IONIZATION MEASUREMENT ON A SPARK CHAMBER TRACK

(Theory)

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1. INTRODUCTION

An energetic incident particle produces in a gas a certain number of electrons along its trajectory. The particle does not show any appreciable change in energy along the length under consideration. Hence the events (i.e. ionizations producing the "primary" electrons) can be considered as independent from each other, and therefore one has to expect a Poisson distribution for the production loci of the primary electrons. A primary electron might possess a kinetic energy. If this energy exceeds the ionization energy of the atoms of the chamber gas, the primary electrons have the possibility of producing further so-called secondary electrons. The average number of secondary electrons produced in this way is at least a factor of three larger than that of the primary electrons. If the energy of a primary electron is higher than 10 keV, it will have a range of about 1 mm in a gas of 1 atm. The secondary electrons, which form a so-called $\delta$ ray can be easily distinguished from electrons located along the trajectory of the energetic particle. However, if the energy of a primary electron is much smaller than 10 keV, its associated secondary electrons are located in the vicinity of the production locus of the primary one, and it will become difficult to distinguish them from primary electrons of the trace. If the distance between the production locus of a primary electron and its secondary electrons is small compared with the average distance of primary electron production loci, the following idealized mathematical picture describes the situation:

Immediately after the passage of the energetic particle, the electrons along its trajectory are "Poisson" distributed. A
single "event", however, might be composed of several events (primary electron and its associated secondaries). Due to diffusion of the manifold events, the Poisson distribution will be perturbed.

2. DISTRIBUTION OF ELECTRONS ALONG THE TRAJECTORY AS A FUNCTION OF TIME

The probability $W(x,t)$ for the occurrence of a "distance" $\geq x$ is to be evaluated. This function can be checked experimentally and used to determine the ionization of the incident particle ("distance" should be understood as the projection in trajectory direction of the distance between two adjoining events).

According to Fig. 1, an interval $L$ (of the trajectory) is considered in which is embedded the interval $B = x$.

![Diagram](image)

We are looking for the probability that no electron will be left after the time $T = t$ within the interval $B$, if a certain number of primary and secondary electrons have been produced at $T = 0$ within the interval $L$.

This problem has been solved in detail elsewhere [see Schneider 1)] for a particular form of the differential cross-section $(\partial\sigma/\partial U)$. However, for the present purposes it is more convenient to leave this cross-section in the general form. Furthermore, by letting $t \to \infty$ one obtains a result which is valid for a long track.
The average number of primary electrons per unit of length $1/x_p$ is given by Eq. (I.1) of Ref. 1:

$$\frac{1}{x_p} = \int_{0}^{\infty} \frac{d\sigma}{dU} \, dU,$$

and the average total number of electrons per unit of length $1/x_t$ [Eq. (I.4)]:

$$\frac{1}{x_t} = \sum_{n=0}^{\infty} (n+1)p_n(U) \frac{d\sigma}{dU} \, dU$$

where $\frac{d\sigma}{dU}$ = primary differential ionization, whereby the electron has received the energy $U$;

$p_n(U)$ = probability for the production of $n$ secondary electrons from a primary one having the energy $U$.

With the help of Eqs. (II.1) to (II.6) [Ref. 1] one finds easily:

$$W(x,t) = \left[ \exp \left\{ -\frac{t}{x_p} + \int_{-\infty}^{\infty} \left( \sum_{n=0}^{\infty} p_n \left( 1 - \omega_1 \right)^{n+1} \frac{d\sigma}{dU} \, dU \right) \, dx_1 \right\} \right]_{t \to \infty}$$

where $\omega_1(x_1,t)$ represents the probability for the presence of one electron after the time $\tau = t$ within the interval $B$ if it has been produced at $\tau = 0$ at a point $x_1$ within $L$.

