This needs the dimension $D$ higher than 5, i.e. the circle should be replaced by a compact space of $R$ dimensions ($R$ depends on the rank of the group $G$) with $G$-symmetry.

$$x^M = (x^u, y^m), \quad m = 1, 2, \ldots, R.$$ 

For example: $R$-sphere $S^R$ with $SO(R+1)$ symmetry.

Now gravity in $D = 4 + R$ dimensions leads to gravity plus non-abelian gauge interactions in $D = 4$:

$$\hat{g}_{MN} = \begin{cases} 
g_{\mu\nu} & \text{graviton} 
g_{\mu n} & Y - M \text{ gauge bosons} 
g_{\mu n} & \text{scalar particles} 
\end{cases} + \text{heavy (> } M_{Pl} \text{) states}$$

(Higgs bosons)

These ideas find their realization in superstring models giving rise to the universal description of gravitational ($N = 1$ supergravity) and gauge ($N = 1, 2$ super Yang-Mills) interactions.

String Interactions

In the superstring theory, the string is considered as a fundamental object at very small distances. The string interactions are then the origin of all point-like particle interactions at low energy. As for the string interactions themselves, they have a topological origin, being a result of the complicated topology of the world sheet.

Consider the scattering of two open strings. The world sheet corresponding to this process is shown in Fig.29. Topologically it is equivalent to a sphere with four points A, B, C, and D associated with emission of strings. A sphere has a trivial topology. Any contour on the surface of a sphere can be continuously deformed into a point (see Fig.29). It is said that a sphere has a genus (it is roughly speaking the number of handles) equal zero. However, to describe the scattering of strings, one should also take into account more complicated world sheet topologies with higher genus. For example, a torus has genus one. A torus is a sphere with one handle. There are two contours on the surface of a torus that cannot be continuously deformed into a point (see Fig.30). (Of course, what is shown is a three dimensional torus. The author is apologizing for not being able to draw a ten-dimensional one.)

If one attentively looks at Fig.30, one finds that the world sheet corresponding to the interacting strings can be divided into several parts consisting of two elements: the triple vertices (the trousers) and the pipes. There is a rigorous mathematical statement that any closed oriented surface can be constructed of these two elements. This means
that the string interactions are always cubic. What is essential is that, in spite of the non-local nature of strings, their interactions are local. However, the interaction point is not well defined. It depends on the position of an observable (see Fig.31). This explains, in particular, why superstring models have no severe ultraviolet divergences. This local cubic interaction gives rise to the local particle interactions of the corresponding low energy effective theory. Higher order contact terms appear there as effective ones, just like in the SM we get an effective four-fermion interaction.

5.3 Superstring Models

At the beginning of the "string revolution" the number of string models was very limited. Mathematical self-consistency happened to be a very restrictive requirement that was considered in favour of the string approach. Later on it has been realized that there are much more possibilities, and investigation is still continuing. Our aim now is to mention some of the most popular models. To begin with, we consider a block-scheme
of fundamental interactions which is associated mainly with the heterotic 10-dimensional string theory, however, its main features remain true in general (see Fig.32).

The fundamental object of the theory is a superstring. It lives in a $D$-dimensional space-time ($D = 10$ for the model at hand). Extra dimensions are compactified on some scale giving rise to a tower of states with a quantized mass. Another result of compactification is the appearance of a local gauge group. Going down with energy after decoupling of heavy massive modes, we end up with a point-like quantum field theory which is supergravity and super GUT. As a result of symmetry breaking on lower energy scales, they lead to Einstein gravity and a supersymmetrical generalization of the SM, respectively. Some extra gauge symmetries may also appear. Further on the situation is more or less standard.

We see that the advocated scheme contains some new particles and forces (shown in italic) which are absent in the SM. This is a usual prediction of superstring theories. We consider now some examples of string models in more detail.

**10-D Heterotic String**

This is one of the first and still reliable models based on a heterotic construction. Left and right moving states in this approach are considered separately. The left movers are those of a 26-dimensional bosonic string theory and right movers belong to a 10-dimensional fermionic string. Compactification occurs in two stages as shown in Fig.33. At first stage the 26 dimensions of the left movers are compactified to ten with the remaining 16 dimensions being internal ones, thus giving rise to $E_8 \otimes E_8$ gauge group.
(\(D = 26\) Bosonic String)\(_{Left\ Movers}\) \(\times\) (\(D = 10\) Fermionic String)\(_{Right\ Movers}\)

\[\downarrow\]

Compactification

\[\downarrow\]

\(D = 16\) "Internal" + \(D = 10\) Bosonic)\(_{LM}\) \(\times\) (\(D = 10\) Fermionic)\(_{RM}\)

\[\downarrow\]

Anomaly Cancellation ⇒ Modular Invariance

\[\downarrow\]

\((E_8 \otimes E_8\ Internal\ Symmetry) \times (D = 10\ Superstring\ Theory)\)

\[\downarrow\]

Compactification (Calabi – Yau, Orbifold)

\[\downarrow\]

\((E_6 \otimes E_8\ Internal\ Symmetry) \times (N = 1\ Supersymmetry\ in\ D = 4)\)

Figure 33: Calabi-Yau type compactification

\[\downarrow\]

Compactification

\[\downarrow\]

\(D = 22\) "Internal" + \(D = 4\) Bosonic)\(_{LM}\) \(\times\) (\(D = 6\) "Internal" + \(D = 4\) Fermionic)\(_{RM}\)

\[\downarrow\]

Anomaly Cancellation ⇒ Modular Invariance

\[\downarrow\]

\(SO(44) \times (N = 4\ Supersymmetry\ in\ D = 4\ or\ smaller\)\)

Figure 34: 4 - D superstring construction

The resulting theory is the anomaly-free 10-dimensional supersymmetric string model. At the subsequent stage of compactification to 4 space-time dimensions the residual gauge group becomes \(E_6 \otimes E_8\) (for the Calabi-Yau compact spaces) and \(N = 1\) supersymmetry is preserved. The "predicted" gauge group of associated GUT is \(E_6\), while another \(E_8\) factor is supposed to be confining on a very high energy scale and to describe the shadow matter.

