AN ANALYSIS OF THE BEAM-BEAM INTERACTION IN HIGH ENERGY $p\bar{p}$ ($pp$) COLLIDERS

DAVID NEUFFER
Physics Department, Texas A & M University, College Station, TX 77843

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Beam–beam interaction simulation results of Neuffer, Riddiford, and Ruggiero are discussed and analyzed. A simple resonance overlap model is developed to estimate the density of chaotic motion observed in 2-D simulations. Tune modulation results are also discussed, and a threshold for beam blowup is calculated using a resonance overlap criterion and compared with simulations. Implications for Superconducting Super Collider (SSC) stability are developed and long-time SSC beam–beam stability is predicted. It is found that SSC beam–beam stability requires a minimum value of the synchroton tune $\nu_s$, which is estimated as $\nu_s > 0.001$.

1. INTRODUCTION

Beam–beam interaction is an important process in $p\bar{p}$ (and $pp$) colliders and is believed to place fundamental limitations on collider performance. However, the intrinsic nonlinearity of this interaction has made understanding these limitations difficult. Reference 1 describes the understanding and speculation concerning beam–beam limitations in 1979, which included the speculation that long-time nonlinear instability in the beam–beam interaction may make useful luminosity impossible in $p\bar{p}$ colliders. Following that report, the author, in collaboration with A. Ruggiero and A. Riddiford, undertook a series of numerical studies of the beam–beam interaction in $p\bar{p}$ colliders. The results of these studies, henceforth referred to as the NRR studies, are buried in a series of laboratory reports and summarized in the national accelerator conference proceedings.2–15 The studies reached important conclusions concerning long-time stability and demonstrated the existence of nonlinear motion properties such as chaotic motion and modulation instability. Ruggiero is also publishing an excellent discussion of some aspects of the NRR simulations.16

The purpose of this paper is to review and reemphasize some of the important features of the NRR studies, to reassess the results in terms of simple resonance models, and to extrapolate the results to estimate beam–beam limitations for the next generation of hadron–hadron colliders, such as the Superconducting Super Collider (SSC).17

In a proton-antiproton ($p\bar{p}$) collider, such as the CERN $Sp\bar{p}S$18 and Fermilab’s Tevatron,19 the beams must remain stably in collision for a total of about 24 hours, which corresponds to $4 \times 10^9$ turns. For the SSC, a similar lifetime is required, but the larger circumference reduces the circulation requirement to 1 to $2 \times 10^8$ turns. Over such large time scales, it is possible that a long-time
instability, such as Arnol’d diffusion, may appear, disrupting collider performance. The NRR studies were undertaken to simulate these long-time, many-turn conditions and to search for instability conditions.

2. BEAM–BEAM SIMULATION PROCEDURE

The basic procedure in the simulations was to propagate a set of particle trajectories through many turns of the collider. Transport around one turn is simulated as the product of two matrix multiplications in both \((x\) and \(y\)) transverse dimensions:

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix}_{\text{after}} = \begin{pmatrix}
\cos(2\pi v_x) & \beta_0 \sin(2\pi v_x) \\
-1 & \beta_0 \\
\beta_0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-4\pi \Delta v \frac{F(x, y)}{\beta_0} & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y'
\end{pmatrix}_{\text{before}}.
\]

(1)

The first matrix represents the linear focusing transport about the ring and is determined by the transverse tunes \((v_x, v_y)\) and Courant–Snyder (C–S) betatron functions \(\beta_x, \beta_y\).\(^{20}\) We choose \(\beta_x = \beta_y = \beta_0 = 2\) m simulating \(p\bar{p}\) collisions. The second matrix is the nonlinear beam–beam interaction caused by passage through the electromagnetic field of the opposite beam bunch in a collision region. The opposite bunch is approximated by a round Gaussian-shaped beam of size \(\sigma\), which is unchanged from turn to turn so that

\[
F = \frac{1 - \exp\left[-\frac{(x^2 + y^2)}{2\sigma^2}\right]}{\left(\frac{x^2 + y^2}{2\sigma^2}\right)}.
\]

(2)

The beam–beam tune shift \(\Delta v\) determines the strength of the interaction, and we choose \(\sigma = 0.0816\) mm to duplicate \(p\bar{p}\) collider values.\(^{19}\) The simulation procedure approximates conditions in a \(p\bar{p}\) collider: zero-length, weak-strong 2-D collisions of round beams without synchrotron radiation. Collisions are weak-strong since the opposite strong beam is unchanged from turn to turn, as in the collisions of a lower-density \(\bar{p}\) bunch with protons, and they are zero-length in that collisions are approximated by a simple velocity change. (Bunches are short compared to \(\beta_0\).)

We note that in this representation of beam–beam collisions the parameters \(\sigma, \beta_0\) serve only to set the units of the description. The physically significant parameters are \(v_x, v_y\), the strength of the nonlinearity \(\Delta v\), and the nonlinear force shape function obtained through \(F(x, y)\).

We emphasize that the \(p\bar{p}\) collider as modeled here is a fully deterministic Hamiltonian system. Stochasticity observed in this system is intrinsic to the dynamics. This is unlike the case for \(e^+e^-\) collisions, analyzed elsewhere,\(^{21-23}\)
where the dynamics are dominated by randomness introduced by quantum fluctuations and the nonconservative process of radiation damping.

Tune modulation is an important process in \( p\bar{p} \) colliders and is simulated by changing the tunes from turn to turn following

\[
\begin{align*}
\nu_x &= \nu_{x0} + \Delta_x \sin \left( \frac{2\pi n}{N_x} \right), \\
\nu_y &= \nu_{y0} + \Delta_y \sin \left( \frac{2\pi n}{N_y} \right),
\end{align*}
\]

(3)

Where \( \Delta_x, \Delta_y \) are the modulation amplitudes, \( N_x, N_y \) are the modulation periods, and \( n \) is the turn number. In modulation simulations, NRR chose \( \Delta_x = \pm \Delta_y \) and varied \( N_m = N_x = N_y \). We note that \( N_m = 1000 \) is near the expected modulation frequencies of power supply ripple and synchrotron oscillations. \( \Delta_x, \Delta_y \) are chosen with \( \Delta \leq 0.01 \), in agreement with expected collider magnitudes.