It is easily verified that:

for $t = 0$

$$\omega_1 = 1 \quad \text{if } x_1 \text{ is within } B$$

$$\omega_1 = 0 \quad \text{if } x_1 \text{ is outside of } B$$

and therefore:

$$W(x) = \exp \left\{ -\frac{x}{x_p} \right\};$$

for $t \to \infty$

$$\omega_1 \to 0$$
and therefore:

\[ W(x) \rightarrow \exp\left(-\frac{x}{x_t}\right). \]

For finite values of \( t \), \( \omega_1 \) is given by the free diffusion of the electron. If the electron temperature is constant one obtains simply:

\[ \omega_1 = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \exp\left[-(x_0 - x_1)^2 / 4D t\right] dx_0 , \]

and for \( x >> \sqrt{4Dt} \) it follows that:

\[ W(x,t) = \exp\left(-\frac{x}{x_p} - f(t)\right) \]

where \( f(t) \) is independent of \( x \), and finally for small distances with \( 1 >> x \cdot (1/\sqrt{4\pi D t}) \exp(-x_1^2/4D t) \) fulfilled:

\[ W(x,t) = \exp\left(-\frac{x}{x_t}\right). \]

A sketch of \( \ln W(x) \) for different \( t \) is represented in Fig. 2. The determination of the primary ionization by counting the events generally leads to a wrong result (the case \( t = 0 \) is the only exception). However, the logarithm of the distribution function is always a straight line for sufficiently large \( x \), so that the primary ionization may be determined from the slope. Also the total ionization is measurable with the help of the small distance distribution, if \( t \) is long enough.

3. DEGENERATION OF THE ORIGINAL DISTRIBUTION DUE TO OVERLAPPING OF AVALANCHES

A certain number of events are lost, if the distance \( \Delta \) between adjoining avalanches is smaller than their size. Assuming the photographic image of an avalanche is a disc of diameter \( D \), then two events can be distinguished from each other if \( \Delta > 0.4D \). In
order to calculate the degenerate distribution, we assume a Poisson distribution and introduce dimensionless quantities:

\[ \delta = \frac{A}{x_p} \text{ and } y = \frac{x}{x_p}. \]

A distance \( y \) between two events should be divided in intervals of length \( \delta \). In general, we will have:

\[ y = n \cdot \delta + \epsilon \] (n integer; \( 0 \leq \epsilon < \delta \); see Fig. 3).

\[ \delta \]
\[ \epsilon \]
\[ B \]
\[ D \]
\[ C \]
\[ A \]

\text{Fig. 3}

It is assumed that \( A \) and \( B \) are loci of events and that the scanning is done from left to right. The conditions to be fulfilled so that the distance \( AB = y \) will be recognized as the distance between two events are the following:

1) there is no event within the interval \( AB \);
2) there are events within the interval \( AC \), but the largest distance between two adjoining events is \( < \delta \);
3) no event is allowed to take place within \( DB \leq \delta \).

Taking into account these restrictions, the probability densities are:

for \( n = 0 \):

\[ \frac{\partial W_0}{\partial y} = 0 \]

for \( n = 1 \):

\[ \frac{\partial W_1}{\partial y} = e^{-(\delta+\epsilon)} + e^{-\delta}(1 - e^{-\epsilon}) = e^{-\delta} \]

for \( n = 2 \):

\[ \frac{\partial W_2}{\partial y} = e^{-(2\delta+\epsilon)} + e^{-(\delta+\epsilon)}(1 - e^{-\delta}) + (1 - e^{-\epsilon} - e^{-\delta})e^{-\delta} \]

\[ = e^{-\delta} - \epsilon e^{-2\delta} \]

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and generally:

\[ \frac{\partial W_n}{\partial y} = e^{-\delta} + e^{-2\delta} \sum_{i=1}^{n-1} \frac{(-\epsilon)^i}{i!} \sum_{j=0}^{i-1} \frac{\delta^j}{j!} - \epsilon \sum_{i=2}^{n-1} e^{-1\delta} \sum_{j=1}^{i-1} \frac{\delta^j}{j!} \]

and for:

\[ n \to \infty : \quad \frac{\partial W_n}{\partial y} \to e^{-\delta n} \left[ \frac{1}{e^{\delta n} - 1} \left( e^{\delta n} - 1 \right) \right] = \frac{\partial W_\infty}{\partial y} . \]

The following table shows the excellent agreement of this asymptotic approximation with the exact solution for \( n = 5 \).