4-D Superstrings

Recently alternative schemes of compactification have been proposed which do not require two steps of compactification. They are known as 4D strings (see Fig.34).

In this case the left moving states and right moving states are compactified simultaneously to the required 4 flat dimensions. The gauge symmetry which has appeared as a result of this procedure is much larger than in the previous case. Groups up to \(SO(44)\) with four supersymmetries are possible. Thus, the uniqueness of the gauge group prediction which was treated as an achievement of the string theory is lost now. It is hoped that some other criteria for making a definite choice will be found in future.

Although the investigations towards the building of realistic models are in progress, the modern phenomenology is essentially based on \(E_6\) symmetry. However, in view of the
above discussion this should be considered as a representative rather than an exhaustive example.

5.4 Superstring Inspired GUTs

Now at least three viable three-generation models have been developed based on different compactification schemes. Remarkably, these schemes give rise to very similar phenomenological signals, basically the supersymmetric SM with a few additional neutrino or lepton-like states and possibly a richer Higgs structure. The problems of inhibiting nucleon decay and generating an acceptable mass pattern have been solved in these models either via new discrete symmetries or by additional gauge symmetries which are expected to be broken on a high scale.

\[ E_8 \otimes E_8 \xrightarrow{\text{Calabi-Yau compactification}} E_6 \otimes E_8 \]

The subgroup of \( E_6 \) is \([SU(3)]^3\), so the group of the SM must be embedded in this group structure

\[
E_6 \supset SU_C(3) \otimes SU_L(3) \otimes SU_R(3)
\]

\[
\supset SU_C(3) \otimes SU_L(2) \otimes U_Y(1)
\]

We show below the \([SU(3)]^3\) decomposition of \( E_6 \) supermultiplets \( \mathbf{78} \) and \( \mathbf{27} \) which should contain the known gauge bosons and quarks and leptons, respectively. (The underlined states are not in the supersymmetric SM.)

**Gauge boson - Gaugino supermultiplets**

\[
\mathbf{78} = (8,1,1) + (1,8,1) + (1,1,8)
\]

Gluons | \( W^\pm, Z, \gamma \) | \( 8 \) new states | (+ Gauginos)

\[
(3,3,3) + (3^*,3^*,3^*)
\]

\( J = 1 \) | Leptoquarks | \( \cdots \) Must be heavy | (+ Inos)

**Fermion - Sfermion supermultiplets**

\[
\mathbf{27} = (1,3,3^*) + (3^*,3^*,1) + (3,1,3)
\]

\[
\begin{align*}
Higgs + Higgsinos H_{1,2} & : (H_1^0, H_2^0, e^+) \\
Leptons + Sleptons e, \nu, N & : \left( \begin{array}{ccc} H_1^- & H_2^- & \bar{\nu}_e \\
\nu_e & N \end{array} \right) \\
Quarks + Squarks u, d, D & : \left( \begin{array}{ccc} u \\
d \\
D \end{array} \right)_L + \left( \begin{array}{c} u \\
d \\
D \end{array} \right)_R
\end{align*}
\]

A specific realization of the compactification procedure will give more details about the low energy structure of the model: the number of families, the low energy gauge group, the masses and couplings.

**Minimal Rank 5 Model**

This is the model where \( E_6 \) is broken on the compactification scale by Wilson loops, i.e. by topologically non-trivial configurations of the gauge fields. In this case one finds...
\( M_c \sim 10^{18} \text{Gev} \)

\( E_8 \otimes E_8 \quad \Rightarrow \quad E_8 \otimes SU(3) \otimes SU(2) \otimes U(1) \otimes U'(1) \times [N = 1 \text{ SUSY}] \)

Breaking via \( \downarrow \)

Wilson loops \( \downarrow \)

Confining \( \downarrow \)

\( m \sim 10^{18} \text{Gev} \)? \( \downarrow \)

\( \Rightarrow \text{Standard Model} + \)

\( \downarrow \text{A new neutral current} \)

Radiative SUSY \( \downarrow \quad < H_1^0 >, < H_2^0 > = O(10^{2} \text{Gev}) \)

Breaking \( O(10^{3} \text{Gev})? \) \( \downarrow \text{Electroweak Breaking} \)

\( SU_c(3) \otimes U_{EM}(1) \)

Figure 35: Minimal rank 5 superstring inspired model

that only 2\( T \) representation survive and the \( E_8 \) group breaks, but not further than to the rank 5 group \( SU_C(3) \otimes SU_L(2) \otimes U(1) \otimes U(1)' \) with one additional \( Z' \) boson whose mass should be of the order of the electroweak breaking scale. The breaking pattern is shown in Fig.35.