In a typical NRR simulation, initial positions for a set of 100 trajectories are chosen randomly from a 4-D Gaussian phase space distribution determined by \( \beta_0 \) and \( \sigma \) and tracked through many thousands or millions of turns. Individual trajectories are inspected for significant changes, as well as the amplitudes for the particle set, the \( x \) and \( y \) emittances

\[
\epsilon_x = 6\sqrt{\langle x^2 \rangle \langle x'^2 \rangle}
\]

and the sum

\[
\epsilon_R = \sqrt{\epsilon_x^2 + \epsilon_y^2}.
\]

In previous publications and internal reports, NRR have reported results of these simulations for a variety of cases corresponding to different beam–beam collision conditions. In the following sections, we summarize some features of these results and relate them to theoretical models.

3. RESULTS OF LONG-TIME SIMULATIONS

Perhaps the most important results of the NRR studies is what was not observed: Long-time simulations of up to \( 1.2 \times 10^8 \) turns, with high accuracy \( (10^{-28}/\text{turn}) \), showed no intrinsic instability in the 2-D beam–beam interaction at collider parameters. This prediction of long-time intrinsic stability in \( p\bar{p} \) collisions was convincingly confirmed in the successful operation of the CERN collider. The speculation that Arnol’d diffusion, in which particle trajectories follow stochastic motion along many intersecting resonances to large amplitudes, may be a dominant limiting process proved unfounded. Three cases were explored in detail in this long-time mode:

**Case A:** \((\nu_x = 0.245, \ \nu_y = 0.245, \ \Delta \nu = 0.01)\) on the diagonal \( \nu_x = \nu_y \) with tune spread covering the \( \nu_x = \nu_y = 0.25 \) resonances.

**Case B:** \((\nu_x = 0.245, \ \nu_y = 0.12, \ \Delta \nu = 0.01)\) with low-order resonances associated with \( \nu_x = 0.25, \ \nu_y = 0.125 \).

**Case C:** \((\nu_x = 0.3439, \ \nu_y = 0.1772, \ \Delta \nu = 0.01)\) free of low-order resonances.
Cases A and C showed no changes in beam sizes (less than 0.01%) or particle trajectory amplitudes in over $1.2 \times 10^8$ turns. Case B did show small exchanges between $x$ and $y$ emittances ($<1\%$) associated with chaotic motion (see below). However, even this case showed no beam blowup on the $10^8$-turn scale. (See Refs. 3, 4, 11, and 16 for more detailed discussions of these results.) Chaotic motion may generate exponential beam growth. Even if an exponential beam blowup were occurring within the simulation error limits, it could still not endanger luminosity when extrapolated to the $10^9$-turn scale of the Tevatron/Sp$ar{p}$S.

The stability prediction for the present generation of hadron colliders (Sp$ar{p}$S, Tevatron), with $\sim 10^9$ turns per cycle, requires some extrapolation from the NRR results at $10^8$ turns. However, the simulations are at precisely the same scale as the SSC. We can expect that beam–beam collisions in the SSC will be intrinsically stable, with perhaps greater confidence than in the present cases.

4. THE BEAM–BEAM INTERACTION IN THE RESONANCE APPROXIMATION

In this section, we follow usual techniques\textsuperscript{24} to approximate the beam–beam interaction by a resonance expansion. The resonance expansion is explicitly developed so that the notation and approximations are clear. The resonance approximation reduces the beam–beam interaction to a pendulum Hamiltonian, and the pendulum is a system commonly studied in nonlinear dynamics as one of the simplest nonlinear systems. Similar features should and do appear; the reader is cautioned, however, that an approximate analysis may mask significant differences.

The matrix multiplications of Eq. (1) are equivalent to integration of the following equations of motion:

$$
\begin{align*}
&x'' + k_x(s)x = -\frac{4\pi \Delta \nu}{\beta_0} \frac{1 - e^{-r^2/2\sigma^2}}{(r^2/2\sigma^2)} x \delta_p(s), \\
y'' + k_y(s)y = -\frac{4\pi \Delta \nu}{\beta_0} \frac{1 - e^{-r^2/2\sigma^2}}{(r^2/2\sigma^2)} y \delta_p(s),
\end{align*}
$$

where $\beta_0$, $\Delta \nu$ are the interaction region C–S betatron amplitudes and beam–beam tune shift, $r^2 = x^2 + y^2$, $\delta_p(s)$ is a periodic delta function representing the short-distance beam–beam crossing, and $k_x(s)$, $k_y(s)$ are periodic focusing functions representing the storage ring. The distance traveled along the ring, $s$, is the independent variable. The beam–beam interaction is that due to a beam with a Gaussian density

$$
\rho(r) = \rho_0 e^{-r^2/2\sigma^2} = \frac{N_{\text{tot}}}{2\pi \sigma^2} e^{-r^2/2\sigma^2}.
$$
The Hamiltonian corresponding to these equations of motion is

\[ H = \frac{1}{2}(p_x^2 + k_x(s)x^2) + \frac{1}{2}(p_y^2 + k_y(s)y^2) + \left(\frac{4\pi \Delta v}{\beta_0}\right) \sigma^2 U(x, y) \delta_p(s), \tag{6} \]

where

\[ U(x, y) = E_{in}(r^2/2\sigma^2) \]

is the potential function of the beam–beam force, and \( E_{in}(u) \) indicates the exponential integral function. Following the definitions in Lieberman and Lichtenberg, \( H \) is a nonautonomous Hamiltonian in two degrees of freedom.