<table>
<thead>
<tr>
<th>( \epsilon )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial W_5}{\partial y} )</td>
<td>1</td>
<td>0.6741</td>
<td>0.4598</td>
<td>0.2245</td>
<td>0.1197</td>
<td>0.0718</td>
</tr>
<tr>
<td>( \frac{\partial W_\infty}{\partial y} )</td>
<td>1</td>
<td>0.6741</td>
<td>0.4598</td>
<td>0.2246</td>
<td>0.1202</td>
<td>0.0737</td>
</tr>
</tbody>
</table>

One has to evaluate the integrals:

\[ \int_0^\delta \frac{\partial W_n}{\partial y} \, d\epsilon = I_n \]

and one obtains:

\[ I_0 = 0 \]
\[ I_1 = \delta e^{-\delta} \]
\[ I_2 = \delta e^{-\delta} - \frac{\delta^2}{2} e^{-2\delta} \]
\[ I_n = \delta e^{-\delta} - e^{-2\delta} \sum_{i=2}^{n} \frac{(-\epsilon)^i}{i!} \sum_{j=0}^{i-1} \frac{\delta^j}{j!} - (1 - e^{-\delta}) \sum_{i=2}^{n-1} e^{-1\delta} \sum_{j=1}^{i-1} \frac{\delta^j}{j!} \]
\[ I_\infty = \int_0^\infty \frac{\partial W_\infty}{\partial y} \, dy = e^{-(n+1)\delta} \cdot \frac{1}{e^{\delta n} - 1} \left( e^{\delta n} - 1 \right) , \]
and finally \( W(y) \) (probability for the occurrence of a distance larger than \( y \)) where \( y = n \cdot \delta \):

\[
W(y) = \sum_{i=m}^{n} I_i + I_{n+1}.
\]

In Fig. 4, \( \ln W(y) \) is represented for different overlapping coefficients \( \delta \). For \( x \) sufficiently large, the curves approach straight lines with a slope independent of \( \delta \). The final slopes are in agreement with that of the original Poisson distribution, namely \( 1/x_p \).

4. EVALUATION METHOD FOR THE PRIMARY IONIZATION

We have seen that in spite of the perturbing effects described above, the large gap distribution will be "Poisson-like". To determine the best fitting value for the primary ionization, we will apply the "Maximum Likelihood" method [see Fortune\(^2\)], but in a simplified form. [A comparison of our method with that used in Ref. 2) has shown that the curve fitting errors are the same.]

Assuming that for \( x \geq n \Delta x \) the distribution becomes sufficiently close to a Poisson distribution, then the probability for the occurrence of an event within a length \( i \Delta x \leq x \leq (i+1)\Delta x \) is given by:

\[
P_i = (1 - y) y^i
\]

where \( y = \exp \left( -\frac{x}{x_p} \right) \)

and the probability for \( n \Delta x \leq x \leq \infty \):

\[
P_n = y^n.
\]
The normalized probabilities are:

\[ p_i = (1-y)^{i-m} \]

\[ p_n = y^{n-m} \]

because:

\[ \sum_{i=m}^{n} p_i = y^m. \]

The so-called extended log likelihood function (logarithm of the joint normalized probabilities) is given by:

\[ L = \sum_{i=m}^{n-1} N_i \ln (1-y) + \sum_{i=m}^{n} (i-m)N_i \ln y + \text{const} \]

where \( N_i \) is the number of events in the interval \( i \Delta x \leq x < (i+1)\Delta x \).

Best fitting values and their errors are given by [see Orear\textsuperscript{3})]:

\[ \frac{\partial L}{\partial y} = 0 \rightarrow y^* = \frac{S_2}{S_1 + S_2}; \]

\[ \left( \frac{1}{x_p} \right)^* = -\frac{\ln y^*}{\Delta x} \]

\[ \Delta \left( \frac{1}{x_p} \right)^* \equiv \left( -\frac{\partial^2 L}{\partial \left( \frac{1}{x_p^2} \right)} \right)^{-1/2} \bigg|_{x_p = x_p^*} = \frac{1}{\Delta x} \sqrt{\frac{S_1}{S_2 (S_1 + S_2)}} \]

where:

\[ S_1 = \sum_{i=m}^{n-1} N_i ; \quad S_2 = \sum_{i=m}^{n} (i-m)N_i. \]

Preliminary measurements of the primary ionization carried out on tracks of cosmic-ray particles of about minimum ionization are found to be in agreement with cloud chamber measurements. The maximum likelihood error for a single 10 cm long track is about 10%.

Extended measurements on a velocity separated beam will be reported in the near future.
REFERENCES

1) F. Schneider: Primary and total ionization and the single track efficiency of the spark chamber, AR/Int. GS/63-7 (1963).

2) R.D. Fortune: The maximum likelihood method applied to the determination of the best value for the primary ionization in track chambers, CERN 62-31 (1962).