The light states \( (m \leq O(M_W)) \) are

\[
3 \times \begin{pmatrix}
\begin{pmatrix} u \\ d \\ D \end{pmatrix}_L + \\
\begin{pmatrix} u \\ d \\ D \end{pmatrix}_R + \\
\begin{pmatrix} H_1^0 \\ H_2^{+} \\ e^+ \end{pmatrix}_R + \\
\begin{pmatrix} H_1^{-} \\ H_2^0 \\ \bar{\nu}_\tau \end{pmatrix}_R
\end{pmatrix}
\]

The new states are underlined.

Three-Generation Calabi-Yau Model

Another possibility is to allow Higgs fields to acquire very large intermediate scale vacuum expectation values, thus breaking the gauge group. It happens to be energetically favourable in the three-generation Calabi-Yau model. Because this model has the known Calabi-Yau manifold, many details of its properties are known, including the multiplet structure, the number of generations, etc. The pattern of symmetry breaking leads now to the standard SUSY model without new \( Z' \) boson (see Fig.36).

The light states \( (m \leq O(M_W)) \) here are

\[
3 \times \begin{pmatrix}
\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} \nu \\ l \end{pmatrix}_L, \begin{pmatrix} t \end{pmatrix}_L + \\
\begin{pmatrix} H_1^+ \\ H_0^0 \end{pmatrix}_L + \\
\begin{pmatrix} H_2^0 \end{pmatrix}_L + \\
\begin{pmatrix} \text{Three neutral} \\ \text{One charged} \\ \text{Leptons} \end{pmatrix}
\end{pmatrix}
\]

The Yukawa couplings in this case can be calculated as integrals over the Calabi-Yau manifold. It is argued that the pattern of symmetry breaking is possible leading to quark and lepton masses consistent with experiment.

We list below some general properties of the superstring inspired models:

1. \( N \geq 1 \) supersymmetry;

2. The number of generations \( N_F \) (of \( E_8 \)) is defined by the topology of the compact space

\[
N_F = \frac{1}{2} |Euler \ characteristics| \]

The "realistic" Calabi-Yau spaces have \( N_F \sim 3, 4; \)

180
$M_c \sim 10^{18}\text{GeV}$

$E_8 \otimes E_8 \quad \Longrightarrow \quad E_8 \otimes SU(3) \otimes SU(3) \quad \otimes \quad SU(3) \times [N = 1 \text{ SUSY}]$

Wilson loops

↓

↓

$< \nu_R > \sim 10^{17}\text{GeV}$

$< N > \sim 10^{15}\text{GeV}$

$\downarrow$

Intermediate

$\downarrow$

Breaking

$SU(3) \otimes SU(2) \otimes U(1)$

$\downarrow$

Standard Model

$\downarrow$

$H_1^0 >, < H_2^0 > \neq 0$

$\downarrow$

Breaking $O(1\text{GeV})$

$SU_c(3)$

$\otimes$

$U_{EM}(1)$

Figure 36: Three-generation Calabi-Yau superstring model

3. Symmetry breaking pattern may lead to some additional symmetries (new neutral currents);

4. All Yukawa couplings of low energy theory are defined by topology and can be calculated;

5. There should exist stable heavy particles with $M \sim M_{Pl}$ and rational electric charges $e/R$ and magnetic monopoles with $g = \frac{2\pi}{e} R$.

It would be ridiculous, at this stage, to claim that any model is the effective low energy theory following from the superstring. But its existence does show that the low energy structure emerging from the underlying compactified theory can closely resemble the observed world.

It was believed some time ago that with the superstring theory we have already found a unique fundamental theory of Nature. Now it is realized that this is not a single theory, rather we have opened a window in a new wonderful and exiting world of superstrings. You are welcome!

IV References

Suggested Reading:
EXPERIMENTAL PROGRAMME AT IHEP ACCELERATOR:
STATUS AND DEVELOPMENT

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Contents
1. Meson spectroscopy (GAMS, LEPTON, VES, SPHYNKS, SVD).
2. Polarization study (PROSA).
3. Neutrino experiments (SCAT, IHEP-JINR ND, TNF).
4. Rare decays etc. (ISTRA, HYPERON, FODS-2).
5. UNK (3 TeV f.t., pp-, pp-collider, status).
6. UNK experimental facilities.
7. UNK first priority experiments.

1. Meson spectroscopy

At the 70 GeV IHEP proton synchrotron a search for new particles and their study is carried out with the help of such the facilities as multi-gamma spectrometer GAMS-2000, spectrometer LEPTON, VERtex Spectrometer (VES), spectrometer SPHYNKS, Spectrometer with Vertex Detector (SVD). Two more spectrometers BIS-2 and MIS-2 are also searching for new particles. Their results have been presented in the lecture "JINR Programme".

GAMS programme is realized with GAMS-2000 at IHEP and GAMS-4000 at CERN. The main goal of the experiments is to look for and to study neutral final states produced in charge exchange reactions

\[ \pi^- p \rightarrow N^0_n \]

\[ \rightarrow \pi^0 \pi^0, \omega \omega, \eta \eta, \eta' \eta \rightarrow \gamma' \gamma' \]

and in central collisions

\[ \pi^- p \rightarrow \pi^- p N^0 \]

\[ \rightarrow p_1 p_2 \]

where \( p_1 \) and \( p_2 \) are pseudoscalar mesons.

The layout of GAMS-2000 is shown in Fig.1.
A number of new particles have been observed in the experiments at 38 GeV (IHEP) and at 100 and 300 GeV (CERN). They are:

\textit{r(2510)-meson} \ [f_6(2510)] \ with \ M=2510\pm30 \ MeV, \ \Gamma=240\pm60 \ MeV, \ \ J^{PC}=6^{++}, \ I^0=0^+.