The Hamiltonian can be transformed into action-angle coordinates by the generating function

\[ G(x, y, \phi_x, \phi_y) = -\frac{x^2}{2\beta_x} \left[ \tan \phi_x - \frac{\beta'_x(s)}{2} \right] - \frac{y^2}{2\beta_y} \left[ \tan \phi_y - \frac{\beta'_y(s)}{2} \right], \tag{7} \]

with

\[ \phi_x = \phi_{x0} + \int_0^s ds' \left( \frac{1}{\beta_x(s')} - \frac{v_x}{R} \right) \]

and where \( \phi_y \) is given by the same equation as \( \phi_x \), with the exchange of \( x \) for \( y \). After the canonical transformation and a change of independent variable from \( s \) to \( \theta = s/R \), we find

\[ H = v_x I_x + v_y I_y + \delta_p(\theta) A U(x, y), \tag{8} \]

with

\[ A = (4\pi \Delta v/\beta_0) \sigma^2 \quad \text{and} \quad x = \sqrt{2I_x/\beta_0} \cos \phi_x. \]

(Note that with round beams \( \beta_x = \beta_y = \beta_0 \).)

Expansion of the beam–beam interaction in a Fourier series gives:

\[ AU \delta_p(\theta) = \frac{A}{2\pi} \sum_{m,n,p} f_{mn}(I_x, I_y) e^{i(m\phi_x + n\phi_y - p\theta)}, \tag{9} \]

where

\[ f_{mn}(I_x, I_y) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\phi_x d\phi_y e^{-i(m\phi_x + n\phi_y)} E_{in} \left( \frac{r^2}{2\sigma^2} \right). \]

The average tune as a function of \( I_x, I_y \) can be obtained by ignoring the time varying part of Eq. (8):

\[ \bar{v}_x = \frac{\partial H}{\partial I_x} = v_x + \frac{A}{2\pi} \frac{\partial f_{00}}{\partial I_x} = v_x + \Delta v_x \left( \frac{I_x}{I_0}, \frac{I_y}{I_0} \right), \tag{10} \]

where \( I_0 = 2/\beta_0 \); \( g_x (= \Delta v_x/\Delta v_{\text{max}}) \) is a function of the amplitudes with values between 0 and 1 and is displayed graphically in Fig. 1. Note that

\[ g_x = g_y, \quad \text{for most} \ I_x, I_y; \]

\[ g_x \rightarrow 1, \quad \text{as} \ I_x \text{ and} \ I_y \rightarrow 0; \]

\[ g_x \rightarrow 0, \quad \text{as} \ I_x \text{ or} \ I_y \rightarrow \infty. \]
A resonance is obtained where

$$mv_x + nv_y = p,$$

with $m, n, p$ integers. In the resonance approximation, we keep only the lowest-order resonant terms of $H$:

$$H = v_x I_x + v_y I_y + \frac{A_{f_{00}}}{2\pi} + \frac{A_{f_{mn}}}{\pi} \cos (m\phi_x + n\phi_y - p\theta).$$

The time variable can be removed from Eq. (12) by transforming to new variables:

$$\Psi_x = \phi_x - v_{x0}\theta,$$

$$\Psi_y = \phi_y - v_{y0}\theta,$$

where $v_{x0}, v_{y0}$ are resonant tunes within the tune spread obeying Eq. (11), and $I_x$ and $I_y$ can be rotated to resonant and nonresonant components $I^+, I^-$ with the nonresonant $I^-$ decoupled.

The new resonant Hamiltonian is:

$$\hat{H} = \frac{1}{\sqrt{n^2 + m^2}} [m(v_x - v_{x0}) + n(v_y - v_{y0})]I^+ + \frac{1}{\sqrt{n^2 + m^2}} [n(v_x - v_{x0}) - m(v_y - v_{y0})]I^- + \frac{A_{f_{00}}}{2\pi} + \frac{A_{f_{mn}}}{\pi} \cos (\sqrt{n^2 + m^2}\psi^+).$$

FIGURE 1 Beam-beam tune shifts $\Delta \nu_x, \Delta \nu_y$ for individual particle trajectories with amplitudes $I_x, I_y$ for the bigaussian round beam used in Eq. 2. The limit $\Delta \nu_{\text{max}}$ corresponds to $\Delta \nu$ in Eq. 2.
where

\[ I^+ = \frac{m I_x + n I_y}{\sqrt{m^2 + n^2}}. \]

Resonant amplitudes \( I_x, I_y \) or \( I_0^+, I_0^- \) are found from the solutions of

\[ \frac{1}{\sqrt{n^2 + m^2}} [m(v_x - v_{x0}) + n(v_y - v_{y0})] + \frac{A}{2\pi} \frac{d f_{00}}{d I^+} = 0. \]  \( \text{(14)} \)

A resonance width can be obtained around these reference points by expanding the Hamiltonian, obtaining

\[ \tilde{H} \equiv \frac{A}{4\pi} \frac{\partial^2 f_{00}}{\partial I^+} (I^+ - I_0^+)^2 + \frac{Af_{nm}}{\pi} \cos (\sqrt{n^2 + m^2} \psi^+) + \text{constant}, \]  \( \text{(15)} \)

which is recognizable as the Hamiltonian of a pendulum. The resonance width (full width) is

\[ \Delta I^+ = 2\sqrt{\frac{8f_{nm}}{f_{00}''}} \]  \( \text{(16)} \)

where

\[ f_{00}'' = \frac{\partial^2 f_{00}}{\partial I^+}. \]

Resonance width decreases with increasing resonance order \((|m| + |n|)\) and also depends on the amplitudes \((I_{x0}, I_{y0})\) at which it is evaluated. For sample cases, we choose \( I_{x0} = I_{y0} \) within the center of the nonlinear tune spread at \( g_x = g_y = 1/2 \) (see Fig. 1). We find, numerically, that this occurs at \( I_x = I_y = 1.35I_0 \). In Table I we tabulate calculated values of the resonance integrals and widths in this case for even \( n, m \) with \(|n| + |m| < 8\). (These integrals are zero if \( n \) or \( m \) is odd.)

| TABLE I |

Resonance Parameters at \( I_{x0} = I_{y0} = 1.35I_0, \Delta \nu = 0.01 \)

| \( m \) | \( n \) | \( \frac{|f_{nm}|}{I_0} \) | \( \frac{|f_{00}'}{I_0} \) | \( \Delta I^+ \) | \( \omega_{x0} \) |
|---|---|---|---|---|---|
| 2 | 2 | 4th order | 0.27 | 0.103 | 2.88 | 0.0042 |
| 4 | 0 | | 0.014 | 0.065 | 2.60 | 0.0034 |
| 4 | 2 | | 0.0027 | 0.096 | 0.95 | 0.00202 |
| 4 | -2 | 6th order | 0.0027 | 0.034 | 1.59 | 0.0012 |
| 6 | 0 | | 0.00091 | 0.065 | 0.34 | 0.0013 |
| 4 | 4 | | 0.00036 | 0.103 | 0.33 | 0.00097 |
| 4 | -4 | | 0.00036 | 0.027 | 0.66 | 0.00005 |
| 6 | 2 | 8th order | 0.00024 | 0.079 | 0.31 | 0.00078 |
| 6 | -2 | | 0.00024 | 0.051 | 0.39 | 0.00063 |
| 8 | 0 | | 0.000064 | 0.065 | 0.18 | 0.00046 |
| 6 | 4 | 10th order | 0.000021 | 0.100 | 0.082 | 0.00030 |
| 10 | 0 | | 0.0000015 | 0.065 | 0.027 | 0.00009 |
In Refs. 5 and 9, NRR reported that chaotic motion occurs at the intersection of two independent resonances. The Hamiltonian at another resonance,

\[ j\nu_{x0} + k\nu_{y0} = p, \]

is the same as Eq. (11) with new resonance variables \( (J^+, \Psi_{jk}^+) \). We can choose \( \nu_{x0}, \nu_{y0} \) at the intersection of the two resonances and expand the Hamiltonian about the intersection to obtain a coupled pendulums Hamiltonian about the resonance amplitudes \( I_{x0}, I_{y0} \).

5. CHAOTIC MOTION IN THE 2-D BEAM–BEAM INTERACTION

NRR determined that large numbers of chaotic trajectories can occur in simulations of the 2-D beam–beam interaction. The definition of chaotic motion which we use is developed by Sinai in his textbook on ergodic theory.\(^{26}\) According to that definition, chaotic trajectories diverge exponentially in distance from nearby trajectories. From this property, they cannot be distinguished from trajectories with a random component, even though the dynamics are fully deterministic. The exponential growth of distances is measured by the Lyapunov exponent; distances grow following

\[ \delta = \delta_0 e^{aN}, \]

where \( \delta_0 \) is a constant depending on the initial separation from a nearby trajectory and \( a \) is the Lyapunov exponent. Chaotic trajectories have nonzero Lyapunov exponents.

The numerical method developed by NRR to distinguish chaotic trajectories is closely related to the ergodic theory definition and is a repeatability test. In this test, an individual particle trajectory is tracked forward in time for many turns. The motion is then reversed by reversing the velocity and the transport matrices determining the motion. Forward and return positions are compared, and the 4-D phase space distance is calculated as a function of the number of turns.

Figure 2 shows results of reversibility tests for sample trajectories of Case B \( (\nu_x = 0.245, \, \nu_y = 0.12, \, \Delta v = 0.01) \), with \( 6.0 \times 10^7 \) turns forward and return \( (1.2 \times 10^8 \) turns total). Initial and final positions agree to 14 decimal places in a double precision test \( (10^{-28} \) error in one turn). Similar results are obtained in our simulations for all nonchaotic “normal” trajectories, and this indicates the amount of accumulated error in our simulations. Reversability tests of some other trajectories show substantially different behavior, and a typical case is shown in Fig. 3. These trajectories, which we label “chaotic,” develop errors of order unity in a few tens of \( \text{thousands} \) of turns in an exponential manner.

In normal trajectories, the errors grow as simple powers of the number of turns \( N \);

\[ \delta = \delta_0 N^\alpha, \]

where \( \delta_0 \) is a single-turn error size \( (10^{-28}) \) and \( \alpha < 2 \) in our simulations. Chaotic
FIGURE 2 Results of reversibility tests for three nonchaotic orbits of Case B ($v_x = 0.245$, $v_y = 0.12$, $\Delta v = 0.01$). The three orbits originate at $x = y = 0.5a$, $1a$, $2a$ and $x' = y' = 0$. Forward and return orbits agree to $\sim 10^{-14}$ after 60 million turns.

FIGURE 3 Results of a reversibility test for a chaotic orbit of Case B. This orbit originates at $x = y = 1.5a$ and $x' = y' = 0$. Error increases exponentially to order unity after $\sim 15,000$ return turns, at which point all information concerning the initial orbit is lost. A clear difference between chaotic and nonchaotic orbits is obtained by comparing with Fig. 2.
trajectory errors grow exponentially

$$\delta = \delta_0 e^{aN},$$

where $a$ is identifiable with the Lyapunov exponent of the trajectory and depends on the details of the transformation. In a typical 2-D beam-beam case with $\Delta v = 0.01$, $a = O(10^{-3})$.

In Case B, $\sim20\%$ of randomly selected trajectories are chaotic and develop exponentially growing errors with Lyapunov exponents in the range of $10^{-3}$ to $10^{-4}$ per turns. The remainder are nonchaotic, with $a < 10^{-9}$; no intermediate cases appear. The repeatability test, or some similar numerical measurement of the Lyapunov exponent, is necessary to determine chaotic motion in this 2-D motion (4-D phase space). Unlike 1-D motion, 2-D chaotic motion does not develop recognizably random patterns in phase space projections. Chaotic orbits do appear near nonchaotic ones. Figure 4 shows phase space projections of some
FIGURE 5 Summary of the results of searches for chaotic motion, where $v_x$, $v_y$ are varied. The numbers indicate the number of chaotic trajectories (out of 100) as a function of the fractional tunes; $v_x$, $v_y$ are chosen near the visible resonance crossings. In each case $\Delta v = 0.01$ is taken, and $v_x$, $v_y$ are chosen such that the crossing is in the center of the tune spread ($v_x = v_{x0} - 0.005$, $v_y = v_{y0} - 0.005$).