\textit{G(1590)-meson} \ [f_0(1590)] \ with \ M=1592\pm25 \ MeV, \ \Gamma=210\pm40 \ MeV, \ \ J^{PC}=0^{++}, \ I^0=0^+.

The G-meson is observed in the \(\eta\eta\), \(\eta'\eta\), \(4\pi^0\) decay modes at the following ratios:

\[ BR(G \to \eta'\eta)/BR(G \to \eta\eta) = 2.7 \pm 0.8 \]
\[ BR(G \to \pi^0\pi^0) < 0.3 \cdot BR(G \to \eta\eta) \]
\[ BR(G \to 4 \pi^0)/BR(G \to \eta\eta) = 0.7 \pm 0.2 \]

The quantum numbers and decay ratios show that G(1590) is a candidate for a glueball.

\textit{X(1810)-meson} \ [f_2(1810)] \ with \ M=1808\pm10 \ MeV, \ \Gamma=185\pm20 \ MeV, \ \ J^{PC}=2^{++}, \ I^0=0^+.

\[ BR(X \to 2 \pi^0) < 0.2 \cdot BR(X \to 4 \pi^0) \]

X(1810) with such properties is a candidate for a tensor glueball.

A search for other new particles decaying into \(\eta\eta\), \(\eta'\eta\) and \(\omega\) was also carried out. The results are:

\textit{X(1750)} \ [f_0(1750)] \ \to \ \eta\eta, \ M=1755\pm 8 \ MeV, \ \Gamma<50 \ MeV, \ J^{PC}=0^{++} \ or \ 2^{++}, \ I^0=0^+.

\textit{X(2220)} \ \to \ \eta'\eta, \ M=2220\pm10 \ MeV, \ \Gamma<80 \ MeV.

If \(X(2220)\) is \(qq\)-meson, \(J^{PC}=2^{++}\) or \(4^{++}\).

\textit{X(1640)} \ \to \ \omega\omega, \ M=1643\pm7 \ MeV, \ \Gamma<70 \ MeV,

\textit{X(1960)} \ \to \ \omega\omega, \ M=1956\pm20 \ MeV, \ \Gamma=220\pm60 \ MeV, \ \ J^{PC}=2^{++}, \ I^0=0^+.

One of possible interpretations of \(X(1640)\) and \(X(1960)\) is that they are excited \(f_2(1270)\).

In central collisions the \(X(1920)-\)meson was observed. Its parameters are: \(M=1917\pm 15 \ MeV, \ \Gamma=90^{+35}_{-50}, \ \ J^{PC}=0^{++}, \ 1^{+} \ or \ 2^{++}, \ I^0=0^+.

\[ BR(X \to \eta\eta, \pi^0\pi^0, K_SK_S) < 0.10 \cdot BR(X \to \eta'\eta). \] \(X(1920)\) is a candidate for exotic state.

Experiment with \textit{LEPTON} spectrometer was aimed to search for and to study rare decays of the known mesons. The data were also analysed to study the reaction \(\pi^-p \to (K^+K^-\pi^0)n\) at 32.5 GeV. In the mass spectrum of \(K^+K^-\) a peak which corresponds to \(\varphi\)-meson was observed. To study the reaction \(\pi^-p \to (\varphi\pi^0)n\) the events with \(1016 < M_{K^+K^-} <\)
1024 MeV were selected. In the $\phi\pi^0$ mass spectrum a peak with $M=1480\pm40$ MeV and $\Gamma=130\pm60$ was observed. The analysis of this state, called $0(1480)$, yielded $J^{PC}=1^{--}$. Its most probable interpretation — an exotic state.

The aim the Vertex Spectrometer is to investigate the multiparticle reaction $\pi^- p \rightarrow n \pi^+\pi^-\pi^+\pi^- m\gamma$, where $m=0,1,2,3,4$.

This reaction is of particular interest for heavy meson spectroscopy. Here heavy glueballs ($\eta\eta'$, $\eta'\eta'$), four-quark mesons ($\rho\rho$, $\rho\omega$, $\omega\omega$), radially excited mesons ($\rho\eta'$, $\omega\eta$ ...) and other objects may be looked for.

The layout of the experiment is shown in Fig.2. The main elements of the set up are liquid hydrogen target, forward block of proportional chambers, multihannel threshold Cerenkov counter, hodoscopes and multicell photon detector of the GAMS type.

At $10^6 \pi^-$/cycle about 150 useful events will be produced in the hydrogen target 30 cm long (the cross section is equal to 100–150 $\mu$b). The geometrical efficiency of the set up is about 50%. About $10^7$ useful events will be detected during 600 hour run.

In 1989 the experiment started to take data.

The experiment with the spectrometer SPHYNKS is aimed to search for exotic states such as multiquark mesons ($qqqq$) and baryons ($qqqqq$), as well as mixed states of hybrid type ($qqg$, $qqgq$).

An exposure with the 70 GeV proton beam is planned to study the diffraction type reaction $p + N \rightarrow R + N$, where $R$ is an exotic baryon or a nonresonance baryon system decaying into exotic meson or baryon.

The $\pi$-meson exposure will logically continue the studies carried out with LEPTON set up. The $\phi\pi^0$-system in the effective mass range up to 3 GeV will be studied and isotopic partners of $0(1480)$, exotic $U$-meson, i.e. the states of the $(uudd)\rightarrow p\pi^0$, $(uuus)\rightarrow AA$, $A\bar{A}\pi^0$ type will be looked for.