Chaotic and nonchaotic initial conditions, indicating the difficulty of geometrical separation. However, the repeatability test easily separates them.

NRR reported results of a systematic search for chaotic motion in 2-D simulations\textsuperscript{10,12} as functions of $v_x$, $v_y$, $\Delta v$, and we now summarize these results. Chaotic motion occurs when the tune spread $\Delta v$ contains the intersection of low-order resonances. A resonance is determined by a relationship between the tunes:

$$mv_x + nv_y = p,$$

where $m$, $n$, and $p$ are integers and our symmetry requires that $m$ and $n$ be even (or zero) for a nonzero resonance width. The order $\Omega$ of the resonance is $\Omega = |m| + |n|$.

The dependence of chaotic motion on resonance order is shown in Fig. 5 (see also Ref. 16). In each case, the resonance intersection is placed in the center of
the tune spread, and 100 trajectories are tested for chaotic motion by a reversibility test. The results are:

(i) Intersections of fourth-, and sixth-order resonances show large regions of chaotic motion (10–30%).
(ii) Intersections of fourth- or sixth-order with eighth-order also show some chaotic motion (<5%).
(iii) Higher-order intersections show little or no chaotic motion (<1%).
(iv) Cases with \( v_x = v_y \) (or \( v_x = v_y \pm 0.5 \)) show no chaotic motion. As shown by Ruggiero,\(^{27}\) these cases have a kinematic invariant (the angular momentum \( p_0 = xy' - yx' \), in these cases) which, with energy invariant, makes the motion integrable.\(^{28}\) This reduces the motion to 1-D with diminished chaotic behavior.

Dependence on \( \Delta v \) has also been explored by varying \( \Delta v \) from 0.005 to 0.02, while keeping the location of the resonance within the center of the tune spread. The density of chaotic motion shows no dependence on \( \Delta v \); however, the mean Lyapunov exponent \( \alpha \) is directly proportional to \( \Delta v \).

The 2-D motion shows no amplitude instability, even in a very long time scale simulation (120 million turns). This indicates that the appearance of chaotic motion need not lead to amplitude instability.

As reported by Chirikov,\(^{29}\) a resonance overlap criterion can predict the existence of chaotic motion. In this paper, we observe that the overlap criterion can be extended to obtain a quantitative measurement of the region of chaotic motion in the beam–beam simulations. The size of this chaotic region can be estimated by calculating this overlapping area in \( I_x, I_y \) space:

\[
\text{Chaotic area} = \Delta I^+ \Delta J^+ |\cos(U_y)|, \tag{17}
\]

where

\[
|\cos U_y| = \frac{|mk - nj|}{\sqrt{m^2 + n^2} \sqrt{j^2 + k^2}}
\]

depends on the relative orientation of the resonances in \( I_x, I_y \) space (\( \cos U_y = 1 \) if the resonances are orthogonal), and \( \Delta I^+ \), \( \Delta J^+ \) are the full widths of the two resonances. The probability that a randomly selected trajectory is chaotic can be estimated by the expression

\[
P(\text{chaotic}) = f(I_{x0}, I_{y0}) \Delta I^+ \Delta J^+ |\cos U_y|, \tag{18}
\]

where \( f(I_{x0}, I_{y0}) \) is the central probability density.

In Refs. 9 and 10, NRR report the probability of chaotic motion for various cases of resonance intersections in simulation. For these cases, \( I_{x0} = I_{y0} = 1.35I_0 \) and

\[
f(I_x, I_y) = \frac{\exp \left[ -\frac{(I_x + I_y)}{I_0} \right]}{I_0^2}
\]

In Table II we compare these simulation results in which chaotic motion in randomly selected trajectories is observed with the calculated probability of chaotic motion (Eq. (18)) for various resonance intersections. Qualitative and quantitative agreement is good, indicating that the above estimation method is
fairly accurate. The results show all distinguishable crossings of fourth- and sixth-order resonances, eliminating cases with \( \nu_x = \nu_y \), which are integrable.

Eighth-order resonances have widths above one-third that of sixth-order ones at our parameters, and therefore should show about one-third as much chaotic motion in intersections with fourth- or sixth-order resonances as do corresponding sixth-order intersections. Simulation results show this same dependence (~1–5% chaotic motion), in agreement with the theoretical model. Higher-order resonances have still smaller widths and should produce very little chaotic motion, in agreement with simulations. The agreements are in fact fortuitously close. In the model, we have ignored higher-order resonances, which would increase the chaotic region and regions of nonchaotic motion within the resonance overlap ("islands of stability"). The errors have canceled.

An important parameter in chaotic motion is the rate of divergence of adjacent trajectories. This is given by the Lyapunov exponent \( a \). In a beam–beam resonance, the equations of motion are

\[
\frac{d\psi^+}{d\theta} = \frac{A}{2\pi f''_{\infty}} I^+, \\
\frac{dI^+}{d\theta} = \frac{A}{\pi} f_{nm} \sqrt{n^2 + m^2} \cdot \sin (\sqrt{n^2 + m^2} \psi^+),
\]

which gives a small oscillation frequency near fixed points (where \( I^+ = J^+ = 0 \)) of \( \omega_{s0} \), with

\[
\omega_{s0} = \sqrt{\frac{A^2}{2\pi^2} (n^2 + m^2) f''_{\infty} f_{nm}}.
\]
and is also tabulated in Table I for the low-order resonances. For the chaotic trajectories of Case B \((v_x = 0.245, v_y = 0.12, \Delta v = 0.01)\), the mean Lyapunov exponent was

\[ a = 0.00037 \]

in the present units. The intersecting low-order resonances have \(\omega_{x0} = 0.0034, 0.002, \) and \(0.0012\). The exponent \(a\) is of the same magnitude but somewhat smaller. This indicates that the rate of chaotic motion is at the same frequency scale as the intersecting resonances.

In Ref. 10 NRR showed that \(a \propto \Delta v\) in simulations. This may be expected, since \(\omega_{x0} \propto \Delta v\) in Eq. (21). In the same simulations, the probability of chaotic motion was independent of \(\Delta v\). This is in agreement with this resonance overlap model, since resonance widths in the beam–beam interaction are independent of \(\Delta v\).

An important result in the NRR simulations is that the appearance of chaotic motion does not lead to rms amplitude growth, even in simulations of greater than \(10^8\) turns. This indicates that the chaotic motion remains confined to the intersecting resonance region at moderate amplitudes. The motion does not diffuse to higher amplitude resonances, as would occur in Arnol’d diffusion.

We now summarize the important results reported in this section for 2-D chaotic motion in the beam–beam interaction:

(i) A reversibility test on particle trajectories can be used to establish the existence of chaotic motion.

(ii) Chaotic motion does occur in the 2-D beam–beam interaction much more easily than in 1-D simulations. The chaotic motion occurs at the intersection of low-order resonances.

(iii) A resonance overlap criterion can be used to estimate the phase space volume of chaotic motion. The frequency scale of the chaotic orbits is the same as that of the overlapping resonances.

(iv) Amplitude instability need not accompany chaotic motion and does not at 2-D collider parameters \((\Delta v = 0.01, \) round beams\). Future research could usefully explore this relationship in greater detail, extending these parameters.

6. TUNE MODULATION AND THE BEAM–BEAM INTERACTION

NRR completed several studies of the beam–beam interaction with tune modulation, and we summarize here some results of these simulations. NRR found that tune modulation can lead to large numbers of chaotic trajectories and to beam blowup if the modulation carries the beam across low-order resonances.\(^9,11,14,16\) Figure 6 shows the modulation in a case \((v_x = 0.3439, v_y = 0.1772, \Delta v = 0.01, \Delta_x = -\Delta_y = 0.005)\) which carries the beam across sixth-order resonances. At \(N_m = 1000, \sim 50\%\) of the orbits are chaotic and a fast beam blowup occurs with a mean growth time of \(\sim 250,000\) turns. This growth time is about the same magnitude as the Lyapunov exponents of the chaotic orbits,
FIGURE 6 The region in tune space swept by the beam in the case \( v_x = 0.3439, v_y = 0.1772, \Delta v = 0.01, \text{ and } \Delta x = -\Delta y = 0.005 \). Crossings of two low-order resonances are shown.

indicating in this case a direct correlation between chaotic motion and beam blowup. This is unlike the 2-D motion discussed in the previous section.

Figure 7 shows initial particle positions in tune space \( \Delta v_x, \Delta v_y \) (compare with Figure 1). Three different types of motion are observed:

(i) Nonchaotic trajectories which show no change in particle amplitudes (large \( \Delta v_x, \Delta v_y \)).

(ii) Chaotic trajectories which show no rms growth in particle amplitudes. These are concentrated at intermediate amplitudes.

(iii) Chaotic trajectories which grow to large amplitude. These particles which show amplitude instability have initial amplitudes at relatively larger values.

The instability observed here is limited to trajectories which are initially at large amplitudes, a condition which corresponds in a collider to the loss of the halo of the beam. This same type of beam loss is empirically observed at the onset of instability in actual collider operation, suggesting a correlation between simulation and operational instability conditions.

The case has been studied in greater detail by varying the modulation period \( N_m \) from 8 to 100,000. Some of the results are displayed in Fig. 8. No chaotic motion was seen for \( N_m < 32 \). For \( 32 < N_m < 200 \) a few chaotic trajectories appear without beam blowup. For \( N_m > 300 \), chaotic motion accompanied by “fast” beam blowup occurs. As \( N_m \to \infty \), the existence of chaotic motion with emittance
growth appears to persist. The fraction of phase space containing chaotic motion remains approximately constant but the Lyapunov exponents and the emittance growth rates decrease as $N_m \to \infty$. (See Refs. 9, 11, 14, and 16 for detailed discussions of the NRR simulation results, including a description of the dependence on $\Delta$.)

Our resonance model can be extended to obtain a qualitative understanding of the NRR results. The Hamiltonian can be modified to include tune modulation by the substitution [see Eq. (3)]

$$v_i \to v_i + \Delta_i \cos (v_i \theta + \phi_i),$$

where $v_s$ is the modulation frequency, $\Delta_i$ is the modulation amplitude, and $\phi_i$ is the phase. To simplify discussion, we choose $\phi_x = \phi_y = 0$ and $v_{sx} = v_{sy} = v_s$. The single resonance Hamiltonian [Eq. (13)] becomes

$$H = \frac{I^*}{\sqrt{m^2 + n^2}} [\delta v_{mn} + \Delta \cos (v_s \theta)] + \frac{A_{f0}}{2\pi} + \frac{A}{\pi} f_{mn} \cos (\sqrt{n^2 + m^2} \psi^*),$$

(22)
where $\Delta = m\Delta_x + n\Delta_y$ and $\delta v_{mn} = m(v_x - v_{x0}) + n(v_y - v_{y0})$, and where we have removed $I^-, \Psi^-$ from the Hamiltonian as nonresonant. We choose a new variable

$$\phi^+ = \psi^+ - \frac{\Delta \sin (v_s \theta)}{v_s \sqrt{n^2 + m^2}},$$

which removes the time-varying term from the first term of Eq. (22), and change the cosine term to

$$\cos \left( \sqrt{n^2 + m^2} \phi^+ + \frac{\Delta}{v_s} \sin v_s \theta \right) = \sum_{k = -\infty}^{\infty} J_k \left( \frac{\Delta}{v_s} \right) \cos \left( \sqrt{n^2 + m^2} \phi^+ + kv_s \theta \right). \quad (23)$$