The layout of the set up is shown in Fig.3. Spectrometer includes a target with guard system, proportional chambers, a differential wide-aperture multihannel Cerenkov counter for particle identification, magnetic spectrometer, hodoscope multihannel Cerenkov counters $O_1$ and $O_2$.

Data taking was started in 1989.
2. Polarization studies

Polarization studies carried out at IHEP in the recent years, are mainly connected with the PROZA set up (IHEP-JINR-TSU) presented in Fig.4. The very first results of these studies gave significant polarization effects in the charge exchange reactions $\pi^- p \rightarrow X^0 n$, where $X^0 = \pi^0, \eta, \eta', \omega, \phi$. They turned out to be quite unexpected and no existing theoretical models could describe them. The polarization dependence on the momentum transfer has a complicated structure and differs for the reactions with different particles in the final state. Fig.5 presents the polarization dependence on the momentum transfer for the charge exchange reaction $\pi^- \rightarrow \pi^0$.

At present PROZA is used for polarization measurements in the inclusive reactions of $\pi^-$ meson scattering on polarized protons and deuterons. The results on measured asymmetry in $\pi^0$ inclusive production are given in Fig.6. The average asymmetry at $p_t > 2$ GeV/c is equal to (37±11)%. The systematic errors are of order of 15-20%.

3. Neutrino experiments at IHEP accelerator

The current neutrino programme at IHEP is now being realized with bubble chamber SKAT and IHEP-JINR Neutrino Detector. The experiments with Tagged Neutrino Facility (TNF) will start this fall.

The bubble chamber SKAT is a heavy liquid bubble chamber of a classical type with fiducial volume of 2 m$^3$, the mass of the useful target (heavy freon CFP$_3$Br) is 3 t. 12000 CC neutrino events and 3000 CC antineutrino events are accumulated in the experiment. These data were used to measure the total cross sections, inclusive distributions, structure functions, NC to CC ratio, etc. These results have been published. Among recent results are charmed particle production on basis of $\mu e$ events, the test of $\nu_\mu - \nu_e$ universality and the measurements of the Weinberg angle.

The relative cross section for $\mu^+ e^-$ pair production in neutrino interaction at 3-30 GeV (available neutrino energy range at the IHEP accelerator) is equal to $(2.0 \pm 0.4) \times 10^{-3}$. The source of these pairs is semileptonic charmed particle decays. The analysis of the $\mu^- e^+$-events gave the cross section for quasi-elastic $\Lambda_c$ production equal to $(6.2 \pm 3.1)\%$ and oross
section for inclusive $D$-meson production equal to $(3.0 \pm 0.9)\%$ relative to the total cross section.

The ratio of the number of $\nu_\mu N$ events to the number of $\nu_e N$ events normalized to the corresponding fluxes is equal to $(0.98 \pm 0.15)$ which does not contradict $\nu_\mu - \nu_e$ universality.

The following limitation on the mixing angle of $\nu_\mu \leftrightarrow \nu_e$ oscillation has been obtained from the analysis of the $\nu_\mu N$, $\nu_e N$-events

$$\sin^2 \theta < 2.5 \times 10^{-3} \text{ at } \Delta m^2 \approx 60 \text{ eV}^2 \text{ (90\% C.L.).}$$

The experiment to study charmed baryon production by neutrinos has been started using bubble chamber SKAT filled with light propane-freon mixture. This mixture will provide a "pure" proton target and a possibility to perform a kinematic analysis. 3000 events with charmed baryons ($\Lambda_c^+, \Sigma_c^+, \Sigma_c^+, \Xi_c^{++}, \Sigma_c^{++}$) are expected at $3 \times 10^{13}$ proton beam intensity on the neutrino target and $5 \times 10^3$ pictures. Almost half of these events will be produced by neutrino interaction with protons.

**IHEP-JINR Neutrino Detector** (Fig.7) consists of two parts: a target and a muon spectrometer. The target part from 40 alternating layers of liquid scintillation counters and drift chambers is surrounded with frame magnets. The useful mass of the target part is 100 t. The muon spectrometer consists of 18 toroidal magnets 4m in diameter and drift chamber planes between them.

The experimental programme for the Neutrino Detector includes high statistic measurements of the structure functions in the $Q^2$ intermediate range, precise measurement of the Weinberg angle, search for neutrino oscillations, beam-dump experiment. Data taking was started in 1986.

**Tagged Neutrino Facility** (Fig.8) is based on the idea of tagging neutrinos by the decay products of parent $K$-mesons. Its main parts are a high intensity hadron beam channel with a possibility to select kaons, a tagging station to measure the energy and momenta of $K^+$ meson decay products, which would allow one to define neutrino parameters, and liquid argon spectrometer BARS with useful mass 400 t. The coincidence system provides a possibility to detect neutrino interaction in the BARS and to determine its production vertex, energy and type using the tagging station data.
The possibilities of the TNF will permit to carry out unique experimental programme including detailed comparison of $\nu_\mu$ and $\nu_e$ interactions (CC and NC), neutrino oscillation and "classical" neutrino physics with the precision 2-3 times better.

TNF will start to run with neutrino beam this fall.

4. Rare decays, etc.

A number of experiments on search for rare decays have been carried out at the IHEP accelerator. At present the experiments with ISTRA (INR USSR AS) and HYPERON (IHEP-JINR) aimed to search for rare kaon decays are in progress.