We have exchanged our single-resonance Hamiltonian for one containing an infinite number of subresonances (see Courant31 for a similar treatment of a different case):

$$H = \frac{I^+}{\sqrt{m^2 + n^2}} \delta v_{mn} + \frac{Af_{00}}{2\pi} + \frac{Af_{mn}}{\pi} \sum_k J_k \left( \frac{\Delta}{v_s} \right) \cos \left( \sqrt{n^2 + m^2} \phi^+ + kv_s \theta \right). \quad (24)$$

The subresonances are spaced $v_s/\sqrt{m^2 + n^2}$ apart in tune, with their central amplitudes, found from the solution of

$$\frac{(\delta v_{mn} + kv_s)}{\sqrt{m^2 + n^2}} + \frac{Af_{00}}{2\pi} = 0, \quad (25)$$
space in amplitude by

$$\delta I^+ = \frac{2\pi \nu_s}{Af_0'' \sqrt{m^2 + n^2}}.$$  \hfill (26)

If the resonance width is greater than the resonance spacing, then the Chirikov overlap criterion is satisfied,\textsuperscript{14} and we may expect stochastic motion and particle trajectories which travel from resonance to resonance. This overlap criterion is

$$\Delta I^+ = 2 \left[ \frac{8f_{nm}J_k(\frac{\Delta}{\nu_s})}{f''_0} \right]^{1/2} \geq \frac{2\pi \nu_s}{Af_0'' \sqrt{m^2 + n^2}}$$  \hfill (27)

or (with $A = 4\pi \Delta \nu$)

$$\nu_s \leq 8 \Delta \nu \left[ \frac{2f_{nm}f''_0(n^2 + m^2)J_k(\frac{\Delta}{\nu_s})}{\nu_s} \right]^{1/2}.$$  \hfill (28)

Absolute values are taken in all expressions.

Similar overlap criteria have been developed in other concurrent studies of beam–beam interaction.\textsuperscript{20,29} We have extended the analysis to two dimensions and directly applied it to the NRR simulation results.

In Ref. 11 NRR reported simulations of beam–beam interaction with tune modulation in which $\nu_s$ was varied while the other parameters ($\nu_x, \nu_y, \Delta \nu, \Delta_x$) were fixed at values (0.3439, 0.1772, 0.01, 0.005) which swept the beam across sixth-order resonances. Figure 6 shows the modulation graphically, indicating the resonances which are crossed. Figure 8 shows the dependence of modulation chaotic motion on modulation period. It was found that no chaotic motion occurs for modulation period $N_m = 1/\nu_s < 32$ turns. For $32 < N_m < 100$ a few chaotic trajectories appear, while for $N_m > 300$ ($\nu_s < 0.003$), large numbers of chaotic trajectories appear and beam blowup (rms amplitude increase) occurs. We can estimate the threshold in $\nu_s$ for this case, using parameters obtained from Table I in Eq. (28). With $\Delta = 0.01$, $f_{nm} = 0.0027$, $f''_0 = 0.065$, and $n^2 + m^2 = 2^2 + 4^2 = 20$ (and nothing that for $\Delta/\nu_s$ small, $J_k(\Delta/\nu_s)$ is maximum for $k = 0$ with a value of $\sim 1$), we obtain $\nu_s = 0.0067$ ($N_m = 150$) as the threshold modulation frequency for resonance overlap, which is in reasonable agreement with the beam blowup threshold observed in simulations. A more accurate evaluation of $J_k$ and the other parameters obtains still closer agreement.

Another simulation result is that there is no lower limit in $\nu_s$ for stochastic beam blowup, although the magnitude of Lyapunov exponents decreases. This result can be explained in terms of the resonance overlap criterion, noting first that

$$J_k(\Delta/\nu_s) \approx \sqrt{2\nu_s/\pi \Delta}, \quad \text{as} \quad \nu_s \to 0.$$

Thus, both subresonance width ($\propto \nu_s^{1/4}$) and subresonance spacing ($\propto \nu_s$) decrease as $\nu_s \to 0$. However, since the spacing decreases more rapidly, the overlap criterion (width > spacing) remains satisfied as $\nu_s \to 0$ ($N_m \to \infty$). However, the blowup rate and the Lyapunov exponents do decrease as the modulation
frequency $v_s \to 0$. Both of these are approximately proportional to $v_s$ as $v_s \to 0$. Figure 8 shows the blowup rate as a function of $N_m = 1/v_s$, and Fig. 9 displays the mean Lyapunov exponent of chaotic trajectories.

An important difference between modulational and 2-D chaotic motion is that the modulational motion leads to amplitude instability, while 2-D motion does not. This can be qualitatively understood in terms of the resonance overlap criterion by comparing Eqs. (24) and (25), from which modulational overlap is developed, with a two-resonance version of Eq. (13). Modulational overlap occurs over an infinite family of subresonances, all centered at different amplitudes, and stochastic motion among these resonances leads to a diffusion like increase in rms amplitudes. Resonance overlap in the 2-D case occurs only between two resonances centered at similar amplitudes and empirically does not feed up into higher-order, different-amplitude resonances.

We can now summarize some of the important results for modulation with the beam-beam interaction:

(i) Chaotic motion with beam blowup can occur if the modulation carries the beam across low-order resonances.

(ii) A resonance overlap criterion can be developed to estimate the threshold in modulation frequency for chaotic motion with beam blowup. The NRR simulation results agree with this overlap criterion.
(iii) The time scale of chaotic motion and beam blowup is consistent with the
resonance frequencies modified by the modulational frequencies.

The reader should not conclude that the NRR results and the overlap model
constitute a complete understanding of modulational effects. The six-dimensional
space \((v_x, v_y, \Delta v, \Delta x, \Delta y, v_z)\) of tune modulation is only partially explored. More
complete analyses and simulations may discover other significant effects and more
accurate measurements of stability conditions.

7. APPLICATIONS TO THE SSC

The basic properties of the beam–beam interaction in the SSC are the same as
that of the NRR simulations: collisions of (nearly) round bunched beams with
\(\Delta v < 0.01\). The SSC has some differences, and it is important to consider the
significance of these differences before developing performance limitations from
analysis of the NRR results:

(i) The SSC is a pp rather than \(\bar{p}p\) collider. This changes the sign of the
beam-beam force. However, a simultaneous change in sign of \(x', y',\) and
\(v_x, v_y\) [or \(v_x \rightarrow (1 - v_x), \ v_y \rightarrow (1 - v_y)\)] leads to identical transformations in
Eq. 1, so after redefinition of \(v_x, v_y,\) the NRR results apply directly to the
pp collider.

(ii) The simulations are explicitly "weak-strong" interactions and do not
precisely duplicate the "strong–strong" collisions of a high-luminosity SSC
pp collider in which both beams are equally affected. In our analysis, we
assume that the total strength of the nonlinearity is the dominant
consideration. Collective strong–strong effects are a topic suitable for
future study.

(iii) The NRR results were obtained with collisions of round beams. The SSC
may choose to collide "flat" beams instead. In the comparison, we choose
round beams, expecting that overall nonlinearity strength \((\Delta v)\), rather
than the beam shape \([F(x, y)]\), is the dominant consideration. This
requires some simulation and experimental confirmation.

(iv) The SSC will have beams with fairly closely spaced bunches crossing at an
angle. In addition to interaction-point head-on collisions, a number of
near collisions will also occur. This could be significant. We require that
the strength of the nonlinearity of near-collisions be much less than the
head-on beam–beam tune shift:

\[\Delta v_{\text{nc}} \ll \Delta v_{\text{B-B}}\]

to insure that the nonlinearity is dominated by the normal beam–beam
effect. This criterion may require further investigation.

(v) Both SSC and \(\bar{p}p\) colliders require luminosity lifetimes of about one day.
However, the larger size of the SSC means that this corresponds to far
fewer turns \((\sim 10^8\) rather than several times \(10^9\)). The SSC should
therefore have superior long-term stability and be less sensitive to ultralong-time-scale effects.

The NRR results directly show that there is no long-time-scale intrinsic nonlinear instability in the beam–beam interaction at SSC parameters. The time scale of the SSC ($10^8$ turns) has been directly explored in the NRR simulations.

The NRR simulations show an important dependence of particle behavior in nonlinear motion on initial position. Simulations testing SSC stability which are based on testing only a few initial particle positions may completely miss important nearby regions of instability. Particle trajectories which appear to be exploring the entire $xy$ plane are in fact confined to a small region in 4-D (or 6-D) phase space. Exploration of a fairly large number of initial conditions is necessary to determine SSC stability in nonlinear motion, whether the nonlinearity is due to multipole fields or the beam–beam interaction.

The SSC is expected to experience tune modulation, and the analysis of the previous section may be applied to estimate limitations. Tune modulation sources include synchrotron oscillations and power supply ripple. The SSC synchrotron frequency will be of the order of $v_s = 0.001$, while power supply ripple should occur at harmonics of 60 Hz, or about $v_s = 0.01$.

As described above, instability is expected if modulation subresonances overlap. This occurs if

$$v_s \leq 8\Delta \nu \sqrt{2f_{nm}f_{00}^{\ast}J_k(\Delta/v_s)(n^2 + m^2)}$$

In the limit where $v_s$ is small, this can be rewritten as

$$(v_s)^{3/4} \leq 8\Delta \nu \left[2(n^2 + m^2)f_{nm}(I)f_{00}^{\ast} \sqrt{\frac{2}{\pi\Delta}} \right]^{1/2}$$

This sets a lower limit on $v_s$; slow oscillations are unstable. A possible strategy for insuring stability for a given $v_s$ is to exclude all resonance orders less than the overlap threshold order from the tune spread. To indicate the scaling of this criterion, we display calculations of the threshold $v_s$ with resonance order for $v_s = 0.001$ and with $f_{nm}$, $f_{00}$ evaluated at $\sim 2\sigma$ amplitudes (see Table III). We find that at $v_s = 0.001$ avoidance of eight and lower orders is desired. This criterion is moderately imprecise; $f_{nm}$ is a sensitive function of amplitude, and the overlap criterion is imprecise. The frequency scale, however, does indicate that power supply ripple can not feed this instability in the SSC, unlike the $Sp\ddot{p}S$, but that synchrotron oscillations are significant, as in the $Sp\ddot{p}S$.

Following a similar analysis and comparison with $Sp\ddot{p}S$ results, Evans and Gareyte$^{29}$ indicate that avoidance of tenth-order resonances ($\Delta \nu_T < 0.025$, where $\Delta \nu_T = 6\Delta \nu$ is the total tune shift per turn) is necessary for stable, $3 \times 10^8$ turn operation of the $Sp\ddot{p}S$. Our analysis of the NRR results indicates that an overlap threshold instability does occur as predicted by the overlap criterion. Comparisons with SSC parameters obtain similar thresholds but at a higher level due to the shorter lifetime ($10^8$ turns). Experimental results at the $Sp\ddot{p}S$ do indicate a lifetime of $>10^8$ turns at injection with a tune spread $\Delta \nu_T = 0.075$. (The tune
spread is dominated by space charge rather than beam–beam interaction at injection, and $v_r$ is larger.

In summary, this analysis indicates similar or superior stability for beam–beam interactions in the SSC, compared with $p\bar{p}$ colliders. The synchrotron tune $v_s$ should be greater than $\sim 0.001$ to insure stability with $\Delta v_T \approx 0.02$.

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