ISTRA programme includes the study of the $K^+ \to \pi^+ \pi^0 \gamma$ decay (test of CP violation), $K \to \pi \nu \bar{\nu}$ decay (number of lepton families), $K \to \pi e^+ e^-$ and $K \to \pi \mu^+ \mu^-$ decays (neutral currents, CP violation, "horizontal" interactions etc.). Data taking has been started this year.

The study of positive kaon decays with HYPERON set up has been also started. The programme includes

\begin{align*}
K^+ & \to \pi^+ \pi^0 \gamma \quad (1), \\
K^+ & \to \pi^+ e^+ e^- \quad (2), \\
K^+ & \to \pi^+ \gamma \gamma, \quad (3), \\
K^+ & \to \pi^+ \pi^0 \pi^0 \quad (4) \text{ and} \\
K^+ & \to \pi^+ \pi^0 \pi^0 \quad (5) \text{ decays.}
\end{align*}

A 200 h run is approved as the first stage of the experiment to study decays (1) and (5). The beam intensity is equal to $10^9$ kaons/s. It will allow to increase the existing statistics by one order of magnitude. The study of other decays will depend on these results.

A wide programme on the study of high $p_t$ hadron production by 70 GeV protons (hard scattering processes) has been carried out at IHEP with the double arm focusing spectrometer FODS. A further study is now carrying out with the up-graded spectrometer FODS-2. It is designed to run in proton and pion beams of higher intensity, has larger acceptance and provided with muon identifier. Data taking was started in 1988.

5. IHEP 3 TeV Accelerating and Storage Complex (UNK)

The UNK project foresees the construction of the 3 TeV accelerator for fixed target experiments and 3 X 3 TeV colliding proton-proton (proton-antiproton) beams.
Three stages of the complex will be located inside an underground tunnel 5.1 m in diameter. The circumference of the tunnel is equal to 20 772 m. The first stage of UNK is a 400-600 GeV accelerator with conventional iron magnets (UNK-1), the second stage - a 3 TeV superconducting accelerator for fixed target experiments (UNK-2), the third stage - a 3 TeV superconducting accelerator (UNK-3) which will be used together with UNK-2 to organize colliding proton-proton beams.

Fig. 9 shows the scheme of the magnetic lattice of UNK. There are six straight sections. Two of them (SS 1 and SS 4), each 800 m long, will house technological accelerator equipment (injection, beam loss localization, beam abort, extraction systems, basic facilities to protect SC magnets from irradiation, etc.).

In the remaining four straight sections, each 490 m long, facilities will be installed for experiments with the internal beam and the colliding beams.

The IHEP 70 GeV proton synchrotron, whose intensity is planned to be increased up to $5 \times 10^{13}$ ppp (now it is $2 \times 10^{13}$ ppp), will be used as the UNK injector. The beam from PS-70 is injected into UNK-1, whose circumference is 14 times larger. It allows beam stacking through multipulse injection. For this the 70 GeV proton beam is preliminarily rebunched at UNK accelerating frequency of 200 MHz and is stacked in UNK-1 up to $6 \times 10^{14}$ protons through 12 consequent pulses. The part of the UNK ring remains unfilled thus providing intervals to make easier the operation of the injection and extraction systems.

After stacking the beam in UNK-1 is accelerated up to 400 GeV and in one turn is transported into UNK-2, where it is further accelerated up to 3 TeV.

For the fixed target experiments three extraction modes are foreseen: 40 s slow extraction, fast resonance extraction (ten 1-2 ms long pulses with 3 s interval) and one turn fast extraction of the whole beam. Fast resonance extraction may be combined with the slow one. The total UNK cycle is 120 s long with 40 s field ramp, 40 s flattop and 40 s field drop.

Basic parameters of the UNK as a 3 TeV accelerator are given in Table 1.

UNK-3 is similar in its design to UNK-2. It is also a 3 TeV superconducting accelerator and together with UNK-2 will provide colliding proton-proton beams. UNK-1 will be an injector to UNK-3,
<table>
<thead>
<tr>
<th>Parameter</th>
<th>UNK-1</th>
<th>UNK-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Injection energy, GeV</td>
<td>70</td>
<td>400</td>
</tr>
<tr>
<td>Maximum energy, GeV</td>
<td>400</td>
<td>3000</td>
</tr>
<tr>
<td>Circumference, m</td>
<td>20771.8</td>
<td>20771.8</td>
</tr>
<tr>
<td>Maximum field, T</td>
<td>0.67</td>
<td>5.0</td>
</tr>
<tr>
<td>Number of dipoles</td>
<td>2256</td>
<td>2176</td>
</tr>
<tr>
<td>Length of dipole, m</td>
<td>5.8</td>
<td>5.8</td>
</tr>
<tr>
<td>Number of quadrupoles</td>
<td>560</td>
<td>496</td>
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<tr>
<td>Length of quadrupole, m</td>
<td>3.46</td>
<td>3.03</td>
</tr>
<tr>
<td>Acceleration time, s</td>
<td>11</td>
<td>40</td>
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<tr>
<td>Accelerating voltage, MV</td>
<td>8</td>
<td>11</td>
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<tr>
<td>Energy gain per turn, MeV</td>
<td>2.1</td>
<td>4.5</td>
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</tbody>
</table>

however the injection from PS-70 to UNK-1 and then to UNK-3 will be done in opposite direction. Traces of UNK-2 and UNK-3 intersect in the centres of straight sections SS 2, SS 3, SS 5 and SS 6.

Last fall CERN proposed to use its $\bar{p}$-source for the construction of the **pp UNK collider** on the basis of the first UNK superconducting ring (UNK-2). CERN and IHEP accelerator physicists have started joint study this possibility. Preliminary estimations show that the luminosity of $10^{31} \text{cm}^{-2}\text{s}^{-1}$ looks quite reasonable. The basic parameters of UNK colliding beams (pp and $\bar{p}p$ choices) are given in Table 2.

**UNK Status**

UNK civil engineering is going on. By the first of July more than 14 km of the ring tunnel (67.5%) and 2360 m of the injection channel (95.5%) were bored. Fig.10 shows the tunnel construction status. It is planned to complete the tunneling of the injection channel and to start the installation of the equipment in 1989. Fig.11 shows the part of the tunnel of the injection channel. Boring of the ring tunnel will be finished in 1990.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>PP</th>
<th></th>
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<tr>
<td>Maximum energy, TeV</td>
<td>3 x 3</td>
<td></td>
<td>3 x 3</td>
<td></td>
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<tr>
<td>Beam intensity</td>
<td>2.4 x 10^{14}</td>
<td>6 x 10^{11}</td>
<td>15 x 10^{12}</td>
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<tr>
<td>Number of bunches</td>
<td>8600</td>
<td>66</td>
<td>66</td>
<td></td>
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<tr>
<td>Number of particles per bunch</td>
<td>2.8 x 10^{10}</td>
<td>9 x 10^{9}</td>
<td>2 x 10^{11}</td>
<td></td>
</tr>
<tr>
<td>Invariant transverse emittance, mm*mrad</td>
<td>5</td>
<td>3</td>
<td>5</td>
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<tr>
<td>Invariant longitudinal emittance, MeV*m</td>
<td>10</td>
<td>30</td>
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<tr>
<td>Bunch length, cm</td>
<td>12</td>
<td>20</td>
<td>20</td>
<td></td>
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<tr>
<td>Bunch spacing, m</td>
<td>1.5</td>
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<td>0.2</td>
<td>20</td>
<td>0.2</td>
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<tr>
<td>Colliding time, h</td>
<td>1</td>
<td>24</td>
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<tr>
<td>β-function at int.p., m</td>
<td>1</td>
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<tr>
<td>Luminosity, cm^{-2}s^{-1}</td>
<td>4 x 10^{32}</td>
<td>1 x 10^{31}</td>
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<tr>
<td>Number of events per crossing</td>
<td>0.3</td>
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<td>1</td>
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<td>Free space for detector, m</td>
<td>±20</td>
<td>±20</td>
<td>±20</td>
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</table>

The construction of all the surface utility buildings is also going on. Their completion is planned for 1990.

Fig. 12 shows the cross section of the UNK regular tunnel with equipment of all three stages. The full-scale 30 m long mock-up of the tunnel (Fig. 13) has been made at IHEP. It is used to clarify all the problems concerning the equipment arrangement and mounting in the real tunnel.

UNK-1 equipment is being manufactured in industry and at IHEP. All contracts have been placed.

A wide scale production of the ring magnets is in progress at Efremov Institute in Leningrad. In 1988 125 magnets were delivered.
at IHEP. The contract for production of 750 magnet during 1989 has been signed. The rest of magnets has to be manufactured in 1990. Fig. 14 shows the UNK-1 dipole magnets at IHEP.

In 1988 the vacuum system for injection channel has completely been manufactured and delivered. The vacuum equipment for 30 periods (2.7 km) of the UNK-1 regular structure has been also delivered. During 1989 the vacuum equipment for 60 periods (5.4 km) will be manufactured.

In 1988 the preserial sample of the power supply for the ring magnet was tested at IHEP. The mass production in industry is planned for 1989-1990.

The mass production of RF cavities and waveguide feeders has been started at IHEP.

Designing and modeling UNK-2 (UNK-3) equipment are going on. The main units of the second and third stages are superconducting magnets. Several tens of short (one meter) and full-scale (six meter) magnets have been designed, manufactured and tested at IHEP. Fig.15 shows IHEP test facility for SC dipole magnets. As a result the design of the magnet with cold iron has been chosen for preserial production. A chain of these magnets will be assembled in the UNK tunnel together with the cryogenic system module for testing close to the real conditions.

Industrial superconducting wires are used in manufacturing the UNK magnets. As a result of joint efforts of IHEP and industry SC wires containing 8910 NbTi 6 micron filaments in copper matrix were produced. The diameter of the wire is 0.85 mm. The critical current density for it is equal to \(2.3 \times 10^5\) A/cm\(^2\) at 4.2 K in 5 T field.

Preparation job for the mass production of the SC magnets is in progress at IHEP. Fig.16 shows the installation of the cryogenic equipment for serial magnets test facility.

6. UNK Experimental Area

UNK Experimental Area includes Hadron Area, Neutrino Area and underground halls for experiments with internal and colliding beams. The layout of the UNK experimental area is shown on Fig.17.

The slow extracted proton beam from straight section 4 is transported to the first splitting station, where it is splitted into two parts. The proton beam for the Hadron Area is splitted once again into two parts at the second splitting station. Then these two beams are transported to the targets TH1 and TH2.
A wide spectrum of particle beams, including hadron beams of moderate and high intensity, electron and photon beams, polarized proton and antiproton beams, a hyperon beam, is planned for the Hadron Area.

Beam lines to Experimental Hall H1 (24 X 240 m²) are assigned to form high intensity hadron (up to $4 \times 10^9$ particles/s) and electron (up to $7 \times 10^8$ particles/s) beams, polarized proton beam from hyperon decays up to $5 \times 10^7$ particles/s with 40% average transverse polarization.

Beam lines to Experimental Hall H2 (18 X 350 m²) are assigned to form electron beam with maximum possible intensity up to $10^{10}$ e⁻/s and $\Sigma^-$-hyperon beam with intensity $10^7$ particles/s at 2700 GeV/c.

Beam lines to Experimental Hall H3 (24 X 300 m²) are assigned to form hadron beams of moderate intensity like $3 \times 10^3$ particles/s for two subsequently running experiments.

There exists a principle possibility to form low energy beams (50-300 GeV) by using the target TH2.

The construction of a universal Neutrino Area with wide and narrow neutrino beams, tagged neutrino beam and prompt neutrino beam (beam-dump) is planned at UNK.

The second part of the slow extracted or whole fast extracted proton beam is transported to one of the neutrino targets (TN1, TN2).

Neutrino flux with wide spectrum in the design beam line geometry and a system of lithium or quadrupole lenses is equal to $2 \times 10^{-3}$ particles/(m²*proton).

Neutrino fluxes with a narrow spectrum are produced by the monochromatized pion and kaon beams, which are provided by the separation of high intensity beams of secondaries within momentum interval from 500 to 2250 GeV/c. This gives a possibility to obtain a neutrino beam with a relative width of (7-20)% from pion decays and (3-6)% from kaon decays at the ±2.5% momentum spread of the proton beam. The expected background level is 1.5%, antineutrino contamination in the neutrino flux is about 0.5%.

An experimental area for the jet target investigations with UNK internal beam is being constructed in straight section SS3. A possibility to carry out experiments with the UNK-1 and UNK-2 beams in the 400-3000 GeV energy range has been foreseen.

In the straight sections SS2 and SS5 the areas for colliding beam experiments at 6 TeV are being constructed.
7. UNK First Priority Experiments.

In 1988 IHEP Scientific Coordinating Council has considered UNK Proposals and Letters of Intent and approved five of them:
1) "Investigation of spin effects at 400–3000 GeV using jet target on the UNK internal beam" (NEPTUN project, SERP-UNK-E-001);
2) "Experimental studies of gluon interactions and glueball production in the central region of hadron collisions at 500–3000 GeV at UNK" (GLUON project, SERP-UNK-E-002);
3) "Multiparticle Spectrometer. Heavy quark physics" (MPS project, SERP-UNK-E-003);
4) "The UNK colliding beam experiments using Universal Calorimetric Detector" (UOD project, SERP-UNK-E-004);
5) "Studies of soft hadron interactions on the UNK colliding beams using streamer chambers" (TSD project, SERP-UNK-E-005).

The layouts of the approved experiments are shown in Figs.18, 19, 20, 21 and 22.

In March 1989 the second workshop "Physics at UNK" took place at IHEP. During this workshop the status of the approved UNK experiments and new Proposals and Letters of Intent were presented. Among them proposals for experiments with neutrino beams, charged leptons, b-physics, tau neutrino, etc. A number of groups from USSR, Europe, USA and Japan have expressed their interest to participate in the UNK experiments.

In order to have an international opinion on the UNK experimental programme as a whole and proposals of the experiments as well as to work out the recommendations on these issues International UNK Experimental Programme Advisory Committee (UNK EPAC) has been organized at IHEP. The members of UNK EPAC are physicists from Soviet Institutes, JINR, CERN, DESY, Saolay, from USA and Japan.
Fig. 1. Overall view of GAMS-2000 set-up at IHEP accelerator.

Fig. 2. Layout of the Vertex Spectrometer (VES).
Fig. 3. Layout of the SPHYNKS set-up to search for exotic states.

Fig. 4. Layout of the PROZA set-up for polarization studies.
Fig. 5. Polarization $P(t)$ in the reaction $\pi^- p \rightarrow \pi^0 n$ at 40 GeV/c.

Fig. 6. Measured $\pi^0$ asymmetries on the polarized protons and deuterons at 40 GeV/c.
Fig. 7. General view of the IHEP–JINR Neutrino Detector.

Fig. 8. Layout of the Tagged Neutrino Facility at IHEP.
Fig. 9. The magnetic lattice of UNK.

Fig. 10. Status of UNK tunneling.
Fig. 11. View of the UNK tunnel ready for equipment installation.

Fig. 12. Cross section of the UNK regular structure.
Fig. 13. The full-scale 30 m long model of the UNK tunnel.

Fig. 14. The UNK-1 iron dipole magnets at IHEP.
Fig. 15. Preserial SC magnets test facility at IHEP.

Fig. 16. Preparation of the IHEP test facility for serial SC magnets.
Fig. 17. Layout of the UNK Experimental Area for fixed target physics.

Fig. 18. Layout of the NEPTUN set-up in the experimental hall.
Fig. 19. Diagram of the GLUON set-up to study the central production of glueballs at UNK.

Fig. 20. Layout of the Multiparticle Spectrometer (MPS) set-up.
Fig. 21. General view of the Universal Calorimetric Detector (UCD) for UNK collider.

Fig. 22. Layout of the Track Streamer Detector (TSD) to study soft hadron interactions with UNK colliding beams.
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3. The frozen wastes of Wierum

4. Typical house in Friesland
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